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Vector space of a Graph

V. Manjula[#1],

Basic Engineering Department,

DVR& Dr. HS MIC College of Technology, Kanchikacherla
manju_adiraju@yahoo.co.in

Abstract

This paper was designed to provide an introduction to the vector spaces used in graph theory. In graph theory, an area of mathematics, a cycle space is a vector space defined from an undirected graph; elements of the cycle space represent formal combinations of cycles in the graph. Cycle spaces allow one to use the tools of linear algebra to study graphs. A cycle basis is a set of cycles that generates the cycle space. In the mathematical discipline of graph theory, the edge space and vertex space of an undirected graph are vector spaces defined in terms of the edge and vertex sets, respectively. These vector spaces make it possible to use techniques of linear algebra in studying the graph.

Introduction

Let $G := (V, E)$ be a finite undirected graph. The vertex space $\mathcal{V}(G)$ of G is the vector space over the finite field of two elements $\mathbb{Z}/2\mathbb{Z} := \{0, 1\}$ of all functions $V \to \mathbb{Z}/2\mathbb{Z}$. Every element of $\mathcal{V}(G)$ naturally corresponds the subset of V which assigns a 1 to its vertices. Also every subset of V is uniquely represented in $\mathcal{V}(G)$ by its characteristic function. The edge space $\mathcal{E}(G)$ is the $\mathbb{Z}/2\mathbb{Z}$-vector space freely generated by the edge set E. The dimension of the vertex space is thus the number of vertices of the graph, while the dimension of the edge space is the number of edges.

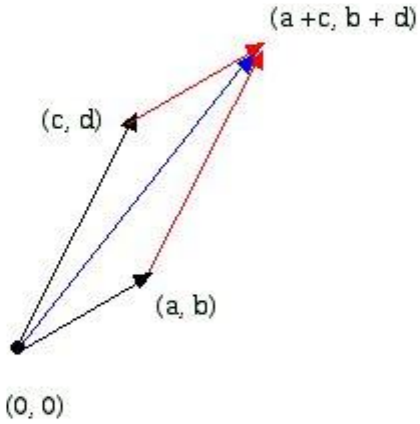
These definitions can be made more explicit. For example, we can describe the edge space as follows:

- elements of the vector space are subsets of E, that is, as a set $\mathcal{E}(G)$ is the power set of E
- vector addition is defined as the symmetric difference: $P + Q := P \triangle Q \qquad P, Q \in \mathcal{E}(G)$
- scalar multiplication is defined by:
  - $0 \cdot P := \emptyset \qquad P \in \mathcal{E}(G)$
  - $1 \cdot P := P \qquad P \in \mathcal{E}(G)$

The singleton subsets of E form a basis for $\mathcal{E}(G)$.

and $\mathcal{V}(G)$ is the power set of V made into a vector space with similar vector addition and scalar multiplication as defined for $\mathcal{E}(G)$.

We can represent velocities, which have a magnitude and direction, by line segments with arrows. In the diagram below, the two original vectors are shown in black. Vectors with the same magnitude and direction as these two are shown drawn in red. The sum of the original two vectors is found by taking the tail of one of the vectors and placing it at the head of the other vector.



A new vector, shown in blue, indicates the sum or resultant of the original two. This is the *parallelogram law* of vector addition which provides the foundation for a geometry of vectors.

Origin and growth

During the 19th century mathematicians and scientists were developing new tools for trying to maximize the way mathematics could be used to get insight into the concepts of location, velocity, and force. These ideas were the outgrowth of a long progression of mathematics set in motion by Newton's spectacular synthesis of using mathematical tools for the benefit of physics. Besides vectors, another natural place where vector spaces arise is in the theory of equations of first degree in several variables. First degree equations, the equations of straight lines in 2-space, of planes in 3-space and the analogues of lines and planes in spaces of higher dimension, are of interest for many reasons.

Cycle Space

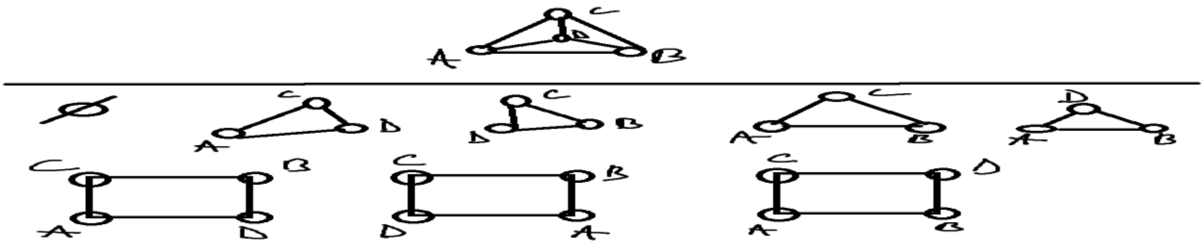
The Cycle Space of a Graph is a subspace of Edge Space. It is the smallest possible set of edge disjoint cycles and the null set .In order to show that a subset of a vector space is a subspace, it is necessary to show three conditions: the subset contains the zero-vector, closure under scalar multiplication, and closure under vector addition.

While Considering scalar multiplication the Edge Space is defined over is {0, 1},:

- $0 * \mathbf{x} = \{\}$, which shows that the zero-vector (the null set) is in Cycle Space
- $1 * \mathbf{x} = \mathbf{x}$, which is the identity.

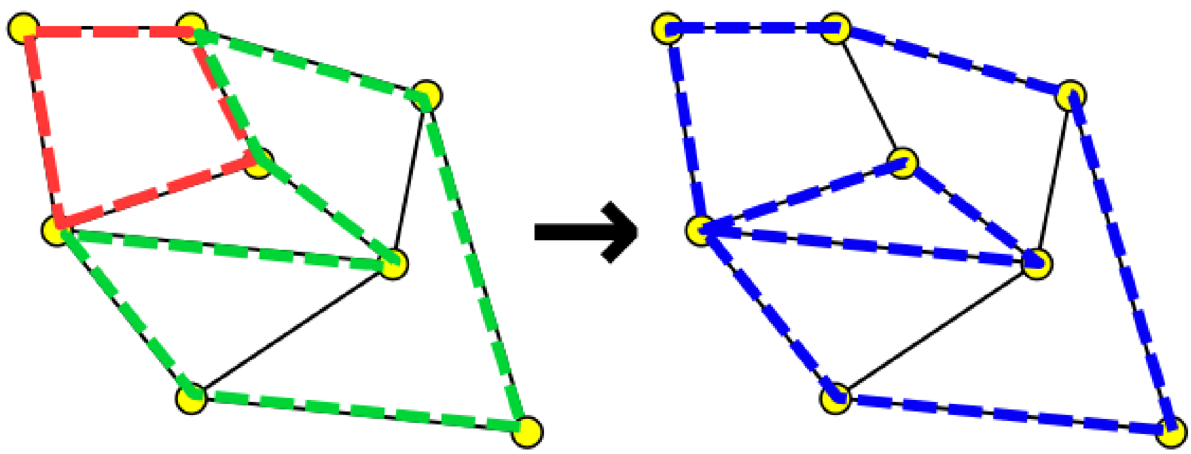
Thus, Cycle Space is closed under scalar multiplication. Now under vector addition, the operation of vector addition is defined as the symmetric difference of the two vectors since Cycle Space is a subspace of Edge Space. The symmetric difference operator will simply act as a union operator. However, if there are two vectors in the Cycle Space: $\mathbf{x_1}$ and $\mathbf{x_2}$, such that $\mathbf{x_1}$ and $\mathbf{x_2}$ share edges, then they also share cycles in common. Thus, the operation $\mathbf{x_1} + \mathbf{x_2}$ will remove the common edges and thus the common cycles. Thus, $\mathbf{x_1} + \mathbf{x_2}$ returns the union of two cycles that don't share edges.

Therefore, since Cycle Space contains the zero-vector (the null set), is closed under vector addition, and is closed under scalar multiplication, it is a vector space and subspace. Now that Cycle Space has been proven to be a subspace, for an example. Consider the Wheel graph on four vertices, the four instances of C_3, as well as the three instances of C_4. The C_4 graphs were obtained by taking the symmetric difference of each pair of the C_3 graphs. Since in simple graphs, no cycle can have fewer than 3 edges, the C_3 cycles are the starting points for this cycle space. As a vector space is a set of vectors, no duplicate cycles were included. Also W_4 is not in its own cycle space.



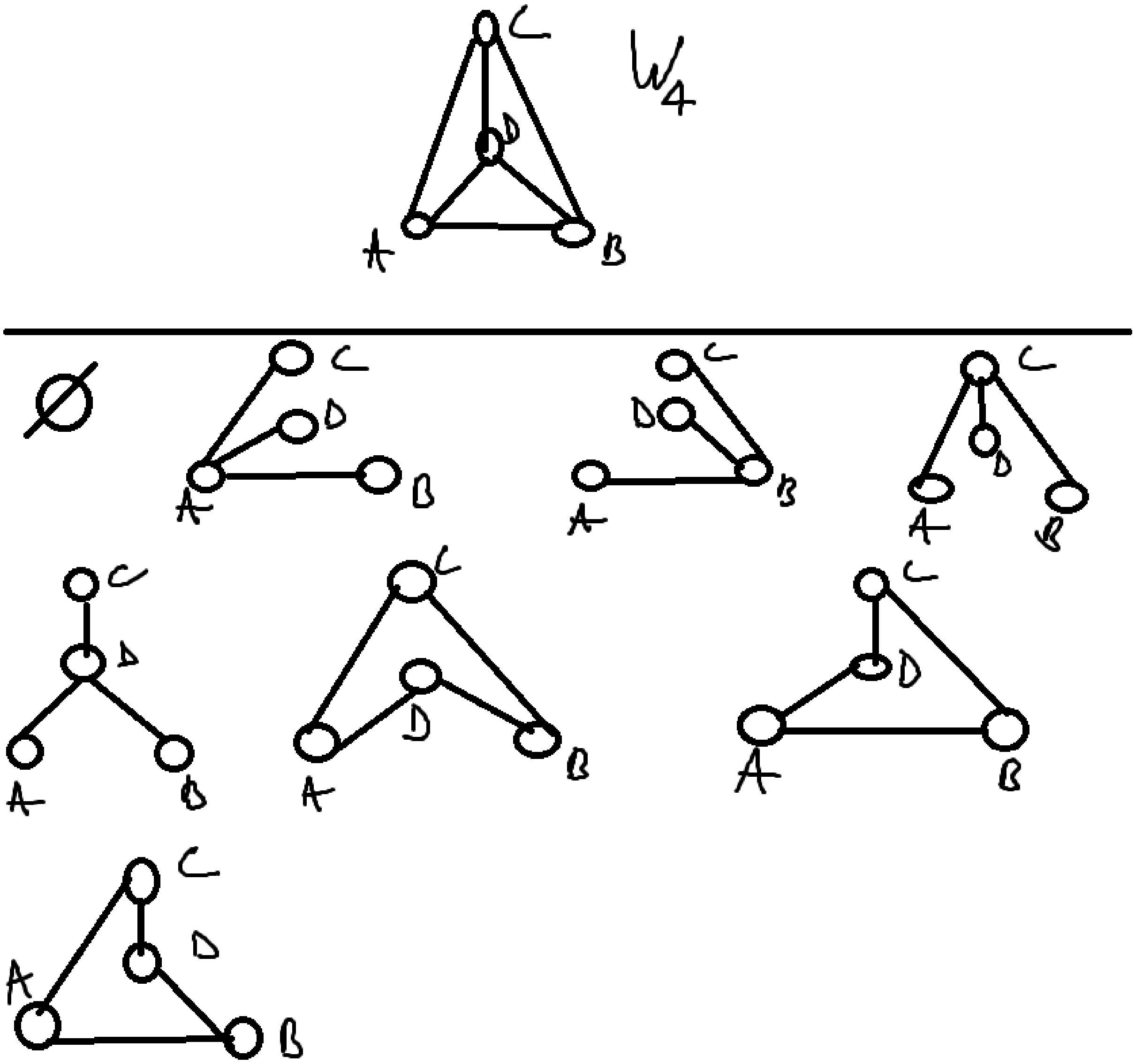
Let G be a finite simple undirected graph with edge set E. The power set of E becomes a Z_2-vector space symmetric difference as addition, identity function as negation, and empty set as zero. The one-element sets form a basis, so its dimension is equal to the number of edges of G. Because every element of this vector space is a subset of E, it can be regarded as an indicator function of type $E \to \mathbb{Z}_2$, then this vector space coincide with the free Z_2-module with basis E. This is the binary edge space of G.

An important subspace of the edge space is the binary cycle space. It is by definition the subspace generated by the edge sets of all the simple cycles of the graph. The addition of two cycles shown dashed is in the figure



Cut Space

Cut Space is another subspace of Edge Space Cut Space describes all the edge-disjoint cuts of G. That is, if the removal of a set of edges from G partitions a component in G into two components, that set of edges is called a cut set. The Cut Space of a Graph contains all edge-disjoint cuts and the null seta Cut Space contains Fundamental Cuts and a Fundamental System of Cuts. Given a spanning tree T on graph G, the removal of any edge in T constitutes a fundamental cut, since T is immediately partitioned into two components. The Fundamental System of Cuts consists of all subsets of E(T) minus a single edge. So if |E(T)| = n, then there are n edge sets in the fundamental system of cuts, each with n-1 elements. Each edge removed from T creates an additional component. Thus, removing all the edges in T produces |V| components, where |V| is the number of vertices in G. Just as the Fundamental System of Cycles forms a basis for the Cycle Space, the Fundamental System of Cuts forms a basis for the Cut Space of the graph. For W_4, any of the cuts with three edges can be used to find a fundamental system of cuts. As an example, the cut {C-A, A-D, A-B} will be used. Thus, removing any arbitrary edge produces a fundamental cut. So the fundamental system of cuts Consists of { {C-A, A-D}, {A-D, A-B}, {A-C, A-B} }.



Applications

Applications of cycle spaces include networking theory for computing and distributed systems, scheduling, and economics. Cycle spaces are also applied in circuit theory within certain aspects of physics, computer engineering, and computer science. Cut Space has applications in networking theory, including flow maximization . It also comes into play with Electricity and Magnetism, specifically in circuit theory and Kirchhoff's law

Conclusion

Graph theory is playing an increasingly important role in the design, analysis, and testing of computer programs In mathematics and computer science, study of *graphs* used to model pair wise relations between objects from a certain collection.Thus vector spaces can be used in graph theory. Each of these vector spaces can be studied further using a linear algebra toolset to discuss rank, nullity, dimension, linear transformations, eigen theory,etc

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