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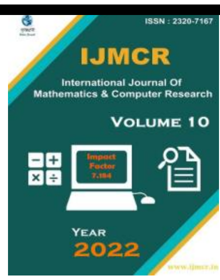

A Note on Natural Transformation & Meijer's G-Function

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ARTICLE INFO	ABSTRACT
Published Online: 24 March 2022 Corresponding Author: **Farooq Ahmad**	The objective of this paper is to investigate natural transform of Meijer's G-Function in which Riemann–Liouville integrals are replaced by more general Prabhakar integrals. We analyze and discuss its properties in terms of Mittag-Leffler functions. Further, we show some applications of these natural transform in classical equations of mathematical physics, like the heat and the free electron laser equations, and in difference-differential equations governing the dynamics of generalized renewal stochastic processes.

KEYWORDS: Meijer's G-Function, Natural transform, Riemann Liouville fractional Integrals, Mittag-Leffler Function.

INTRODUCTION

Partial differential equations and their applications arise frequently in many branches of physics,
Engineering, and other sciences. There are many works that provided using integral transform method to solve some types of partial differential equations, for example in [06, 7] integral transform is used to solve boundary value problems and integral equations.
Laplace transform is one of the most used in the mathematical and engineering community. Definition of the Laplace transform, notations, and Laplace transforms of some elementary functions can be found in [8].
Furthermore, one dimensional Laplace transform was extended to two dimensional and called as double Laplace transform. The first introduction of double Laplace transform was in [7]. Some operation calculus of double Laplace transform can be found in [9]. Double Laplace transform was used to solve heat, wave, and Laplace's equations with convolution terms (see [10]), telegraph and partial integro differential equations. Sumudu transform was first introduced by [14] and some of its applications were given by [15]. The aims of this study are to generalize the definition of single Natural transform to double Natural transform and achieve its main properties, in order to solve telegraph, wave and partial integro-differential equations.

DEFINITIONS AND PRELIMINARIES USED IN THIS PAPER:

1. Meijer's G-Function

The G-function was introduced by Cornels Simon Meijer (1936) as a very general function intended to include most of the known special functions as particular cases. This was not the only attempt of its kind. The generalized hyper geometric function and Macrobert E-function had the same aim, but Meijer's G-function was able to include these as particular case as well. The majority of the special functions can be represented in terms of the G-functions.
The first definition was made by Meijer using a series, but now a day the accepted and more general definition is in terms of Mellin-Barnes type integral. Meijer's G-functions provides an interpretation of the symbol pFq when p>q+1.
The Meijer's G-function is defined as [10]

$$G_{p,q}^{mn}\left[z\left|\begin{matrix}a_1,a_2,\dots,a_p\\ b_1,b_2,\dots,b_q\end{matrix}\right.\right]=G\begin{matrix}m,n\\ p,q\end{matrix}\left[z\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right]=G_{p,q}^{m,n}(z)$$

$$=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)z^s ds}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}$$

where an empty product is interpreted as 1. In above equation , 0< m < q, 0< n< p, and the parameters are such, that no pole of $\Gamma(b_j - s)$, j = 1, 2, 3, … , m coincides with any pole of $\Gamma(1- a_k+s)$, k = 1, 2,3,… ,n.

There are three different paths L of integration:

(i) L runs from -i∞ to +i∞ so that all the poles of $\Gamma(b_j\text{-}s)$, j = 1, 2, 3, …, m are to the right side and all the poles of $\Gamma(1-a_k+s)$, k= 1, 2, 3, … , n to the left of L.

The integral converges if (p+q) < 2 (m+n) and |argz|<$[m+n-\frac{p}{2}-\frac{q}{2}]\pi$

(ii) L is a loop starting and ending at +∞ and encircling all poles of $\Gamma(b_j\text{-}s)$, j = 1, 2, 3… m, once in negative direction but none of the poles of $\Gamma(1-a_k+s)$, k= 1, 2, 3… n.

The integral converges if q>1 and either p<q or p=q, and |z|<1,

(iii) L is a loop starting and ending at -∞ and encircling all poles of $\Gamma(1- a_k+s)$, k = 1, 2, 3, … n once in the +ve direction but none of the poles of $\Gamma(b_j\text{-}s)$, j=1, 2, 3 … m.

It is always assumed that the values of parameters and of the variable z are such that at least one of the three definitions makes sense. In cases, when more than one of these definitions makes sense, they lead the same result. Thus no ambiguity arises. The integral converges if (p+q) < 2 (m+n) and |arg z|< $(m+n-\frac{p}{2}-\frac{q}{2})\pi$, ∀ i = 1, 2…r

Natural transform:

The Natural transform earlier called as N transform [7] was extensively surveyed by the authors in [3], where the said transform is sketched out from the Fourier integral to define the complex inverse Natural transform as well. Albeit it combines the futures of Laplace and Sumudu transforms [1, 2] and therefore the region of convergence also includes both of them. Hence the Natural transform of the function $f(t) \in R^2$ is given by the following integral equation [3, 7].

$$N[f(t)] = \int_0^\infty e^{-st} f(ut)dt\ ,\ Re(S) > 0, u(-\tau_1,\tau_2)$$

Provided the function $f(t) \in R^2$ *is defined in the set*

A= { *f(t)* | $\exists M, \tau_1,\tau_2 > 0. |f(t)| < M\ e^{\frac{|t|}{\tau_j}}$ }.

Lemma-I:

For instance the Natural transform of the t^n, $n > -1$ is given by [3]

$$N[t^n] = \int_0^\infty e^{-st}(ut)^n dt\ ,\ Re(S) > 0,$$

$$= u^n \int_0^\infty e^{-st}\, t^n dt$$

$$= u^n \frac{\Gamma(n+1)}{s^{n+1}}$$

MAIN RESULTS

In this section, we consider the natural transform of the generalized function of fractional calculus called S-function and making use of the given above lemma to derive following useful results.

Theorem (1.1): Let *A*= { *f(t)* | $\exists M, \tau_1,\tau_2 > 0. |f(t)| < M\ e^{\frac{|t|}{\tau_j}}$ }, and $N[f(t)]$ be the natural transform associated with S-function . Then there holds the following relationship

$$N\left\{G^{mn}_{p,q}\left[z\left|{}^{a_1,a_2,\dots,a_p}_{b_1,b_2,\dots,b_q}\right.\right]\right\} = \frac{1}{t}\ G^{mn}_{p+1,q}\left[z\left|{}^{a_1,a_2,\dots,a_p}_{b_1,b_2,\dots,b_q}\right.\right]$$

Provided the function $f(t) \in R^2$.

Proof: By using the definition of the generalized S -function of fractional calculus and the natural transform, we get

$$N\left\{G^{mn}_{p,q}\left[z\left|{}^{a_1,a_2,\dots,a_p}_{b_1,b_2,\dots,b_q}\right.\right]\right\} = N\left\{\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)z^s ds}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\right\}$$

$$N\left\{G^{mn}_{p,q}\left[z\left|{}^{a_1,a_2,\dots,a_p}_{b_1,b_2,\dots,b_q}\right.\right]\right\} = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)ds}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}$$

$N\{z^s\}$

By making use of lemma –I in above equation, we get

$$NG^{mn}_{p,q}\left[z\left|{}^{a_1,a_2,\dots,a_p}_{b_1,b_2,\dots,b_q}\right.\right] = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)ds}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\ u^n\frac{\Gamma(s+1)}{t^{s+1}}$$

Or

$$N\left\{G^{mn}_{p,q}\left[z\left|{}^{a_1,a_2,\dots,a_p}_{b_1,b_2,\dots,b_q}\right.\right]\right\} = \frac{1}{t}\ G^{mn}_{p+1,q}\left[z\left|{}^{a_1,a_2,\dots,a_p}_{b_1,b_2,\dots,b_q}\right.\right]$$

CONCLUSION

This work deals with definition of Natural transform and S-function. Fundamental properties of Natural transform are obtained. Further, some examples and applications on Natural transform are presented. Using Natural transform to solve some types of equations with variable coefficients will be a future work.

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