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On Some Relations between the Stirling Numbers of First and Second kind

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ARTICLE INFO	ABSTRACT
Published Online: 19 March 2019 Corresponding Author: **Henrik Stenlund**	The powers of the ordinary differential operator can be expanded in terms of the Cauchy-Euler differential operator and for the opposite case. The expansions involve the Stirling numbers of first and second kind as is well known. Two relations between the Stirling numbers of first and second kind will find their proof in this work, generated by the two expansions. A third relation is obtained by algebraic manipulation from the two known recursion relations.

KEYWORDS: Stirling numbers, Cauchy-Euler operator, differential operator

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1. Introduction

An explicit relationship between the powers of the Cauchy-Euler differential op- erator $(x\frac{\partial}{\partial x})^i$ and the ordinary differential operator $x^i(\frac{d}{dx})^i$ iswellknown.

They are displayed in this communication as an expansion in terms of the other, both ways. Stirling numbers of first and second kind are involved in the expansions. We are using the Stirling numbers of first kind as signed and the second kind is unsigned.
The Stirling numbers are extensively studied for instance in the field of combinatorics [6], [11], [9], [2], [3], [12]. [10] has reviewed some interesting historic facts of the Stirling numbers.

The equations below seem to be practically the only relations known between the Stirling numbers of first and second kind [7], [5].

$$S_{1\,n}^{(m)} = \sum_{k=0}^{n-m} (-1)^k \binom{n-1+k}{n-m+k} \binom{2n-m}{n-m-k} S_{2\,n-m+k}^{(k)} \quad (1)$$

$$S_{2\,n}^{(m)} = \sum_{k=0}^{n-m} (-1)^k \binom{n-1+k}{n-m+k} \binom{2n-m}{n-m-k} S_{1\,n-m+k}^{(k)} \quad (2)$$

$$\delta_{j,k} = \sum_{l=0}^{max(k,j)+1} S_{1\,l}^{(j)} S_{2\,k}^{(l)} \quad (3)$$

$$\delta_{k,j} = \sum_{l=0}^{max(k,j)+1} S_{1\,k}^{(l)} S_{2\,l}^{(j)} \quad (4)$$

In the following, assuming $x \in R$, the differential operators are applied to functions whose derivatives exist. The partial and ordinary differential operators are used interchangeably here and the practice depends on being with a multivariate or univariate case. We mark in the following treatment the differential operator with

$$D = \frac{d}{dx} \quad (5)$$

2. Expansions of the Differential Operators

The commutator of the differential operator is well known.

$$[D, x] = 1 \quad (6)$$

Of great use is also the product of the differential operators

$$x^{i+1}D^{i+1} = xDx^iD^i - ix^iD^i \quad (7)$$

This is straightforward to prove by using the commutator. $S_{1\ i}^{(m)}$ is the Stirling number of first kind. The recurrence relation is the following

$$S_{1\ n+1}^{(m)} = S_{1\ n}^{(m-1)} - nS_{1\ n}^{(m)} \tag{8}$$

and the special values are

$$S_{1\ n}^{(0)} = \delta_n^0, \quad S_{1\ n}^{(n)} = 1, \quad S_{1\ n}^{(1)} = (-1)^{n-1}(n-1)! \tag{9}$$

$S_{2\ i}^{(m)}$ is the Stirling number of second kind. The recursion relation is

$$S_{2\ i+1}^{(m)} = mS_{2\ i}^{(m)} + S_{2\ i}^{(m-1)} \tag{10}$$

and the special values are

$$S_{2\ n}^{(0)} = \delta_n^0 \tag{11}$$

$$S_{2\ n}^{(1)} = 1, \quad S_{2\ n}^{(n)} = 1 \tag{12}$$

For both kinds of numbers, they are determined from the recursion relations above. All other numbers outside the number triangles are zero. For basic properties of the Stirling numbers of first and second kind, see [7], [4], [8]. The expansion below has been proven in [1] and is done for instance by induction and by using recursion relations.

$$x^i D^i = \sum_{m=1}^{i} S_{1\ i}^{(m)} (xD)^m \tag{13}$$

The reverse expansion has as well been proven in [1] in an analogous way.

$$(xD)^i = \sum_{m=1}^{i} S_{2\ i}^{(m)} x^m D^m \tag{14}$$

3. Identities for the Stirling Numbers

The equations (13) and (14) represent real expansions. One is able to connect them by substituting the right sides across

$$x^m D^m = \sum_{j=1}^{m} S_{1\ m}^{(j)} \sum_{k=1}^{j} S_{2\ j}^{(k)} x^k D^k \tag{15}$$

$$(xD)^j = \sum_{m=1}^{j} S_{2\ j}^{(m)} \sum_{k=1}^{m} S_{1\ m}^{(k)} (xD)^k \tag{16}$$

It turns out that these expansions are only apparent, not real. But these relations hold in general since they are plain algebraic equations. Therefore, one can replace the operator $x^m D^m$ with xm and $(xD)^j$ with x^j by $x \in C$ having any value whatsoever. The outcomes will be

$$x^m = \sum_{j=1}^{m} S_{1\ m}^{(j)} \sum_{k=1}^{j} S_{2\ j}^{(k)} x^k \tag{17}$$

$$x^j = \sum_{m=1}^{j} S_{2\ j}^{(m)} \sum_{k=1}^{m} S_{1\ m}^{(k)} x^k \tag{18}$$

If the sequence of operations from (15), (16) to (17) and (18), would appear vague, one can apply the operators to e^x and x, correspondingly, to assure oneself of their validity. Especially, when x = 1

$$1 = \sum_{j=1}^{m} S_{1\ m}^{(j)} \sum_{k=1}^{j} S_{2\ j}^{(k)} \tag{19}$$

$$1 = \sum_{j=1}^{m} S_{2\ m}^{(j)} \sum_{k=1}^{j} S_{1\ j}^{(k)} \tag{20}$$

These form relationships between the Stirling numbers of first and second kind. m determines the dimension of the triangular matrices (number of rows) taken into account. One can observe that this kind of sum is always limited in value to unity when the matrix dimension grows to infinity. Since only the lowest row in the leftmost Stirling number above is involved, we are tempted to complete the matrix multiplication to

$$N = \sum_{m=1}^{N} \sum_{j=1}^{m} S_{1\ m}^{(j)} \sum_{k=1}^{j} S_{2\ j}^{(k)} \tag{21}$$

$$N = \sum_{m=1}^{N} \sum_{j=1}^{m} S_{2\ m}^{(j)} \sum_{k=1}^{j} S_{1\ j}^{(k)} \tag{22}$$

N being the number of rows of the triangular matrices. An additional relation between the two Stirling numbers can be obtained by cross-connecting the recursion equations (8) and (10). The connection is made at the common nontrivial point m, getting a result

$$(S_{2\ i+1}^{(m)} - S_{2\ i}^{(m-1)})S_{1\ m}^{(n)} = (S_{1\ m}^{(n-1)} - S_{1\ m+1}^{(n)})S_{2\ i}^{(m)} \tag{23}$$

One may recall the condition of $1 \leq m \leq n$ for both $S_{1\ n}^{(m)}, S_{2\ n}^{(m)}$. This leads to the following group of equations for the indices. The critical ones for nonzero equations are

$$m \geq n, \quad i \geq m. \tag{24}$$

Less critical for nonzero terms in equations are

$$i + 1 \geq m, \quad m + 1 \geq n. \tag{25}$$

One is able to choose the indices n, m, i within these conditions. One nonzero option is $n = m = i$.

4. Discussion

One can expand an ordinary differential operator’s power in terms of the Cauchy- Euler differential operator powers in equation (13). Correspondingly, a power of the Cauchy-Euler operator can be expanded in terms of the ordinary differ- ential operator powers in equation (14). The expansions have as coefficients the Stirling numbers of first and second kind correspondingly. One is further able to cross-connect them by substitution leading to a pair of relations between the Stirling numbers of first and second kind. A third kind of relation between the Stirling numbers is obtained by cross connecting the two basic recursion relation. Equations (19), (20), (21), (22) and (23) are believed to be new.

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