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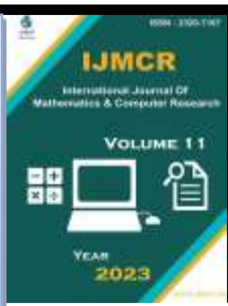


# Boundedness Solutions of a Certain Fourth-Order Nonlinear Differential Equation

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| ARTICLE INFO | ABSTRACT |
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| **Published Online:** **27 April 2023** Corresponding Author: **Muzaffer Ateş** | We consider the boundedness solutions of a certain fourth-order nonlinear and nonhomogeneous differential equation. We prove that the solutions of this equation are bounded by Cauchy formula. |

## 1. INTRODUCTION

In this paper, we study the boundedness of solutions to fourth-order nonlinear differential equation:

$$x^{(4)} + a\dddot{x} + f(x, \dot{x}, \ddot{x}) = p(t) \quad (1)$$

where , $t \in [0, \infty)$, $a > 0$ a constant, the functions $f, p$ and their first derivatives are continuous depending on their arguments. In the relevant literature [1, 2, 3] the same problem was proved that the solution and its derivatives up to order three are bounded.

Generally, the qualitative behavior of particular cases of this equation have been studied by many authors over years. However, this particular form is a generalization of the earlier ones. Moreover, when inserting $h(x)$ into the left hand side of (1), our study generalizes and improves the results of Ogundare [1], Omeike [2], Tunç and Ateş [3].

It should be noted that there exist many papers dealing with boundedness of solutions to certain nonlinear differential equations of third and fourth order in the literature. Some of them cited in [1-10]: For the third order Ateş [5] used Lyapunov's second method to investigate global stability properties and boundedness result of the solutions. For nonlinear differential equations of fourth order, Afuwape and Adesina [6] used the frequency-domain approach to discuss the stability and periodicity of solutions, while Tunç and Tiryaki [7,8] used intrinsic method to study the boundedness and stability of solutions. On the same time, Tunç [9, 10] used the Lyapunov's second method to study the stability and boundedness properties to the solutions of certain fourth order nonlinear differential equations. Further, other papers in this connections include those of Ogundare [1], Omeike [2], Tunç and Ateş [3] and Andres [4], where the Cauchy formula was applied to evaluate the boundedness of solutions to certain third and fourth order differential equation with oscillatory restoring and forcing terms.

Following the approach in [1-4], we shall use the Cauchy formula for the particular solution of the nonhomogeneous linear part of (1), to prove that the solution $x(t)$ and its derivatives $\dot{x}, \ddot{x}, \dddot{x}, x^{(4)}$ and $x^{(5)}$ are bounded. The aim of this work is to improve the previous studies and make some contribution to the literature since there are only a few studies [1-4] on the related subject.

**Remark 1 In** the related literature, the conditions and the results of oscillatory restoring term $h(x)$ and $g(x, \dot{x})$ are the same. Hence, in our study we omit $h(x)$ and $g(x, \dot{x})$ in the left hand side of (1).

## 2. PRELIMINARY RESULTS

We need the following lemmas in the proof of our main result.

**Lemma 1** We assume that there exist positive constants $a, b$ $(a^2 > 4b), c$ and $p$ such that the following conditions hold for all $x \in R$ and $t \geq 0$:

(i) $|p(t)| \leq P$,

(ii) $0 < \frac{f(x, \dot{x}, \ddot{x})}{\ddot{x}} \leq b < \infty$, $(\ddot{x} \neq 0)$.

Then, each solution $x(t)$ of (1) satisfies

$$\limsup_{t \to \infty} |\dddot{x}(t)| \leq \frac{3P}{a},$$

provided that

$$\limsup_{t \to \infty} |\dot{x}(t)| \leq \frac{P}{c},$$

$$\limsup_{t \to \infty} |\ddot{x}(t)| \leq \frac{2P}{b}.$$

Note that a, b, c satisfiy the conditions ensuring that the auxiliary equation

$\lambda^3 + a\lambda^2 + b\lambda + c = 0,$

which has negative real roots.

**Proof.** Substituting $z := \ddot{x}$, we get from (1) that

$\dot{z} + az = p(t) - f(x, \dot{x}, \ddot{x})$

with the solution of the form

$$\ddot{x}(t) = z(t) = Ce^{-at} + \int_{t_x}^{t} e^{-a(t-\tau)}\left[p(\tau) - f\big(x(\tau), \dot{x}(\tau), \ddot{x}(\tau)\big)\right]d\tau,$$

where C is an arbitrary constant and $t \geq t_x$ (a great enough number). Let us assume that the assumptions of Lemma 1, for $t \geq t_x$, hold. Then, we have not only

$$|\ddot{x}(t)| \leq \left|\int_{T_x}^{t} e^{-a(t-\tau)}\left[p(\tau) - f\big(x(\tau), \dot{x}(\tau), \ddot{x}(\tau)\big)\right]d\tau\right|$$

$$\leq \int_{T_x}^{t} \left(|p(\tau)| + \left|\frac{f(x(\tau), \dot{x}(\tau), \ddot{x}(\tau))}{\ddot{x}(\tau)}\right| |\ddot{x}(\tau)|\right) e^{-a(t-\tau)}\, d\tau$$

$$\leq \int_{T_x}^{t} (P + b|\ddot{x}(\tau)|)\, e^{-a(t-\tau)}\, d\tau$$

$$\leq \frac{3p}{a}\left(1 - e^{-a(t-T_x)}\right),$$

but also

$$\limsup_{t\to\infty} |\ddot{x}(t)| \leq \frac{3P}{a}.$$

Lemma 1 is thus proved.

**Lemma 2** In addition to the assumptions of Lemma 1, we assume the following conditions hold:

(i) $|\dot{p}(t)| \leq \dot{P}$,

(ii) $\max\{|f_x(x, \dot{x}, \ddot{x})|, |f_{\dot{x}}(x, \dot{x}, \ddot{x})|, |f_{\ddot{x}}(x, \dot{x}, \ddot{x})|\} \leq b_0$,

where $b_0, \dot{P}$ are suitable positive constants. Then, the fifth derivative of $x(t)$ of (1) is bounded.

**Proof.** First, according to the assumptions of Lemma1 and from (1), we get the inequality.

$$\left|x^{(4)}(t)\right| \leq 6P. \quad (2)$$

Next, from (1) the fifth derivative of $x(t)$ satisfies

$$x^{(5)}(t) = \dot{p}(t) - ax^{(4)}(t) - f_x(x, \dot{x}, \ddot{x})\dot{x}(t) - f_{\dot{x}}(x, \dot{x}, \ddot{x})\ddot{x}(t) - f_{\ddot{x}}(x, \dot{x}, \ddot{x})\,\dddot{x}(t).$$

Then, by the assumptions of Lemma 2 and (2), we have

$$\left|x^{(5)}(t)\right| \leq \dot{P} + a\left|x^{(4)}(t)\right| + b_0(\dddot{x}|(t)| + \ddot{x}|(t)| + \dot{x}|(t)|),$$

$$\leq \dot{P} + 6aP + b_0P\left[\frac{3}{a} + \frac{2}{b} + \frac{1}{c}\right].$$

Thus, Lemma 2 is proved.

## 3. MAIN RESULT

**Theorem** We assume that there exist positive constants $c, K, P_0$ and $t_0$ such that $0 \leq t_0 \leq t$ and the following conditions hold:

(i) $0 < c \leq \frac{f(x,\ \dot{x},\ \ddot{x})}{\dot{x}} < \infty$, $(\dot{x} \neq 0)$,

(ii) $\left|\int_{t_0}^{t} p(\tau)d\tau\right| \leq P_0$,

(iii) $\max\{\ |\dot{x}(t_0)|,\ |\ddot{x}(t_0)|\ \} \leq K$.

Then, the solution $x(t)$ of (1) is bounded and satisfies the inequality,

.

$$|x(t)| \leq |x(t_0)| + \frac{1}{c}\left[P_0 + \frac{3P}{a} + \frac{2aP}{b} + aK + K\right]. \quad (3)$$

**Proof.** If we integrate (1) from $t_0$ to t, we get

$$\ddot{x}(t) - \ddot{x}(t_0) + a[\dot{x}(t) - \dot{x}(t_0)] + \int_{t_0}^{t} f\big(x(\tau), \dot{x}(\tau), \ddot{x}(\tau)\big)\, d\tau = \int_{t_0}^{t} p(\tau)d\tau.$$

Therefore,on replacing $\int_{t_0}^{t} f\big(x(\tau), \dot{x}(\tau), \ddot{x}(\tau)\big)\, d\tau$ with $c[x(t) - x(t_0)]$, and using the results of Lemma 1 and 2 we come to (3).

The proof of the Theorem is completed.

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