[Volume 4 issue 1 January 2016]
PageNo.1220-1229 ISSN : 2320-7167

Picard Sequence And Fixed Point Results On Parametric B-Metric Spaces

***Motiram Likhitker*[1,] *R. D. Daheriya*[2]**

[1,2]*Department of Mathematics, Government J.H. Post Graduate College, Betul, P.O. Box 460001, Madhya Pradesh, India*

♠Correspondingauthor, e-mail: mrlikhitker950@gmail.com

Abstract:
In this article, we obtain some fixed point results single valued mappings with rational expression in the setting of a parametric b-metric space. The presented theorems extend, generalize and improve many existing results in the literature. Also, we introduce some examples the support the validity of our results.

Keywords: *fixed point, Picard sequence, parametric b-metric space,*

1. INTRODUCTION

Fixed point theory has attracted many researchers since 1922 with the admired Banach fixed point theorem. This theorem supplies a method for solving a variety of applied problems in mathematical sciences and engineering. A huge literature on this subject exists and this is a very active area of research at present. The concept of metric spaces has been generalized in many directions. The notion of a b-metric space was studied by Czerwik in [14-15] and a lot of fixed point results for single and multi-valued mappings by many authors have been obtained in (ordered) b-metric spaces (see, e.g., [13,16-17]).The concept of fuzzy set was introduced by Zadeh [3] in 1965. In 1975, Kramosil and Michalek [4] introduced the notion of fuzzy metric space, which can be regarded as a generalization of the statistical (probabilistic) metric space. This work has provided an important basis for the construction of fixed point theory in fuzzy metric spaces. In 2004, Park introduced the notion of intuitionistic fuzzy metric space [5]. He showed that, for each intuitionistic fuzzy metric space$(\mathrm{X},\ \mathrm{M},\ \mathrm{N},\ *,\ \diamond)$, the topology generated by the intuitionistic fuzzy metric (M,N) coincides with the topology generated by the fuzzy metric M. For more details on intuitionistic fuzzy metric space and related results we refer the reader to [5-12]. The notion of parametric metric spaces being a natural generalization of metric spaces was recently introduced and studied by Hussain et al. [12]. Hussain et al. [1] introduced a new type of generalized metric space, called parametric b-metric space, as a generalization of both metric and b-metric spaces. For more details on parametric metric space and related results we refer the reader to [1, 2, 12].

In this paper, we obtain some fixed point results single valued mappings with rational expression in the setting of a parametric b-metric space. These results improve and generalize some important known results in litrature. Some related results and illustrative some examples to highlight the realized improvements are also furnished.

2. PRELIMINARIES

Throughout this paper $\mathbb{R}$ and $\mathbb{R}^+$ will represents the set of real numbers and nonnegative real numbers, respectively.

In 2014, Hussain et al. [12] defined and studied the concept of parametric metric space.

Definition 2.1 Let X be a nonempty set and $\mathcal{P} : X \times X \times (0,+\infty) \to [0,+\infty)$ be a function. We say $\mathcal{P}$ is a parametric metric on X if,

(1) $\mathcal{P}(x,y,t) = 0$ for all $t > 0$ if and only if $x = y$;

(2) $\mathcal{P}(x,y,t) = \mathcal{P}(y,x,t)$ for all $t > 0$;

(3) $\mathcal{P}(x,y,t) \le \mathcal{P}(x,z,t) + \mathcal{P}(z,y,t)$ for all $x,y,z \in X$ and all $t > 0$:

and one says the pair $(X,\mathcal{P})$ is a parametric metric space.

The following definitions are required in the sequel which can be found in [12].

Definition 2.2 Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a parametric metric space $(\mathrm{X}, \mathcal{P})$.

1. $\{x_n\}_{n=1}^{\infty}$ is said to be convergent to $x \in X$, written as $\lim_{n\to\infty} x_n = x$, for all $t > 0$, if $\lim_{n\to\infty} \mathcal{P}(x_n, x, t) = 0$.
2. $\{x_n\}_{n=1}^{\infty}$ is said to be a Cauchy sequence in X if for all $t > 0$, if $\lim_{n,m\to\infty} \mathcal{P}(x_n, x_m, t) = 0$.
3. $(\mathrm{X}, \mathcal{P})$ is said to be complete if every Cauchy sequence is a convergent sequence.

Definition 2.3 Let $(\mathrm{X}, \mathcal{P})$ be a parametric metric space and $T: X \to X$ be a mapping. We say T is a continuous mapping at x in X, if for any sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} Tx_n = Tx$.

Example 2.4 Let X denote the set of all functions $f : (0, +\infty) \to \mathbb{R}$. Define $\mathcal{P} : X \times X \times (0, +\infty) \to [0, +\infty)$ by $\mathcal{P}(f, g, t) = |f(t) - g(t)|\ \forall\, f, g \in X$ and all $t > 0$. Then $\mathcal{P}$ is a parametric metric on X and the pair $(X, \mathcal{P})$is a parametric metric space.

Very recently, Hussain et al. [1] introduced the concept of parametric b-metric space.

Definition 2.5 Let X be a nonempty set, $s \geq 1$ be a real number and $\mathcal{P} : X \times X \times (0, +\infty) \to [0, +\infty)$ be a function. We say $\mathcal{P}$ is a parametric b-metric on X if,

(1) $\mathcal{P}(x, y, t) = 0$ for all $t > 0$ if and only if $x = y$;

(2) $\mathcal{P}(x, y, t) = \mathcal{P}(y, x, t)$ for all $t > 0$;

(3) $\mathcal{P}(x, y, t) \leq s[\mathcal{P}(x, z, t) + \mathcal{P}(z, y, t)]$for all $x, y, z \in X$and all $t > 0$, where $s \geq 1$.

and one says the pair $(X, \mathcal{P}, s)$ is a parametric metric space with parameter $s \geq 1$.

Obviously, for $s = 1$, parametric b-metric reduces to parametric metric.

The following definitions will be needed in the sequel which can be found in [1].

Definition 2.6 Let $\{x_n\}_{n=1}^{\infty}$be a sequence in a parametric b-metric space $(\mathrm{X}, \mathcal{P}, \mathrm{s})$.

1. $\{x_n\}_{n=1}^{\infty}$is said to be convergent to $x \in X$, written as $\lim_{n\to\infty} x_n = x$, for all $t > 0$, if $\lim_{n\to\infty} \mathcal{P}(x_n, x, t) = 0$.
2. $\{x_n\}_{n=1}^{\infty}$is said to be a Cauchy sequence in X if for all $t > 0$, if $\lim_{n,m\to\infty} \mathcal{P}(x_n, x_m, t) = 0$.
3. $(\mathrm{X}, \mathcal{P})$ is said to be complete if every Cauchy sequence is a convergent sequence.

Example 2.6 [18] Let $X = [0, +\infty)$ and define $\mathcal{P} : X \times X \times (0, +\infty) \to [0, +\infty)$ by $\mathcal{P}(x, y, t) = t(x - y)^p$. Then $\mathcal{P}$ is a parametric b-metric with constant $s = 2^p$. In fact, we only need to prove (3) in Definition 2.5 as follows: let $\mathrm{x, y, z} \in \mathrm{X}$. Set $u = x - z, v = z - y$,so $u + v = x - y$.From the inequality$(a + b)^p \leq (2\, max\{a, b\})^p \leq 2^p(a^p + b^p), \forall\, a, b \geq 0$, we have

$$
\begin{aligned}
\mathcal{P}(x, y, t) &= t(x - y)^p \\
&= t(u + v)^p \\
&\leq 2^p t(u^p + v^p) \\
&= 2^p(t(x - z)^p + t(z - y)^p) \\
&= s\big(\mathcal{P}(x, z, t) + \mathcal{P}(z, y, t)\big)
\end{aligned}
$$

with $s = 2^p > 1$.

Definition 2.7 Let $(\mathrm{X}, \mathcal{P}, \mathrm{s})$ be a parametric b-metric space and $T: X \to X$ be a mapping. We say $T$is a continuous mapping at x in X, if for any sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} Tx_n = Tx$.

In general, a parametric b-metric function for $s > 1$ is not jointly continuous in all its variables.

Lemma 2.8[18] Let $(X, \mathcal{P}, s)$ be a b-metric space with the coefficient $s \geq 1$ and let $\{\mathrm{x_n}\}_{\mathrm{n=1}}^{\infty}$ be a sequence in X. if $\{\mathrm{x_n}\}_{\mathrm{n=1}}^{\infty}$ converges to x and also $\{\mathrm{x_n}\}_{\mathrm{n=1}}^{\infty}$ converges to y, then $x = y$. That Is the limit of $\{\mathrm{x_n}\}_{\mathrm{n=1}}^{\infty}$ is unique.

Lemma 2.9[18] Let $(X,\mathcal{P},\mathrm{s})$ be a b-metric space with the coefficient $s \geq 1$ and let $\{\mathrm{x_n}\}_{\mathrm{n=1}}^{\infty}$ be a sequence in X. if $\{\mathrm{x_n}\}_{\mathrm{n=1}}^{\infty}$ converges to x. Then

$$\frac{1}{s}\mathcal{P}(x,y,t) \leq \lim_{n\to+\infty}\mathcal{P}(x_n,y,t) \leq s\,\mathcal{P}(x,y,t)$$

$\forall\, y \in X$and all $t > 0$.

Lemma 2.10[18] Let $(X,\mathcal{P},\mathrm{s})$ be a b-metric space with the coefficient $s \geq 1$ and let $\{\mathrm{x_k}\}_{\mathrm{k=0}}^{\mathrm{n}} \subset \mathrm{X}$. Then

$$\mathcal{P}(\mathrm{x_n},\mathrm{x_0},\mathrm{t}) \leq \mathrm{s}\mathcal{P}(\mathrm{x_0},\mathrm{x_1},\mathrm{t}) + \mathrm{s}^2\mathcal{P}(\mathrm{x_2},\mathrm{x_3},\mathrm{t}) + \cdots + \mathrm{s}^{\mathrm{n}-1}\mathcal{P}(\mathrm{x_{n-2}},\mathrm{x_{n-1}},\mathrm{t})$$
$$+\mathrm{s}^{\mathrm{n}-1}\mathcal{P}(\mathrm{x_{n-1}},\mathrm{x_n},\mathrm{t})$$

Lemma 2.11 [18] Let$(X,\mathcal{P},s)$ be a parametric metric space with the coefficient $s \geq 1$. Let $\{\mathrm{x_n}\}_{\mathrm{n=1}}^{\infty}$be a sequence of points of Xsuch that

$$\mathcal{P}(x_n,x_{n+1},t) \leq \lambda\,\mathcal{P}(x_{n-1},x_n,t)$$

where$\lambda \in \left[0,\frac{1}{s}\right)$and $n = 1,2,\ldots$. Then $\{\mathrm{x_n}\}_{\mathrm{n=1}}^{\infty}$is a Cauchy sequence in $(X,\mathcal{P},s)$.

3. MAIN RESULT

Let $(X,\mathcal{P},s)$ be a parametric b-metric space, let $x_0 \in X$, and let $f: X \to X$ be a given mapping. The sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n = f^n x_0 = f x_{n-1}$ for all $n \in \mathbb{N}$ is called a Picard sequence of initial point x_0. In this section, prove the following fixed point theorem in parametric b-metric space.

Theorem 3.1 Let $(X,\mathcal{P},\mathrm{s})$ be a complete parametric b-metric space with the coefficient $s \geq 1$ and let $f: X \to X$ be a mapping such that

$$s\mathcal{P}(fx,fy,t) \leq \frac{\mathcal{P}(x,fy,t)+\mathcal{P}(y,fx,t)}{\mathcal{P}(x,fx,t)+\mathcal{P}(y,fy,t)+\ell(t)}\mathcal{P}(x,y,t) \tag{3.1}$$

$\forall\, x,y \in X$ and all $t > 0$, where $\ell: (0,\infty) \to (0,\infty)$ is a function. Then

(i). T has at least one fixed point $\mathrm{x}^\star \in X$;

(ii).every Picard sequence of initial point $x_0 \in X$ converges to a fixed point of f;

(iii). if $x^\star, x_\star \in X$ are two distinct fixed points of f, then $\mathcal{P}(x^\star,x_\star,t) \geq \frac{s}{2}$ for all $t > 0$.

Proof Let $x_0 \in X$ be an arbitrary point, and let $\{x_n\}_{n=1}^{\infty}$ be a Picard sequence of initial point x_0, that is, $x_n = f^n x_0 = f x_{n-1}$ for all $n \in \mathbb{N}$. If $x_{n_0} = x_{n_0-1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is fixed point of f and so $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. If $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$, from (3.1), we have

$$\begin{aligned} s\mathcal{P}(x_n,x_{n+1},t) &= s\mathcal{P}(fx_{n-1},fx_n,t) \\ &\leq \frac{\mathcal{P}(x_{n-1},fx_n,t)+\mathcal{P}(x_n,fx_{n-1},t)}{\mathcal{P}(x_{n-1},fx_{n-1},t)+\mathcal{P}(x_n,fx_n,t)+\ell(t)}\mathcal{P}(x_{n-1},x_n,t) \\ &\leq \frac{\mathcal{P}(x_{n-1},x_{n+1},t)}{\mathcal{P}(x_{n-1},x_n,t)+\mathcal{P}(x_n,x_{n+1},t)+\ell(t)}\mathcal{P}(x_{n-1},x_n,t) \\ &\leq \frac{s[\mathcal{P}(x_{n-1},x_n,t)+\mathcal{P}(x_n,x_{n+1},t)]}{\mathcal{P}(x_{n-1},x_n,t)+\mathcal{P}(x_n,x_{n+1},t)+\ell(t)}\mathcal{P}(x_{n-1},x_n,t) \end{aligned} \tag{3.2}$$

The last inequality gives us

$$\mathcal{P}(x_n,x_{n+1},t) \leq \frac{\mathcal{P}(x_{n-1},x_n,t)+\mathcal{P}(x_n,x_{n+1},t)}{\mathcal{P}(x_{n-1},x_n,t)+\mathcal{P}(x_n,x_{n+1},t)+\ell(t)}\mathcal{P}(x_{n-1},x_n,t) \tag{3.3}$$

From (3.3), we deduce that the sequence $\{\mathcal{P}(x_{n-1},x_n,t)\}$ is decreasing for all $t > 0$. Thus there exists a nonnegative real number λ such that $\lim_{n\to\infty}\mathcal{P}(x_{n-1},x_n,t) = \lambda$. Then we claim that $\lambda = 0$. If $\lambda > 0$, on taking limit as $n \to +\infty$ on both sides of (3.3), we obtain

$$\lambda \leq \frac{\lambda+\lambda}{\lambda+\lambda+\ell(t)}\lambda < \lambda$$

which is contradiction. It follows that $\lambda = 0$. Now we prove that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Let $\delta \in \left[0,\frac{1}{s}\right[$. Since $\lambda = 0$, then there exists $n(\delta) \in \mathbb{N}$ such that for all $t > 0$,

$$\frac{\mathcal{P}(x_{n-1},x_n,t)+\mathcal{P}(x_n,x_{n+1},t)}{\mathcal{P}(x_{n-1},x_n,t)+\mathcal{P}(x_n,x_{n+1},t)+\ell(t)} \leq \delta, \forall\ n \geq n(\delta) \tag{3.4}$$

This implies that

$$\mathcal{P}(x_n, x_{n+1}, t) \leq \delta \mathcal{P}(x_{n-1}, x_n, t), \forall\, n \geq n(\delta) \tag{3.5}$$

for all $t > 0$. Repeating (3.5) n-times, we get

$$\mathcal{P}(x_n, x_{n+1}, t) \leq \delta^n \mathcal{P}(x_0, x_1, t), \forall\, n \geq n(\delta) \tag{3.6}$$

Let $m > n$. It follows that

$$\begin{aligned} \mathcal{P}(x_n, x_m, t) &\leq s\mathcal{P}(x_n, x_{n+1}, t) + s^2\mathcal{P}(x_{n+1}, x_{n+2}, t) + \cdots + s^{m-n}\mathcal{P}(x_{m-1}, x_m, t) \\ &\leq (s\delta^n + s^2\delta^{n+1} + \cdots + s^{m-n}\delta^{m-1})\mathcal{P}(x_0, x_1, t) \\ &\leq s\delta^n(1 + s\delta + \ldots + (s\delta)^{m-n-1})\, \mathcal{P}(x_0, x_1, t) \\ &\leq \frac{s\delta^n}{1-s\delta}\, \mathcal{P}(x_0, x_1, t) \end{aligned} \tag{3.7}$$

for all $t > 0$. Since $s\delta < 1$. Assume that $\mathcal{P}(x_0, x_1, t) > 0$. By taking limit as $m, n \to +\infty$ in above inequality we get

$$lim_{n,m\to\infty}\, \mathcal{P}(x_n, x_m, t) = 0. \tag{3.8}$$

Therefore, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Also, if $\mathcal{P}(x_0, x_1, t) = 0$, then $\mathcal{P}(x_n, x_m, t) = 0$ for all $m > n$ and we deduce again that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since X is a complete parametric b-metric space, the sequence $\{x_n\}_{n=1}^{\infty}$converges to exists $x^\star \in X$. Now, we shall prove that $x^\star$ is fixed point of f. Using (3.1) with $x = x_n, y = x^\star$ and all $t > 0$, we obtain

$$\begin{aligned} s\mathcal{P}(x_{n+1}, fx^\star, t) &= s\mathcal{P}(fx_n, fx^\star, t) \\ &\leq \frac{\mathcal{P}(x_n,fx^\star,t)+\mathcal{P}(x^\star,fx_n,t)}{\mathcal{P}(x_n,fx_n,t)+\mathcal{P}(x^\star,fx^\star,t)+\ell(t)}\mathcal{P}(x_n, x^\star, t) \\ &\leq \frac{\mathcal{P}(x_n,fx^\star,t)+\mathcal{P}(x^\star,x_{n+1},t)}{\mathcal{P}(x_n,x_{n+1},t)+\mathcal{P}(x^\star,fx^\star,t)+\ell(t)}\mathcal{P}(x_n, x^\star, t) \end{aligned} \tag{3.9}$$

Moreover, from

$$\mathcal{P}(x^\star, fx^\star, t) \leq s[\mathcal{P}(x^\star, x_n, t) + \mathcal{P}(x_n, fx^\star, t)]$$

We have

$$\begin{aligned} \mathcal{P}(x^\star, fx^\star, t) - s\mathcal{P}(x_n, x^\star, t) &\leq s\mathcal{P}(x_n, fx^\star, t) \\ &\leq s^2[\mathcal{P}(x_n, x^\star, t) + \mathcal{P}(x^\star, fx^\star, t)] \end{aligned} \tag{3.10}$$

as $n \to +\infty$, we deduce that

$$\begin{aligned} \mathcal{P}(x^\star, fx^\star, t) &\leq lim_{n\to\infty}\, inf_{t>0}\, s\mathcal{P}(x_n, fx^\star, t) \\ &\leq lim_{n\to\infty}\, sup_{t>0}\, s\mathcal{P}(x_n, fx^\star, t) \\ &\leq s^2\mathcal{P}(x^\star, fx^\star, t) \end{aligned} \tag{3.11}$$

On letting lim inf, as $n \to +\infty$, on both sides of (3.11) and using (3.9) we obtain

$$\begin{aligned} \mathcal{P}(x^\star, fx^\star, t) &\leq lim_{n\to\infty}\, inf_{t>0}\, s\mathcal{P}(x_{n+1}, fx^\star, t) \\ &\leq \frac{s^2\mathcal{P}(x^\star,fx^\star,t)}{\mathcal{P}(x^\star,fx^\star,t)+\ell(t)} lim_{n\to\infty}\, sup_{t>0}\, s\mathcal{P}(x_n, fx^\star, t) \\ &= 0 \end{aligned} \tag{3.12}$$

This implies that $\mathcal{P}(x^\star, fx^\star, t) = 0$ for all $t > 0$, that is, $fx^\star = x^\star$ and hence $x^\star$ is a fixed point of f. Thus (i) and (ii) hold. If $x_\star \in X$, with $x^\star \neq x_\star$, is another fixed point of f, then using (3.1) with $x = x^\star$ and $y = x_\star$, we get

$$
\begin{aligned}
s\mathcal{P}(fx^\star, fx_\star, t) &\le \frac{\mathcal{P}(x^\star, fx_\star, t)+\mathcal{P}(x_\star, fx^\star, t)}{\mathcal{P}(x^\star, fx^\star, t)+\mathcal{P}(x_\star, fx_\star, t)+\ell(t)}\mathcal{P}(x^\star, x_\star, t) \\
&\le [\mathcal{P}(x^\star, fx_\star, t)+\mathcal{P}(x_\star, fx^\star, t)]\mathcal{P}(x^\star, x_\star, t) \\
&= [\mathcal{P}(x^\star, x_\star, t)+\mathcal{P}(x_\star, x^\star, t)]\mathcal{P}(x^\star, x_\star, t) \\
&= 2\mathcal{P}^2(x^\star, x_\star, t) \qquad (3.13)
\end{aligned}
$$

and hence $\mathcal{P}(x^\star, x_\star, t) \ge \frac{s}{2}$; that is, (iii) holds.

Remark From Theorem 3.1, we obtain Theorem 16 of [12] if $s = 1$.

In the following result we consider a weak contractive condition.

Theorem 3.2 Let $(X, \mathcal{P}, s)$ be a complete parametric b-metric space with the coefficient $s \ge 1$ and let $f: X \to X$ be a mapping such that

$$
s\mathcal{P}(fx, fy, t) \le \frac{\mathcal{P}(x, fy, t)+\mathcal{P}(y, fx, t)}{\mathcal{P}(x, fx, t)+\mathcal{P}(y, fy, t)+\ell(t)}\mathcal{P}(x, y, t) + \mu\mathcal{P}(y, fx, t) \qquad (3.14)
$$

$\forall\, x, y \in X$ and all $t > 0$, where $\ell: (0, \infty) \to (0, \infty)$ is a function and μ is a nonnegative real number. Then

(i). f has at least one fixed point $x^\star \in X$;
(ii).every Picard sequence of initial point $x_0 \in X$ converges to a fixed point of f;
(iii). if $x^\star, x_\star \in X$ are two distinct fixed points of f, then $\mathcal{P}(x^\star, x_\star, t) \ge max\left\{0, \frac{(s-\mu)}{2}\right\}$ for all $t > 0$.

Proof Let $x_0 \in X$ be an arbitrary point, and let $\{x_n\}_{n=1}^\infty$ be a Picard sequence of initial point x_0, that is, $x_n = f^n x_0 = fx_{n-1}$ for all$n \in \mathbb{N}$. If $x_{n_0} = x_{n_0-1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is fixed point of f and so $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence. If $x_n \ne x_{n-1}$ for all $n \in \mathbb{N}$, from (3.14), we have

$$
\begin{aligned}
s\mathcal{P}(x_n, x_{n+1}, t) &= s\mathcal{P}(fx_{n-1}, fx_n, t) \\
&\le \frac{\mathcal{P}(x_{n-1}, fx_n, t)+\mathcal{P}(x_n, fx_{n-1}, t)}{\mathcal{P}(x_{n-1}, fx_{n-1}, t)+\mathcal{P}(x_n, fx_n, t)+\ell(t)}\mathcal{P}(x_{n-1}, x_n, t) + \mu\mathcal{P}(x_n, fx_{n-1}, t) \\
&\le \frac{\mathcal{P}(x_{n-1}, x_{n+1}, t)}{\mathcal{P}(x_{n-1}, x_n, t)+\mathcal{P}(x_n, x_{n+1}, t)+\ell(t)}\mathcal{P}(x_{n-1}, x_n, t) \\
&\le \frac{s[\mathcal{P}(x_{n-1}, x_n, t)+\mathcal{P}(x_n, x_{n+1}, t)]}{\mathcal{P}(x_{n-1}, x_n, t)+\mathcal{P}(x_n, x_{n+1}, t)+\ell(t)}\mathcal{P}(x_{n-1}, x_n, t) \qquad (3.15)
\end{aligned}
$$

The last inequality gives us

$$
\mathcal{P}(x_n, x_{n+1}, t) \le \frac{\mathcal{P}(x_{n-1}, x_n, t)+\mathcal{P}(x_n, x_{n+1}, t)}{\mathcal{P}(x_{n-1}, x_n, t)+\mathcal{P}(x_n, x_{n+1}, t)+\ell(t)}\mathcal{P}(x_{n-1}, x_n, t) \qquad (3.16)
$$

From (3.16), we deduce that the sequence $\{\mathcal{P}(x_{n-1}, x_n, t)\}$ is decreasing for all $t > 0$. Thus there exists a nonnegative real number λ such that $\lim_{n\to\infty} \mathcal{P}(x_{n-1}, x_n, t) = \lambda$. Then we claim that $\lambda = 0$. If $\lambda > 0$, on taking limit as $n \to +\infty$ on both sides of (3.14), we obtain

$$
\lambda \le \frac{\lambda+\lambda}{\lambda+\lambda+\ell(t)}\lambda < \lambda \qquad (3.17)
$$

which is contradiction. It follows that $\lambda = 0$. Now we prove that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence. Let $\delta \in \left[0, \frac{1}{s}\right[$. Since $\lambda = 0$, then there exists $n(\delta) \in \mathbb{N}$ such that for all $t > 0$,

$$
\frac{\mathcal{P}(x_{n-1}, x_n, t)+\mathcal{P}(x_n, x_{n+1}, t)}{\mathcal{P}(x_{n-1}, x_n, t)+\mathcal{P}(x_n, x_{n+1}, t)+\ell(t)} \le \delta, \forall n \ge n(\delta) \qquad (3.18)
$$

This implies that

$$
\mathcal{P}(x_n, x_{n+1}, t) \le \delta\mathcal{P}(x_{n-1}, x_n, t), \forall\, n \ge n(\delta) \qquad (3.19)
$$

for all $t > 0$. Repeating (3.19) n-times, we get

$$
\mathcal{P}(x_n, x_{n+1}, t) \le \delta^n\mathcal{P}(x_0, x_1, t), \forall\, n \ge n(\delta) \qquad (3.20)
$$

Now, it is easy to show $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. The completeness of X ensures that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to some $x^\star \in X$. Now, we shall prove that $x^\star$ is fixed point of f. Using (3.14) with $x = x_n, y = x^\star$ and all $t > 0$, we obtain

$$\begin{aligned} s\mathcal{P}(x_{n+1}, fx^\star, t) &= s\mathcal{P}(fx_n, fx^\star, t) \\ &\leq \frac{\mathcal{P}(x_n, fx^\star, t)+\mathcal{P}(x^\star, fx_n, t)}{\mathcal{P}(x_n, fx_n, t)+\mathcal{P}(x^\star, fx^\star, t)+\ell(t)}\mathcal{P}(x_n, x^\star, t) + \mu\mathcal{P}(x^\star, fx_n, t) \\ &\leq \frac{\mathcal{P}(x_n, fx^\star, t)+\mathcal{P}(x^\star, x_{n+1}, t)}{\mathcal{P}(x_n, x_{n+1}, t)+\mathcal{P}(x^\star, fx^\star, t)+\ell(t)}\mathcal{P}(x_n, x^\star, t) + \mu\mathcal{P}(x^\star, x_{n+1}, t) \quad (3.21) \end{aligned}$$

Moreover, from

$$\mathcal{P}(x^\star, fx^\star, t) \leq s[\mathcal{P}(x^\star, x_n, t) + \mathcal{P}(x_n, fx^\star, t)]$$

We have

$$\begin{aligned} &\mathcal{P}(x^\star, fx^\star, t) - s\mathcal{P}(x_n, x^\star, t) \\ &\qquad \leq s\mathcal{P}(x_n, fx^\star, t) \\ &\qquad \leq s^2[\mathcal{P}(x_n, x^\star, t) + \mathcal{P}(x^\star, fx^\star, t)] \quad (3.22) \end{aligned}$$

as $n \to +\infty$, we deduce that

$$\begin{aligned} \mathcal{P}(x^\star, fx^\star, t) &\leq lim_{n\to\infty}\, inf_{t>0}\, s\mathcal{P}(x_n, fx^\star, t) \\ &\leq lim_{n\to\infty}\, sup_{t>0}\, s\mathcal{P}(x_n, fx^\star, t) \\ &\leq s^2\mathcal{P}(x^\star, fx^\star, t) \quad (3.23) \end{aligned}$$

On letting lim inf, as $n \to +\infty$, on both sides of (3.23) and using (3.21) we obtain

$$\begin{aligned} \mathcal{P}(x^\star, fx^\star, t) &\leq lim_{n\to\infty}\, inf_{t>0}\, s\mathcal{P}(x_{n+1}, fx^\star, t) \\ &\leq \frac{s^2\mathcal{P}(x^\star, fx^\star, t)}{\mathcal{P}(x^\star, fx^\star, t)+\ell(t)} lim_{n\to\infty}\, sup_{t>0}\, s\mathcal{P}(x_n, fx^\star, t) \\ &\quad + lim_{n\to\infty}\, inf_{t>0}\, \mu\mathcal{P}(x^\star, x_{n+1}, t) \\ &= 0 \quad (3.24) \end{aligned}$$

This implies that $\mathcal{P}(x^\star, fx^\star, t) = 0$ for all $t > 0$, that is, $fx^\star = x^\star$ and hence $x^\star$ is a fixed point of f. Thus (i) and (ii) hold. If $x_\star \in X$, with $x^\star \neq x_\star$, is another fixed point of f, then using (3.14) with $x = x^\star$ and $y = x_\star$, we get

$$\begin{aligned} s\mathcal{P}(fx^\star, fx_\star, t) &\leq \frac{\mathcal{P}(x^\star, fx_\star, t)+\mathcal{P}(x_\star, fx^\star, t)}{\mathcal{P}(x^\star, fx^\star, t)+\mathcal{P}(x_\star, fx_\star, t)+\ell(t)}\mathcal{P}(x^\star, x_\star, t) + \mu\mathcal{P}(x_\star, fx^\star, t) \\ &\leq [\mathcal{P}(x^\star, x_\star, t) + \mathcal{P}(x_\star, x^\star, t)]\mathcal{P}(x^\star, x_\star, t) + \mu\mathcal{P}(x_\star, x^\star, t) \\ &= 2\mathcal{P}^2(x^\star, x_\star, t) + \mu\mathcal{P}(x_\star, x^\star, t) \quad (3.25) \end{aligned}$$

and hence $\mathcal{P}(x^\star, x_\star, t) \geq max\left\{0, \frac{(s-\mu)}{2}\right\}$ for all $t > 0$, that is, (iii) holds.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

AUTHOR'S CONTRIBUTIONS

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

ACKNOWLEDGEMENTS

The authors express deep gratitude to the referee for his/her valuable comments and suggestions

REFERENCES

[1] Nawab Hussain, Peyman Salimi, Vahid Parvaneh, Fixed point results for various contractions in parametric and fuzzy b-metric spaces, J. Nonlinear Sci. Appl. 8 (2015), 719-739.

[2] R. Jain, R. D. Daheriya and M. Ughade, Fixed Point, Coincidence Point and Common Fixed Point Theorems under Various Expansive Conditions in b-Metric Spaces, International Journal of Scientific and Innovative Mathematical Research, vol.3, no.9, pp. 26-34, 2015.

[3] L. A. Zadeh, “Fuzzy sets,” Information and Computation, vol. 8, pp. 338–353, 1965.

[4] I. Kramosil and J. Michalek, “Fuzzy metrics and statistical metric spaces,” Kybernetika, vol. 11, no. 5, pp. 336–344, 1975.

[5] J. H. Park, “Intuitionistic fuzzy metric spaces,” Chaos, Solitons and Fractals, vol. 22, no. 5, pp. 1039–1046, 2004.

[6] C. Alaca, D. Turkoglu, and C. Yildiz, “Fixed points in intuitionistic fuzzy metric spaces,” Chaos, Solitons & Fractals, vol.29, no. 5, pp.1073–1078, 2006.

[7] D. Coker, “An introduction to intuitionistic fuzzy topological spaces,” Fuzzy Sets and Systems, vol. 88, no. 1, pp. 81–89, 1997.

[8] J. S. Park, Y. C. Kwun, and J. H. Park, “A fixed point theorem in the intuitionistic fuzzy metric spaces,” Far East Journal of Mathematical Sciences, vol. 16, no. 2, pp. 137–149, 2005.

[9] M. Rafi and M. S. M. Noorani, “Fixed point theorem on intuitionistic fuzzy metric spaces,” Iranian Journal of Fuzzy Systems, vol. 3, no. 1, pp. 23–29, 2006.

[10] B. Schweizer and A. Sklar, “Statistical metric spaces,” Pacific Journal of Mathematics, vol. 10, pp. 313–334, 1960.

[11] C. Ionescu, S. Rezapour, and M. E. Samei, “Fixed points of some new contractions on intuitionistic fuzzy metric spaces,” Fixed Point Theory and Applications, vol. 2013, article 168, 2013.

[12] N. Hussain, S. Khaleghizadeh, P. Salimi, and Afrah A. N. Abdou, “A New Approach to Fixed Point Results in Triangular Intuitionistic Fuzzy Metric Spaces”, Abstract and Applied Analysis, Volume 2014, Article ID 690139, 16 pages.

[13] Marta Demma and Pasquale Vetro, Picard Sequence and Fixed point Results on b-metric spaces, Journal of Function Spaces, vol.2015, Article ID 189861 6 pages.

[14] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inf. Univ. Ostravensis, 1 (1993), 5-11.

[15] S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, Atti Sem. Mat. Fis. Univ. Modena, 46 (1998), 263-276.

[16] M. A. Alghamdi, N. Hussain, P. Salimi, Fixed point and coupled fixed point theorems on b-metric-like spaces, J. Inequal. Appl., 2013 (2013), 25 pages.

[17] N. Hussain, V. Parvaneh, J. R. Roshan, Z. Kadelburg, Fixed points of cyclic weakly (ψ, φ, L, A, B)-contractive mappings in ordered b-metric spaces with applications, Fixed Point Theory Appl., 2013 (2013), 18pages.

[18] R. Jain, R. D. Daheriya and M. Ughade, Fixed Point, Coincidence Point and Common Fixed Point Theorems under Various Expansive Conditions in Parametric Metric Spaces and Parametric b-Metric Spaces, Gazi University Journal of Science, Accepted, 2015.