A Characterization of Zero-Inflated Binomial Model

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Abstract:
Zero-inflated probability models have been applied to a variety of situations in the recent years. Especially they are found very useful in count regression modeling. The zero-inflated binomial model is characterized in this paper through a differential equation which is satisfied by its probability generating function.

Keywords: Zero-inflated binomial model, probability generating function, linear differential equation.

Introduction:
A subfamily of power series distributions, whose probability generating function (pgf) $f(s)$ satisfies the differential equation $(a+bs)f'(s)=cf(s)$ with $f'(s)$ being the first derivative of $f(s)$, has been characterized by Nanjundan (2010). Binomial, Poisson, and negative binomial distributions are members of this family. Also, Nanjundan and Sadiq Pasha (2015) have characterized zero-inflated Poisson distribution through a linear differential equation. Along the same lines, Nagesh et al (2015a, 2015b) have characterized zero-inflated geometric distribution and further extended the characterization to zero-inflated negative binomial distribution. In this paper, the zero-inflated binomial distribution is characterized using a differential equation satisfied by its pgf.

A random variable $X$ is said to have a zero-inflated binomial distribution if its probability mass function is given by

$$p(x) = \begin{cases} \varphi + (1-\varphi)q^n, & x = 0 \\ (1-\varphi) \left(\begin{array}{c} n \\ x \end{array}\right) p^n q^{n-x}, & x = 1, 2, \ldots, n \end{cases}$$

$$= \varphi p_0(x) + (1-\varphi) p_1(x), \quad 0 < \varphi < 1,$$

where $p_0(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$ and $p_1(x) = \left(\begin{array}{c} n \\ x \end{array}\right) p^x q^{n-x}, \quad x = 1, 2, \ldots, n$;

$$0 < p < 1, \quad p + q = 1.$$  

Hence the distribution of $X$ is a mixture of a distribution degenerate at zero and a binomial distribution. The probability generating function (pgf) of $X$ is given by

$$f(s) = E(S^X) = \sum_{x=0}^{\infty} p(x)s^x, \quad 0 < s < 1$$

$$f(s) = \varphi + (1-\varphi)(q + ps)^n. \quad (2)$$
Characterization:
The following theorem characterizes a random variable \( X \) having a zero-inflated binomial distribution.

**Theorem:** Let \( X \) be a random variable taking only a finite number of non-negative integer values \( 0, 1, \ldots, n \) with \( n \geq 1 \). Then \( X \) has a zero-inflated binomial distribution if and only if its pgf \( f(s) \) is such that

\[
f(s) = a + b(c + ds)f'(s)
\]

where \( a \neq 0, b, c, d \) are constants and \( f'(s) \) is the derivative of \( f(s) \).

**Proof:**
1) Suppose that \( X \) has a zero-inflated binomial distribution with the probability mass function (pmf) specified in (1). On differentiating its pgf, we get

\[
f'(s) = (1 - \varphi)np(q + ps)^{n-1}.
\]

Note that \( f'(s) \) satisfies (3) with \( a = 1 - \varphi, b = \frac{1}{np}, c = q, \) and \( d = p \).

2) Suppose that the pgf \( f(s) \) of \( X \) satisfies (3). Writing the differential equation (3) as

\[
y = a + b(c + dx) \frac{dy}{dx},
\]

we see that

\[
\frac{dy}{y - a} = \frac{1}{bd(c + dx)} \frac{1}{d} dx.
\]

Integrating both sides, we obtain

\[
\frac{1}{bd} \log(y - a) = \log(c + dx) + \text{constant}.
\]

That is \( y = k \log(c + dx)bd \), where \( k \) is a constant. Hence the solution of the differential equation (2) becomes

\[
f(s) = a + k(c + ds)bd.
\]

Since \( f(1) = 1 \), we get \( k = (1 - \varphi)(c + d)bd \). Further, either \( b \to 0 \) or \( d \to 0 \) implies that \( f(s) \to 0 \) and hence \( b, d \neq 0 \). Therefore, (3) can be written as

\[
f(s) = a + (1 - \varphi)(c + d)bd(c + ds)bd.
\]

If \( c = 0 \), then \( f'(s) = (1 - \varphi)bd^{-1} \) and \( f'(0) = 0 \). Since \( f(s) \) is a pgf, \( f'(0) = P(X = 1) > 0 \).

Hence \( c \neq 0 \). Since \( X \) takes the values \( 0, 1, \ldots, n \), its pgf is such that

\[
f(s) = p_0 + p_1s + p_2s^2 + \ldots + p_ns^n,
\]

where \( p_x = P(X = x) \). Note that \( f(s) \) in (4) matches with that in (5) if and only if \( \frac{1}{bd} \) is a positive integer.

Take \( \frac{1}{bd} = m \). Then the equation (4) can be expressed as

\[
f(s) = a + (1 - \varphi)(c + d)^{-m}(c + ds)^m.
\]

Note that \( (c + ds)^m = f_1(s) \) on the RHS of \( f(s) \) is the pgf of a binomial distribution and \( f_0(s) = 1 \) is the pgf of a random variable degenerate at 0. Therefore \( f(s) = af_0(s) + (1 - \varphi)(c + d)^{-m}f_1(s) \) can be identified as a convex combination of these two pgfs. Hence \( c + d = 1 \) and the pgf of \( X \) becomes

\[
f(s) = a + (1 - \varphi)(c + ds)^m.
\]
Hence $f(s)$ of (6) satisfies (2) with $a = \varphi$, $c = q$, $d = p$, and $m = n$ and this completes the proof of the theorem.

References: