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Bounds for the Second Hankel Determinant of λ-q-Spirallike Functions

Read. S. A. Qahtan[1], Hamid Shamsan[2], S. Latha[3]

[1,2,3]Department of Mathematics, Yuvaraja's College, University of Mysore, Mysore 570005, INDIA

ARTICLE INFO	ABSTRACT
Published Online: 11 May 2020 Corresponding Author: **Read. S. A. Qahtan**	The object of the present paper is to obtain an upper bounded to the second Hankel determinant $\lvert a_2a_4 - a_2^3\rvert$ for λ-q- spirallike function of f^{-1} belonging to certain subclasses of analytic functions.

KEYWORDS: Analytic function, λ-q-spirallike functions, Upper bound, Second Hankel determinant.

AMS Subject Classification: 30C45

1. INTRODUCTION

In the present investigation, we denote by $\mathcal{A}$ the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (1)$$

defined on the unit disk $E = \{z: z \in \mathbb{C} \ and \ |z| < 1\}$ normalized by $f(0) = 0, f'(0) - 1 = 0$. The set S denotes the class of univalent functions in $\mathcal{A}$. A close observation of the series development of f suggests that finding bound for the coefficient an of functions of the form is an important problem. As early as in 1916, Bieberbach conjectured that the n^{th} coefficient of an univalent function is less than or equal to that of Koebe function. One of the important coefficient estimation is the Hankel determinants. The K^{th} order Hankel determinants $(k \geq 1)$ of $f \in \mathcal{A}$ is defined by

$$\mathbf{H_K(n)} := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+k-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+k} \\ \vdots & \vdots & & \vdots \\ a_{n+k-1} & a_{n+k} & \cdots & a_{n+2k-2} \end{vmatrix}$$

where $(n = 1,2, \ldots and k = 1,2, \ldots)$. For our discussion, in this paper we consider the second Hankel determinant. In recent years, study of q-analogs of subclasses univalent function is adopted among function theonsb, the sequel , we obtain an upper bound to the second Hankel determinant for λ-q-spirallike functions.

Definition 1.1. *[11] The q-analogue of f is given by*

$$\partial_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{z(1-q)}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}, \quad where (0 < q < 1) \qquad (2)$$

Equivalently (2), may be written as

$$\partial_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad z \neq 0$$

where

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1 \\ n, & q = 1 \end{cases}$$

Note that as $q \to 1^-$, $[n] \to n$.

Definition 1.2. *A function $f \in A$ is said to be λ-q-spiral starlike $(|\lambda| \leq \frac{\pi}{2})$, if and only if*

$$\Re\left\{e^{i\lambda} \frac{z\partial_q f(z)}{f(z)}\right\} \geq 0, z \in E. \qquad (3)$$

The class of λ-spiral starlike functions defined and studied by Spacek [29] is denoted by $SPST(\lambda)$. In this paper we study the class of λ-q-spiral starlike functions and denoted by $SPST(\lambda, q)$. It is observed when $\lambda = 0, SPST(0, q) = ST_q$.

Definition 1.3 *A function $f \in A$ is said to be convex λ-q-spiral, where*$\left(\frac{-\pi}{2} \leq \lambda \leq \frac{\pi}{2}\right)$, *if it satisfies the condition*

$$\Re\left[e^{i\lambda}\left\{1 + \frac{zq\partial_q^2 f(z)}{\partial_q f(z)}\right\}\right] \geq 0, z \in E. \quad (4)$$

The class of convex λ-spiral functions defined by Robertson (according to Goodman [9]) is denoted by $CVSP(\lambda)$. In this paper we study the class of convex λ-q-spiral functions and denoted by $CVSP(\lambda, q)$. It is observed when $\lambda = 0, CVSP(0, q) = CV_q$.

Let $\mathcal{P}$ denote the class of functions

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots = \{1 + \textstyle\sum_{n=1}^{\infty} c_n z^n\}, \forall z \in E. \quad (5)$$

Lemma 1.1.*[4] If the function $p \in \mathcal{P}$ is given by the series (1.3) then the following sharp estimate holds:*

$$|c_n| \leq 2 (n = 1,2, \ldots).$$

Lemma 1.2.*[8] If the function $p \in \mathcal{P}$ is given by the series (1.3), then*

$$2c_2 = c_1^2 + x(4 - c_1^2), \quad (6)$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z, \quad (7)$$

for some x, z with $|x| \leq 1$ and $|z| \leq 1$.

2. MAIN RESULTS

Theorem 2.1. *If f given by (1) in the class $\mathcal{C}_\lambda(q)(|\lambda| \leq \frac{\pi}{4})$, and $f^{-1}(\omega) = \omega + \sum_{n=2}^{\infty} t_n \omega^n$ near $\omega = 0$, is the inverse function of f, then*

$$\begin{aligned} |t_2 t_4 - t_3^2| \leq & \frac{\left\{\left(2\left[2[4]_q[2]_q([2]_q - 2q[3]_q) + D\right]\right)^2 + R\right\}\cos^2\lambda}{4q^2[4]_q[3]_q^2[2]_q^4\{q[3]_q(5q[4]_q[3]_q + [2]_q^2 - 5[4]_q[2]_q) - [4]_q([2]_q - 2q[3]_q)^2\}} \\ & - \frac{4\left[2[4]_q[2]_q([2]_q - 2q[3]_q) + D\right]Y\cos\lambda}{4q^2[4]_q[3]_q^2[2]_q^4\{q[3]_q(5[4]_q[3]_q + [2]_q^2 - 5[4]_q[2]_q) - [4]_q([2]_q - 2q[3]_q)^2\}} \\ & + \frac{Y^2}{4q^2[4]_q[3]_q^2[2]_q^4\{q[3]_q(5[4]_q[3]_q + [2]_q^2 - 5[4]_q[2]_q) - [4]_q([2]_q - 2q[3]_q)^2\}}, \end{aligned} \quad (8)$$

where

$$D = q[3]_q\{5[4]_q[2]_q - [2]_q^2([2]_q + 1)\},$$

$$R = 16([4]_q[2]_q^2)\left(q[3]_q(5[4]_q[3]_q + [2]_q^2 - 5[4]_q[2]_q) - [4]_q([2]_q - 2q[3]_q)^2\right),$$

$$Y = \left\{2\left(q[3]_q[2]_q^3 - [4]_q[2]_q^2\right) + [4]_q[2]_q^2 - \left(q[3]_q[2]_q^3 - [4]_q[2]_q^2\right)\right\}^2 .$$

Proof. Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in CVSP(\lambda, q)$, from the Definition 1.3, there exists an analytic function $p \in \mathcal{P}$ in the unit disc E with $p(0) = 1$ and $Rep(z) > 0$ such that

$$\left[e^{i\lambda}\left\{1 + \frac{qz\partial_q^2 f(z)}{\partial_q f(z)}\right\}\right] = p(z) \Leftrightarrow \left[e^{i\lambda}\{\partial_q f(z) + qz\partial_q^2 f(z)\} - i\sin\lambda\,\partial_q f(z)\right] \quad (9)$$

$$= \cos\lambda\{\partial_q f(z) \times p(z)\}.$$

Replacing $\partial_q f(z), z\partial_q^2 f(z))$ and $p(z)$ with their equivalent series expressions in the relation (9), we have

$$\begin{aligned} &\left[\left(e^{i\lambda}\{1 + \textstyle\sum_{n=2}^{\infty} [n]_q a_n z^{n-1}\} + qz\{\sum_{n=2}^{\infty} [n]_q[n-1]_q a_n z^{n-2}\}\right)\right. \\ &\qquad \left. - i\sin\lambda\{1 + \textstyle\sum_{n=2}^{\infty} [n]_q a_n z^{n-1}\}\right] = \left[\cos\lambda\{1 + \textstyle\sum_{n=2}^{\infty} [n]_q a_n z^{n-1}\} \times \{1 + \sum_{n=1}^{\infty} c_n z^n\}\right]. \end{aligned}$$

Upon simplification, we obtain

$$\begin{aligned} & e^{i\lambda}\left[[2]_q a_2 z + [3]_q[2]_q a_3 z^2 + [4]_q[3]_q a_4 z^3 + \ldots\right] = \cos\lambda\left[c_1 z + (c_2 + [2]_q c_1 a_2)z^2\right. \\ & \left. + (c_3 + [2]_q c_2 a_2 + [3]_q c_1 a_3)z^3 + \ldots\right]. \end{aligned} \quad (10)$$

On equating the coefficients of like powers of z, z^2 and z^3 respectively in (10), after simplifying, we get

$$\begin{aligned} & \left[a_2 = \frac{e^{-i\lambda}}{[2]_q} c_1 \cos\lambda, a_3 = \frac{e^{-i\lambda}}{[3]_q[2]_q}\{c_2 + c_1^2 e^{-i\lambda}\cos\lambda\}\cos\lambda,\right. \\ & \left. a_4 = \frac{e^{-i\lambda}}{[4]_q[3]_q[2]_q}\{[2]_q c_3 + ([2]_q + 1)c_1 c_2 e^{-i\lambda}\cos\lambda + c_1^3 e^{-2i\lambda}\cos^2\lambda\}\cos\lambda\right]. \end{aligned} \quad (11$$

Let $f(z)=\sum_{n=2}^{\infty}a_nz^n\in\mathcal{C}_\lambda(q)\ (|\lambda|\le\frac{\pi}{4})$, from the definition of inverse function of f, we have

$$\omega=f\{f^{-1}(\omega)\}. \tag{12}$$

Using the expression $f(z)$ the relation (12) is equivalent to

$$\omega=f\{f^{-1}(\omega)\}$$
$$=[f^{-1}(\omega)+\sum_{n=2}^{\infty}a_n\{f^{-1}(\omega)\}^n][\{f^{-1}(\omega)\}+a_2\{f^{-1}(\omega)\}^2+a_3\{f^{-1}(\omega)\}^3+\ldots]. \tag{13}$$

Using the expression $f^{-1}(\omega)$ in (13), we have

$$\omega=\{(\omega+t_2\omega+t_3\omega+\ldots)+a_2(\omega+t_2\omega^2+t_3\omega^3+\ldots)^2+a_3(\omega+t_2\omega^2+t_3\omega^3+\ldots)^3$$
$$+a_4(\omega+t_2\omega^2+t_3\omega^3+\ldots)^4+\ldots\}.$$

Using simplification, we obtain

$$\begin{Bmatrix}(t_2+a_2)\omega^2+(t_3+2a_2t_2+a_3)\omega^3+\\(t_4+2a_2t_3+a_2t_2^2+3a_3t_2+a_4)\omega^4+\ldots\end{Bmatrix}=0. \tag{14}$$

Equating the coefficients of like powers of ω^2,ω^3, and ω^4 on both sides of (14) respectively, after simplifying, we get

$$\{t_2=-a_2\},t_3=\{-a_3+2a_2^2\},t_4=\{-a_2+5a_2a_3-5a_2^3\}. \tag{15}$$

Using the values of a_2,a_3 and a_4 in (11) along with (15) yields

$$\begin{aligned}t_2&=\frac{-e^{-i\lambda}}{[2]_q}c_1\cos\lambda,\\ t_3&=\frac{-e^{-i\lambda}}{q[3]_q([2]_q^2}\{([2]_qc_2+\big([2]_q-2q[3]_q\big)c_1^2e^{-i\lambda}\cos\lambda\}\cos\lambda,\\ t_4&=\frac{-e^{-i\lambda}}{q[4]_q[3]_q[2]_q^3}\{[2]_q^3c_3-\{\big(5[4]_q[2]_q-[2]_q^2([2]_q+1)\big)c_1c_2\}e^{-i\lambda}\cos\lambda+W\}\cos\lambda,\end{aligned} \tag{16}$$

where

$$W=\big(5q[4]_q[3]_q+[2]_q^2-5[4]_q[2]_q\big)c_1^3e^{-2i\lambda}\cos^2\lambda.$$

Substituting the values of t_2,t_3 and t_4 from (16) in the second Hankel function for the function $f\in\mathcal{C}_\lambda(q)$, we have

$$|t_2t_4-t_3^2|\le\frac{\cos^2\lambda}{q^2[4]_q[3]_q^2[2]_q^4}\times|q[3]_q[2]_q^3c_1c_3-\big[D+2[4]_q[2]_q([2]_q-2q^2[3]_q)\big]c_1^2c_2\cos\lambda$$
$$-[4]_q[2]_q^2c_2^2+\big[q[3]_q(5[4]_q[3]_q+[2]_q^2-5[4]_q[2]_q)-[4]_q([2]_q-2q^2[3]_q)^2\big]c_1^4\cos^2\lambda|,|,$$

where

$$D=q[3]_q\{5[4]_q[2]_q-[2]_q^2([2]_q+1)\}.$$

The above expression is equivalent to

$$|t_2t_4-t_3^2|\le\frac{\cos^2\lambda}{q^2[4]_q[3]_q^2[2]_q^4}\times|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4|, \tag{17}$$

where

$$\begin{Bmatrix}d_1=q[3]_q[2]_q^3,\ d_2=-\big[D+2[4]_q[2]_q([2]_q-2q^2[3]_q)\big]\cos\lambda,\\ d_3=-\big[[4]_q[2]_q^2\big],\ d_4=\big[q[3]_q(5[4]_q[3]_q+[2]_q^2-5[4]_q[2]_q)-[4]_q([2]_q-2q^2[3]_q)^2\big]\cos^2\lambda\end{Bmatrix}, \tag{18}$$

where

$$D=q[3]_q\{5[4]_q[2]_q-[2]_q^2([2]_q+1)\}.$$

Substituting the values of c_2 and c_3 from (6) and (7) respectively from Lemma 1.2, on the right-hand side of (17), we have

$$4|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4|\le|(d_1+2d_2+d_3+4d_4)c_1^4$$
$$+[2d_1c_1+2(d_1+d_2+d_3)|x|-\{(d_1+d_3)c_1^2+2d_1c_1-4d_3\}|x|^2]\times(4-c_1^2)|. \tag{19}$$

Using the value of d_1,d_2,d_3 and d_4 in the relation (18), upon simplification, we obtain

$$\begin{aligned}&\{(d_1+2d_2+d_3+4d_4)\\&=\{\big(4\big[q[3]_q(5[4]_q[3]_q+[2]_q^2-5[4]_q[2]_q)-[4]_q([2]_q-2q^2[3]_q)^2\big]\cos\lambda-(d_2+d_3)\big)(\cos\lambda-1)\},\\&d_1=2q[3]_q[2]_q^3,(d_1+d_2+d_3)=-\big[D+2[4]_q[2]_q([2]_q-2q^2[3]_q)\cos\lambda-(q[3]_q[2]_q^3-[4]_q[2]_q^2)\big]\},\end{aligned} \tag{20}$$

where

$$d_2+d_3=q[3]_q[2]_q^3-[4]_q[2]_q^2,$$
$$D=\{5[4]_q[2]_q-[2]_q^2([2]_q+1)\}.$$

Using the fact $|z|<1$,upon simplification, we get

$$-\{(d_1+d_3)c_1^2+2d_1c_1-4d_3\}\le-[2]_q^2\{\big(q[3]_q[2]_q-[4]_q\big)c_1+2[4]_q\big)(c_1+2)\}. \tag{21}$$

Since $c_1\in[0,2]$, using the result $(c_1+a)(c_1+b)\ge(c_1-a)(c_1-b)$, where $a,b\ge0$, on the right-hand side of the above inequality, we get

$$-\{(d_1+d_3)c_1^2+2d_1c_1-4d_3\} \le -[2]_q^2\{(q[3]_q[2]_q-[4]_q)c_1-2[4]_q)(c_1-2)\}. \quad (22)$$

Substituting the calculated values from (21) and (22), on the right-hand side of (19), upon simplification, we get

$$\begin{aligned}&4|d_1c_1c_3+d_2c_1^2+d_3c_2^2+d_4c_1^4|\\ &\le |\{(\cos\lambda-1)(4[q[3]_q(5[4]_q[3]_q+[2]_q^2-5[4]_q[2]_q)-[4]_q([2]_q-2q^2[3]_q)^2]\cos\lambda-(d_2+d_3))\}c_1^4\\ &+2q^2[3]_q[2]_q^3c_1+2[2[4]_q[2]_q([2]_q-2q^2[3]_q)+D\cos\lambda-(q[3]_q[2]_q^3-[4]_q[2]_q^2)]c_1^2|x|\\ &-[2]_q^2\{(q[3]_q[2]_q-[4]_q)c_1-2[4]_q)(c_1-2)\}|x|^2\times(4-c_1^2)|,\end{aligned} \quad (23)$$

where

$$D=q[3]_q\{5[4]_q[2]_q-[2]_q^2([2]_q+1)\}.$$

Choosing $c_1=c\in[0,2]$, applying triangle inequality and replacing $|x|$ by μ on the right-hand side of the above inequality, we obtain

$$\begin{aligned}&4|d_1c_1c_3+d_2c_1^2+d_3c_2^2+d_4c_1^4|\\ &\le |\{(\cos\lambda-1)(4[q[3]_q(5[4]_q[3]_q+[2]_q^2-5[4]_q[2]_q)-[4]_q([2]_q-2q^2[3]_q)^2]\cos\lambda-(d_2+d_3))\}c^4\\ &+2q^2[3]_q[2]_q^3c+2[2[4]_q[2]_q([2]_q-2q^2[3]_q)+D\cos\lambda-(q[3]_q[2]_q^3-[4]_q[2]_q^2)]c^2\mu\\ &-[2]_q^2\{(q[3]_q[2]_q-[4]_q)c-2[4]_q)(c-2)\}\mu^2\times(4-c^2)|,\\ &=F(c,\mu), for\ 0\le\mu=|x|\le 1,\end{aligned} \quad (24)$$

Where$D=q[3]_q\{5[4]_q[2]_q-[2]_q^2([2]_q+1)\}$.

Now the function $F(c,\mu)$ is maximized on the closed square $[0,2]\times[0,1]$.

Differentiating $F(c,\mu)$ in (24) partially with respect to μ, we get

$$\begin{aligned}\frac{\partial F}{\partial\mu}&=[2\{2[4]_q[2]_q([2]_q-2q^2[3]_q)+q[3]_qD\cos\lambda-(q[3]_q[2]_q^3-[4]_q[2]_q^2)\}c^2\\ &+2\mu\{([2]_q^2(q[3]_q[2]_q-[4]_q)c-2[4]_q)(c-2)\}]\times(4-c^2),\end{aligned} \quad (25)$$

where

$$D=\{5[4]_q[2]_q-[2]_q^2([2]_q+1)\}$$

For $0<\mu<1$, for fixed c with $0<c<2$ and for $\frac{-\pi}{4}\le\lambda\le\frac{\pi}{4}$, from (25), we observe that $\frac{\partial F}{\partial\mu}>0$.Therefore,$F(c,\mu)$ cannot have maximum value at any point in the interior of the closed square$[0,2]\times[0,1]$. Further, for fixed $c\in[0,2]$, we have

$$\max_{0\le\mu\le1}F(c,\mu)=F(c,1)=G(c). \quad (26)$$

Therefore, replacing μ by 1 in $F(c,\mu)$, upon simplification, we obtain

$$\begin{aligned}G(c)&=\{-4[q[3]_q(5[4]_q[3]_q+[2]_q^2-5[4]_q[2]_q)-[4]_q([2]_q-2q^2[3]_q)^2]\cos^2\lambda c^4\\ &+4[2[2[4]_q[2]_q([2]_q-2q^2[3]_q)+D]\cos\lambda-Y]c^2+16([4]_q[2]_q^2)\},\end{aligned} \quad (27)$$

where

$$D=q[3]_q\{5[4]_q[2]_q-[2]_q^2([2]_q+1)\},$$

$$Y=\{2(q[3]_q[2]_q^3-[4]_q[2]_q^2)-[2]_q^2(q[3]_q[2]_q-[4]_q)+[2]_q^2[4]_q\}.$$

$$\begin{aligned}G'(c)&=-16[q[3]_q(5[4]_q[3]_q+[2]_q^2-5[4]_q[2]_q)-[4]_q([2]_q-2q^2[3]_q)^2]\cos^2\lambda c^3\\ &+8[2[2[4]_q[2]_q([2]_q-2q^2[3]_q)+D]\cos\lambda-Y]c,\end{aligned} \quad (28)$$

where

$$D=q[3]_q\{5[4]_q[2]_q[2]_q^2([2]_q+1)\},$$

$$Y=\{2(q[3]_q[2]_q^3-[4]_q[2]_q^2)+[4]_q[2]_q^2-(q[3]_q[2]_q^3-[4]_q[2]_q^2)\}.$$

$$\begin{aligned}G''(c)&=\{-48[q[3]_q(5[4]_q[3]_q+[2]_q^2-5[4]_q[2]_q)-[4]_q([2]_q-2q^2[3]_q)^2]\cos^2\lambda c^2\\ &+8[2[2[4]_q[2]_q([2]_q-2q^2[3]_q)+D]\cos\lambda-Y]\},\end{aligned} \quad (29)$$

Where

$$D=q[3]_q\{5[4]_q[2]_q-[2]_q^2([2]_q+1)\},$$

$$Y=\{2(q[3]_q[2]_q^3-[4]_q[2]_q^2)+[4]_q[2]_q^2-(q[3]_q[2]_q^3-[4]_q[2]_q^2)\}.$$

To obtain extreme values of $G(c)$, consider $G'(c)=0$. From (28), we get

$$\begin{aligned}&-8c[2\{q[3]_q(5[4]_q[3]_q+[2]_q^2-5[4]_q[2]_q)-[4]_q([2]_q-2q^2[3]_q)^2\}\cos^2\lambda c^2\\ &-[2[2[4]_q[2]_q([2]_q-2q^2[3]_q)+D]\cos\lambda-Y]]=0,\end{aligned} \quad (30)$$

Where

$$D=q[3]_q\{5[4]_q[2]_q-[2]_q^2([2]_q+1)\},$$

$$Y=\{2(q[3]_q[2]_q^3-[4]_q[2]_q^2)+[4]_q[2]_q^2-(q[3]_q[2]_q^3-[4]_q[2]_q^2)\}.$$

Let us discuss the following cases:

Case1: If $c=0$, then, from (29), we obtain

$$G''(c)=\left[2\left[2[4]_q[2]_q([2]_q-2q^2[3]_q)+D\right]\cos\lambda-Y\right]\geq 0, for\ |\lambda|\leq\frac{\pi}{4}, \tag{31}$$

Where

$$D=q[3]_q\{5[4]_q[2]_q-[2]_q^2([2]_q+1)\},$$
$$Y=\{2\left(q[3]_q[2]_q^3-[4]_q[2]_q^2\right)+[4]_q[2]_q^2-(q[3]_q[2]_q^3-[4]_q[2]_q^2)\}.$$

From the second derivative test $G(c)$ has minimum value at $c=0$.

Case2: If $c\neq 0$, then, from (30), we get

$$c^2=\left\{\frac{\left[2\left[2[4]_q[2]_q([2]_q-2q^2[3]_q)+D\right]\cos\lambda-Y\right]}{2\{q[3]_q(5[4]_q[3]_q+[2]_q^2-5[4]_q[2]_q)-[4]_q([2]_q-2q^2[3]_q)^2\}\cos^2\lambda}\right\}, \tag{32}$$

Where

$$D=q[3]_q\{5[4]_q[2]_q-[2]_q^2([2]_q+1)\},$$
$$Y=\{2\left(q[3]_q[2]_q^3-[4]_q[2]_q^2\right)+[4]_q[2]_q^2-(q[3]_q[2]_q^3-[4]_q[2]_q^2)\}.$$

Using the value of c^2 obtained from (32) in (30), upon simplification,we obtain

$$G''(c)=-16\left[2\left[2[4]_q[2]_q([2]_q-2q^2[3]_q)+D\right]\cos\lambda-Y\right]<0, for\ |\lambda|\leq\frac{\pi}{4}, \tag{33}$$

Where

$$D=q[3]_q\{5[4]_q[2]_q-[2]_q^2([2]_q+1)\},$$
$$Y=\{2\left(q[3]_q[2]_q^3-[4]_q[2]_q^2\right)+[4]_q[2]_q^2-(q[3]_q[2]_q^3-[4]_q[2]_q^2)\}.$$

By the second derivative test $G(c)$ has maximum value at c, where c^2 is given in (32). Substituting the value of c^2 in (27), after simplifying, we get

$$\begin{aligned}\max_{0\leq c\leq 2}G(c)= &\ \frac{\left\{\left(2\left[2[4]_q[2]_q([2]_q-2q^2[3]_q)+D\right]\right)^2+16([4]_q[2]_q^2)R\right\}\cos^2\lambda}{\{q[3]_q(5[4]_q[3]_q+[2]_q^2-5[4]_q[2]_q)-[4]_q([2]_q-2q^2[3]_q)^2\}\cos^2\lambda}\\ - &\ \frac{4\left[2[4]_q[2]_q([2]_q-2q^2[3]_q)+D\right]Y\cos\lambda}{\{q[3]_q(5[4]_q[3]_q+[2]_q^2-5[4]_q[2]_q)-[4]_q([2]_q-2q^2[3]_q)^2\}\cos^2\lambda}\\ + &\ \frac{Y^2}{\{q[3]_q(5[4]_q[3]_q+[2]_q^2-5[4]_q[2]_q)-[4]_q([2]_q-2q^2[3]_q)^2\}\cos^2\lambda},\end{aligned} \tag{34}$$

Where

$$D=q[3]_q\{5[4]_q[2]_q-[2]_q^2([2]_q+1)\},$$
$$R=\left(q[3]_q(5[4]_q[3]_q+[2]_q^2-5[4]_q[2]_q)-[4]_q([2]_q-2q^2[3]_q)^2\right),$$
$$Y=\{2\left(q[3]_q[2]_q^3-[4]_q[2]_q^2\right)+[4]_q[2]_q^2-(q[3]_q[2]_q^3-[4]_q[2]_q^2)\}^2.$$

Considering the maximum value of $G(c)$ only at c^2, from (24) and (34), upon simplification, we obtain

$$\begin{aligned}|d_1c_1c_3+d_2c_1^2+d_3c_2^2+d_4c_1^4|= &\ \frac{\left\{\left(2\left[2[4]_q[2]_q([2]_q-2q^2[3]_q)+D\right]\right)^2+16([4]_q[2]_q^2)R\right\}\cos^2\lambda}{4\{q[3]_q(5[4]_q[3]_q+[2]_q^2-5[4]_q[2]_q)-[4]_q([2]_q-2q^2[3]_q)^2\}\cos^2\lambda}\\ - &\ \frac{4\left[2[4]_q[2]_q([2]_q-2q^2[3]_q)+D\right]Y\cos\lambda}{4\{q[3]_q(5[4]_q[3]_q+[2]_q^2-5[4]_q[2]_q)-[4]_q([2]_q-2q^2[3]_q)^2\}\cos^2\lambda}\\ + &\ \frac{Y^2}{4\{q[3]_q(5[4]_q[3]_q+[2]_q^2-5[4]_q[2]_q)-[4]_q([2]_q-2q^2[3]_q)^2\}\cos^2\lambda},\end{aligned} \tag{35}$$

Where

$$D=q[3]_q\{5[4]_q[2]_q-[2]_q^2([2]_q+1)\},$$
$$R=\left(q[3]_q(5[4]_q[3]_q+[2]_q^2-5[4]_q[2]_q)-[4]_q([2]_q-2q^2[3]_q)^2\right),$$
$$Y=\{2\left(q[3]_q[2]_q^3-[4]_q[2]_q^2\right)+[4]_q[2]_q^2-(q[3]_q[2]_q^3-[4]_q[2]_q^2)\}^2.$$

From (17) and (35), after simplifying, we get

$$\begin{aligned}|t_2t_4-t_3^2|\leq &\ \frac{\left\{\left(2\left[2[4]_q[2]_q([2]_q-2q^2[3]_q)+D\right]\right)^2+16([4]_q[2]_q^2)R\right\}\cos^2\lambda}{4q^2[4]_q[3]_q^2[2]_q^4\{q[3]_q(5[4]_q[3]_q+[2]_q^2-5[4]_q[2]_q)-[4]_q([2]_q-2q^2[3]_q)^2\}}\\ - &\ \frac{4\left[2[4]_q[2]_q([2]_q-2q^2[3]_q)+D\right]Y\cos\lambda}{4q^2[4]_q[3]_q^2[2]_q^4\{q[3]_q(5[4]_q[3]_q+[2]_q^2-5[4]_q[2]_q)-[4]_q([2]_q-2q^2[3]_q)^2\}}\\ + &\ \frac{Y^2}{4q^2[4]_q[3]_q^2[2]_q^4\{2[3]_q(5[4]_q[3]_q+[2]_q^2-5[4]_q[2]_q)-[4]_q([2]_q-2q^2[3]_q)^2\}},\end{aligned} \tag{36}$$

Where

$$D=q[3]_q\{5[4]_q[2]_q-[2]_q^2([2]_q+1)\},$$
$$R=\left(q[3]_q(5[4]_q[3]_q+[2]_q^2-5[4]_q[2]_q)-[4]_q([2]_q-2q^2[3]_q)^2\right),$$

$$Y = \{2(q[3]_q[2]_q^3 - [4]_q[2]_q^2) + [4]_q[2]_q^2 - (q[3]_q[2]_q^3 - [4]_q[2]_q^2)\}^2.$$

As $q \to 1^-$ in the above Theorem we obtain the following result proved by Krishna [17]

Corollary 2.1. *If f given by (1) in the class $\mathcal{C}_\lambda, q)$ $(|\lambda| \le \frac{\pi}{4})$, and $f^{-1}(\omega) = \omega + \sum_{n=2}^{\infty} t_n \omega^n$ near $\omega = 0$, is the inverse function of f, then*

$$|t_2 t_4 - t_3^2| \le \left[\frac{(57\cos^2\lambda - 30\cos\lambda + 9}{288}\right]. \qquad (37)$$

As $q \to 1^-$, and $\mu = 0$ in the above Theorem we obtain the following result proved by Krishna [17].

Remark 2.1. $|t_2 t_4 - t_3^2| \le \frac{1}{8}$.

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