[Volume 2 issue 10 October 2014]
Page No.712-722 ISSN :2320-7167

INTERNATIONAL JOURNAL OF MATHEMATICS AND COMPUTER RESEARCH

Solutions to Singular Differential Equations with Neumann's Boundary -Value Problems Using Recursive form of B-spline Based Collocation Method

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Abstract: A Recursive form B-spline basis function is used as basis in B-spline collocation method. The method is applied to solve second order singular differential equations with Neumann's boundary conditions. Results of Numerical examples show the efficiency of the method. Stability of present method and accuracy of numerical solution is constantly improved by decreasing the nodal space.

Keywords

Collocation method, B-splines, Singular differential equations, Neumann's boundary conditions

1. Introduction

In recent years, many numerical methods are developed to solve singular differential equations with Neumann- Dirchlet's boundary conditions. The methods include like B-spline collocation method [13], finite difference method [4], kernel space [5, 6] sinc collocation method [7] and predictor and corrector method [8] and many more. The B-spline based collocation method is used to evaluate boundary value problems including singular boundary value problems [9].

However, it is observed from the recent literature that B-spline basis functions are derived using fixed equidistant space for a particular degree only. If the recursive formulation given by Carl. De boor [12] is applied, the basis function evaluation can be generalized and without fixing of degree of the basis function can be used in collocation method for uniform or non uniform mesh sizes.

In this paper, after defining the B-spline basis function recursively, the B-spline collocation method is described and formulated. The efficiency of the method is demonstrated using the second order singular differential equations with Neumann's boundary conditions.

Considering second order linear differential equations with variable coefficients

$$\frac{d^2U}{dx^2} + k_1 P(x)\frac{dU}{dx} + k_2 Q(x)U = R(x) \qquad , \qquad a \le x \le b \;\; \ldots\ldots \quad (1)$$

with Neumann's boundary conditions $U'(a) = d1, U(b) = d2$

where $a, b, d_1, d_2, k_1\, and\ k_2$ are constant $P(x), Q(x)\, and\ R(x)$ are functions of x

Let $U^h(x) = \sum_{i=-2}^{n-1} C_i N_{i,p}(x) \quad \ldots\ldots. \quad (2)$, where C_i's are constants to be determined and $N_{i,p}(x)$ are B-spline basis functions, be the approximate global solution to the exact solution $U(x)$ of the considered second order singular differential equation (1).

2.1B-splines

In this section, definition and properties of B-spline basis functions [1, 2] are given in detail. A zero degree and other than zero degree B-spline basis functions are defined at x_i recursively over the knot vector space $X = \{x_1, x_2, x_3 \ldots\ldots\ldots x_{n-1}, x_n\}$ as

$i)\quad if\quad p = 0$

$$N_{i,p}(x) = 1 \qquad if \quad x \in (x_i, x_{i+i})$$

$$N_{i,p}(x) = 0 \qquad if \quad x \notin (x_i, x_{i+i})$$

$ii)\quad if\quad p \ge 1$

$$N_{i,p}(x) = \frac{x - x_i}{x_{i+p} - x_i} N_{i,p-1}(x) + \frac{x_{i+p+1} - x}{x_{i+p+1} - x_{i+1}} N_{i+1,p-1}(x)$$

$\ldots\ldots\ldots \quad (3)$

where p is the degree of the B-spline basis function and x is the parameter belongs to X .When evaluating these functions, ratios of the form 0/0 are defined as zero

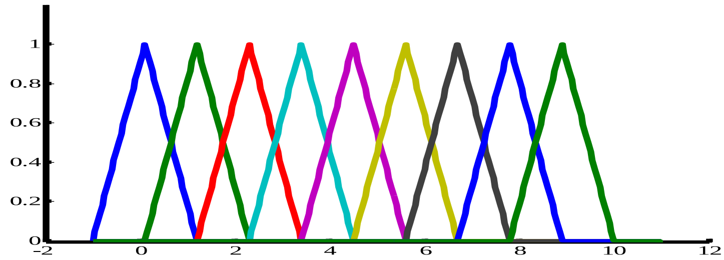

Fig (1), First degree B-spline basis function with uniform Knot vector X= {-1,0,1,2,3,4,5,6,7,8,9,10,11,12}

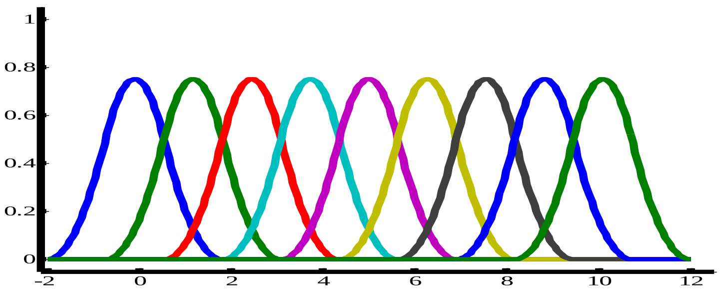

Fig (2), Second degree B-spline basis function with uniform Knot vector X= {-2,-1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

2.2 Derivatives of B-splines

If p=2, we have

$$N'_{i,p}(x) = \frac{x - x_i}{x_{i+p} - x_i} N'_{i,p-1}(x) + \frac{N_{i,p-1}(x)}{x_{i+p} - x_i} + \frac{x_{i+p+1} - x}{x_{i+p+1} - x_{i+1}} N'_{i+1,p-1}(x) - \frac{N_{i+1,p-1}(x)}{x_{i+p+1} - x_{i+1}}$$

$$N''_{i,p}(x) = 2\frac{N'_{i,p-1}(x)}{x_{i+p} - x_i} - 2\frac{N'_{i+1,p-1}(x)}{x_{i+p+1} - x_{i+1}} \qquad \ldots\ldots\ldots\ldots \qquad (4)$$

In the above equations, the basis functions are defined as recursively in terms of previous degree basis function i.e. the p^{th} degree basis function is the combination of ratios of knots and (p-1) degree basis function. Again $(p-1)^{th}$ degree basis function is defined as the combination ratios of knots and (p-2) degree basis function. In a similar way every B-spline basis function of degree up to (p-(p-2)) is expressed as the combination of the ratios of knots and its previous B-spline basis functions.

The B-spline basis functions are defined on knot vectors. Knots are real quantities. Knot vector is a non decreasing set of Real numbers. Knot vectors are classified as non-uniform knot vectors, uniform knot vector and open uniform knot vectors. Uniform knot vector in which difference of any two consecutive knots is constant is used for test problems in this paper. Two knots are required to define the zero degree basis function .In a similar way, a p^{th} degree B-spline basis function at a knot have a domain of influence of (p+2) knots. B-spline basis functions of degree one and degree two over uniform knot vector are shown graphically below in figures (1) and (2).

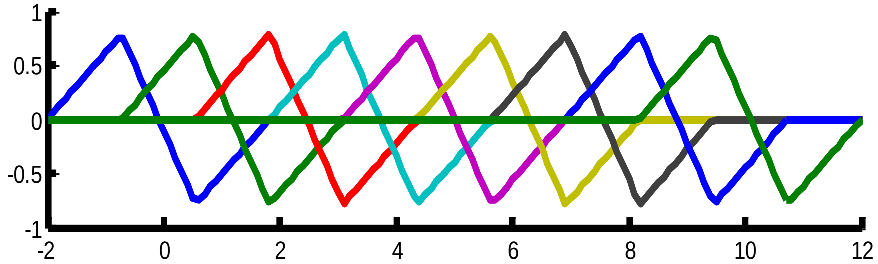

Fig (3), First derivative of second degree B-spline basis function with uniform Knot vector X= {-2,-1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

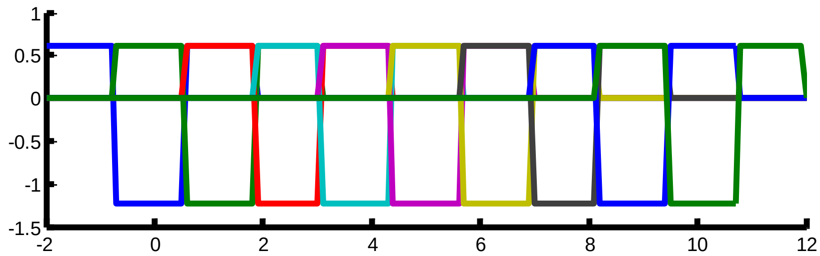

Fig (4), Second derivative of second degree B-spline basis function with uniform Knot vector X= {-2,-1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

2.3 B-spline collocation method

Collocation method is widely used in approximation theory particularly to solve differential equations .In collocation method, the assumed approximate solution is made it exact at some nodal points by equating residue zero at that particular node. B-spline basis functions are used as the basis in B-spline collocation method whereas the base functions which are used in normal collocation method are the polynomials vanishes at the boundary values. Residue which is obtained by substituting equation (2) in equation (1) is made equal to zero at nodes in the given domain to determine unknowns in (2).Let $[a,b]$ be the domain of the governing differential equation and is partitioned as $X = \{a = x_0, x_1, x_2 \ldots\ldots\ldots x_{n-1}, x_n = b\}$ with equal length $h = \frac{b-a}{n}$ of n sub domains. The x_i's are known as nodes, the nodes are treated as knots in collocation B-spline method where B-spline basis functions are defined and these nodes are used to make the residue equal to zero to determine unknowns C_i's in (2).two extra knot vectors are taken into consideration beside the domain of problem both side when evaluating the second degree B-spline basis functions at the nodes.

Substuiting, the approximate solution (2) and its derivatives in (1).

$$\frac{d^2U^h}{dx^2} + k_1P(x)\frac{dU^h}{dx} + k_2Q(x)U^h = R(x) \qquad \text{i.e.}$$

$$\sum_{i=-2}^{n-1} C_i N''_{i,p}(x) + k_1P(x)\sum_{i=-2}^{n-1} C_i N'_{i,p}(x) + k_2Q(x)\sum_{i=-2}^{n-1} C_i N_{i,p}(x) = R(x) \qquad \ldots\ldots (5)$$

Equation (5) which is valuated at x_i's ,i=0, 1, 2,....n-1 gives the system of (n-1) × (n+1) equations in which (n+1) arbitrary constants are involved. Two more equations are needed to have (n+1) × (n+1) square matrix which helps to determine the (n+ 1) arbitrary constants. The remaining two equations are obtained using

$$\sum_{i=-2}^{n-1} C_i N_{i,p}(a) = d_1, \qquad \text{............} \qquad (6)$$

$$\sum_{i=-2}^{n-1} C_i N_{i,p}(b) = d_2, \qquad \text{............} \qquad (7)$$

Now using all the above equations (5), (6), (7) i.e. (n+1) a square matrix is obtained which is diagonally dominated matrix because every second degree basis function has values other than zeros only in three intervals and zeros in the remaining intervals, it is a continuing process like when one function is ending its effect in its surrounding region than other function starts its effectiveness as parameter value changing. In other words, every parameter has at most under the three (p=2) basis functions. The systems of equations are easily solved for arbitrary constants C_i's. Substuiting these constants in (2), the approximation solution is obtained and used to estimate the values at domain points.

Absolute Relative error is evaluated by using the following relationship in exact and approximate solutions

$$\textit{Absolute Relative Error} = \left| \frac{U_{exact} - U_{appro}}{U_{exact}} \right|$$

3. Numerical Experiments

The effectiveness of the present method is demonstrated by considering the various examples

Example 1: Consider a singular boundary value problem given

$$U'' + \frac{2}{x}U' - 4U(x) = -2, \qquad 0 < x \le 1, \qquad U'(0) = 0 \quad , U(1) = 5.5$$

The exact solution is $U(x) = 0.5 + \frac{5\sinh(2x)}{x\sinh 2}$. The domain is divided into equal intervals and associated with knot vector space .Table 1 present's results at selected nodes inside the domain and compares their values with the exact solution for different mesh sizes. It is observed from the table1 that the solution approaches to the exact solution as the mesh size is decreased.

Table 1. Computed value and exact value at different nodes with different mesh sizes 'h'

Node 'x'	Estimated values at h=.002	Estimated values at h=.001	Estimated values at h=.0005	Exact values
.1	3.27751158212234	3.27656807367877	3.27609603880699	3.27562381647618
.2	3.33304140870147	3.33218186338812	3.33175181456266	3.33132158129189
. 3	3.42720499880426	3.42642356799421	3.42603258397339	3.42564142056487
.4	3.56227455309514	3.56156938858884	3.56121654890977	3.56086353732463
.5	3.74152499690348	3.74089850474504	3.74058501716492	3.74027136831943
.6	3.96932810291296	3.96878741630261	3.96851685387418	3.96824614512855
.7	4.25127857063801	4.25083626969027	4.25061493139471	4.25039346768551
.8	4.59435600983891	4.59403112758981	4.59386854227820	4.59370586068823
.9	5.00712791217250	5.00694727976670	5.00685687994258	5.00676642428200

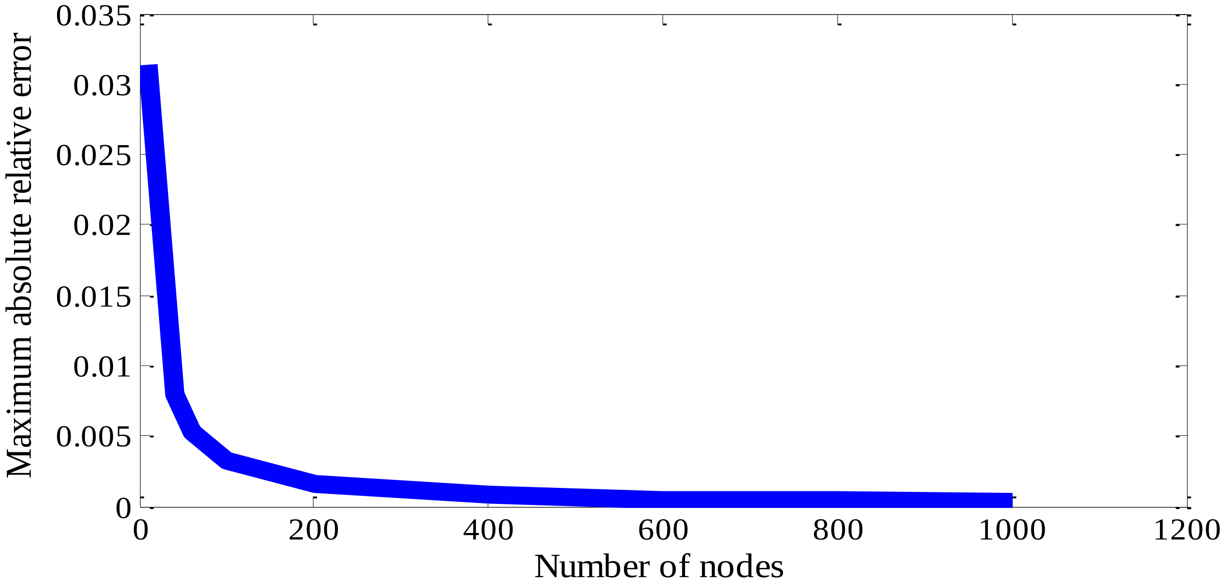

Fig (5), Trend of maximum absolute relative error for different number of nodes

Maximum absolute relative error is approaching the x-axis as moving along the x-axis, i.e falling in values of maximum absolute relative error is happening continuously as number interpolating points are increased. This shows the consistency of the present method even close at singular points.

Example 2: The exact solution of another singular boundary value problem considered below

$$U'' + \frac{1}{x}U' = \frac{8}{(8-x^2)^2} \quad , \qquad U'(0) = 0 \quad and \quad U(1) = 0$$

is $U(x) = 2\log(\frac{7}{8-x^2})$

The following below figures 6, 7and 8 gives the comparison of Approximation solution and exact solution for example.2 for different mesh sizes 'h'. Decreasing mesh size improves the approximation solution as well as consistently moving to converge with exact values very well. This shows the importance of this method in evaluating second order singular differential equations with Neumann's boundary conditions.

Table2 gives the relative error of the nodes which are in the neighborhood of the singular point zero

Nodes close to singular point x=0	.0100	.0050	.0025	.001	.0007	.0005	.0003
Absolute Relative error	0.001001	0.000479	0.000234	.000092	.000061	.000045	2.29309790580385e-05

Absolute Relative error is fall down at points of the neighborhood of the singular point zero like at normal points. Results prove the efficiency of method to singular value boundary problems with Neumann's conditions.

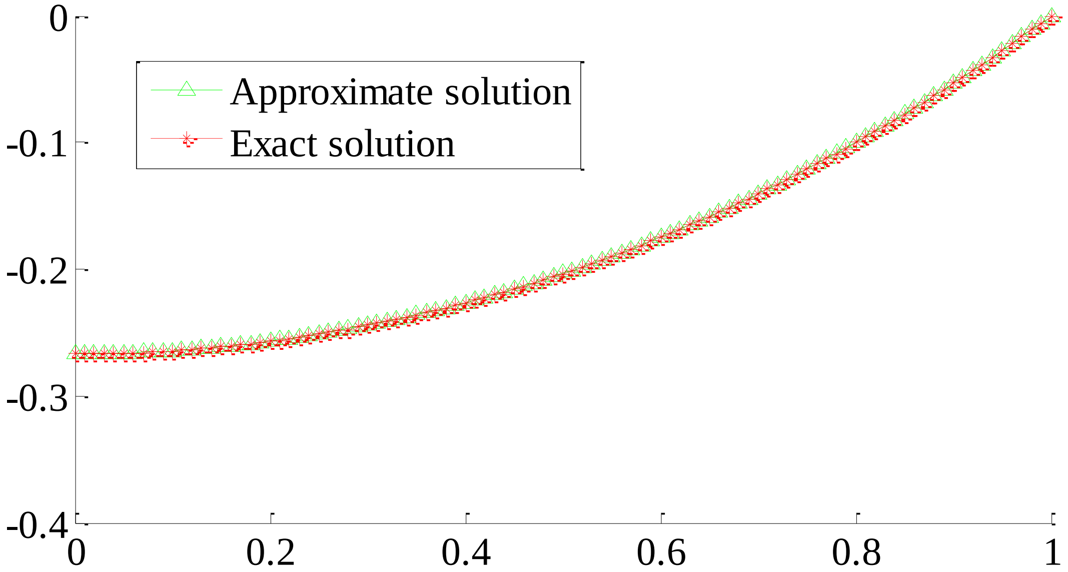

Fig (6). Comparison of approximate and Exact solutions for the Example2 for mesh size h=0.01

Conclusions

The B-spline basis functions defined recursively are incorporated in the collocation method and applied the same to the singular boundary value problems. The effectiveness of the proposed method is illustrated by considering two numerical examples. The solution is compared with exact solution and found to be in good approximation. This method may be applied to different types of singular boundary value problems for its efficiency

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