International Journal of Mathematics and Computer Research
ISSN: 2320-7167
Volume 11 Issue 07 July 2023, Page no. – 3510-3513
Index Copernicus ICV: 57.55, Impact Factor: 7.362
DOI: 10.47191/ijmcr/v11i7.02

On the Projective Algebra of Matsumoto Space

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ARTICLE INFO
Published online:
12 July 2023
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ABSTRACT
In the present paper we are study of Matsumoto space on the projective algebra and Lie Algebra of the projective group. The projective Algebra of Matsumoto space is characterized as certain Lie sub algebra of the projective algebra. Further, which is devoted to studying the condition of Finsler space of constant flag curvature and vanishing S curvature admits a non Riemannian space of affine projective vector field with Matsumoto metric is Berwald space.
2010 AMS Subject Classification: 53B40, 53C60.

KEYWORDS: Matsumoto space, Projective Vector fields, Projective Algebra, Lie Algebra, Lie sub algebra.

1. INTRODUCTION

A Finsler metric on a manifold is a family of norms in tangent spaces, which vary smoothly with the base point. Every Finsler metric determines a spray by its systems of geodesic equations. Thus, Finsler spaces can be viewed as special spray spaces. On the other hand, every Finsler metric defines a distance function by the length of minimial curves. Thus Finsler spaces can be viewed as regular metric spaces. Riemannian spaces are special regular metric spaces. In 1854, B. Riemann mentioned general regular metric spaces, but he thought that there were nothing new in the general case. In fact, it is technically much more difficult to deal with general regular metric spaces. For more than half century, there had been no essential progress in this direction until P. Finsler did his pioneering work in 1918. Finsler studied the variational problems of curves and surfaces in general regular metric spaces. Some dificult problems were solved by him. Since then, such regular metric spaces are called Finsler spaces. In projective Finsler geometry, we study projectively equivalent Finsler metrics on a manifold, namely, geodesics are same up to a parameterization. In this section we studied the projective algebra and Lie algebra of the projective group $P(M,F)$. The projective algebra of Kropina space and Matsumoto space are characterized as a certain Lie sub algebra of the projective algebra $p(M,\alpha)$. Further, we proved that the Kropina space and Matsumoto space of vanishing $S-$ curvaruter admits a non α affine projective vector field is Berwald space.

The notion of Matsumoto space is quite an old topic. The Matsumoto metric is an interesting $(\alpha,\beta)-$ metric introduced by using gradient of slope, speed and gravity. This metric formulates the model of Finsler space. Many authors [1],[6], [9] have studied this metric by different perspectives. Another interesting and important class of Finsler space is the class of Berwald spaces. Berwald space is the Finsler spaces with linear connections. The purpose of the present paper is to investigate the Vanishing $S-$ curvature admits a α- affine projective vector field on the projective algebra and Lie Algebra of the projective group.

2. PRELIMINARIES

Let M be a $n-$ dimensional differentiable manifold endowed with a Riemannian metric g and a differentiable $1-$ form, $wg_{ij}(x)$ and $w_j(x)$ be the components of g and with respect to the local chart (U,φ,R^n) and let L be a real function defined on $\varphi(U)\times R^n$ by

$L(x^i,y^i) = w_i(x)y^i + (gij(x)y^i,y^j)^{1/2}$.

Clearly L is global function on TM given locally by the above expression. Moreover L satisfies the homogeneity property and on an open sub manifold A of TM,

$Rank\left(\dot{\partial}_\iota,\dot{\partial}_J,\frac{L^2}{2}\right)=n.$

Thus L is the fundamental function of Finsler space $F^n = (TM,L)$ and this space is called Randers space. Let $C_{ijk} = \frac{1}{2}[F]^2{}_{y^iy^jy^k} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$.

Define symmetric trilinear form $C = C_{ijk}dx^i \otimes dx^j \otimes dx^k$ on $TM\backslash\{0\}$. We call C the Cartan torison. Let F be a Finsler metric on a $n-$ dimensional manifold M. The canonical geodesic $\sigma(t)$ of F is characterized by $\frac{d^2\dot{\sigma}_t}{dt^2} + 2G^i(\sigma(t), \dot{\sigma}_t) = 0$, where G^i are the geodesic coefficients having the expression $G^i = \frac{1}{4}g^{ij}\{[F^2]_{x^i y^l}y^k - [F^2]_{x^l}\}$ with $(g^{ij}) = (g_{ij})^{-1}$ and $\dot{\sigma} \;\; = \frac{\partial \sigma^i}{\partial t}\frac{\partial}{\partial x^i}$. A spray on M is a globally C^∞ vector field G on $TM\backslash\{0\}$ which is expressed in local coordinates and conventions as follows

$G = y^i \frac{\partial}{\partial x^i} - 2G\frac{\partial}{\partial y^i}$, $G^i_j = \frac{\partial G^i}{\partial y^i}$, $G^i_{jk} = \frac{\partial G^i_j}{\partial y^k}$, $G^i_{jkl} = \frac{\partial G^i_{jk}}{\partial y^l}$.

Definition 2.1. [2] The collection of all projective vector fields on a Finsler space (M, F) is finite dimensional Lie algebra with respect to the usual Lie bracket operation on vector fields, called the projective algebra and is denoted by $p(M, F)$, and is the Lie algebra of the projective
group $P(M, F)$.

Definition 2.2. [2] A projective vector field (projective) is a smooth vector field on a semi
Riemannian manifold M whose flow preserves the geodesic structure of M without necessarily preserving the affine parameter of any geodesic.

3. Projective vector fields on Matsumoto space

In this section, the projective algebra $p(M, F \;\; = \frac{\alpha^2}{\alpha-\beta})$ of a Matsumoto space is characterized as a certain Lie sub algebra of the projective algebra $p(M, \alpha)$. Further, we proved that the Matsumoto space of vanishing $S-$ curvature admits is Berwald space.

Theorem 1. Let $\left(M, F \;\; = \frac{\alpha^2}{\alpha-\beta}\right)$ be a Matsumoto space and V be a vector field on M. Then V is $F-$ projective if and only if V is $\alpha-$ projective and $L_{\hat{v}} = \left\{\frac{\alpha}{1-2\alpha}s^i_0\right\} = 0$.

Proof: Let V is $F-$ projective, $L_{\hat{v}}\, D^i_{jkl} = 0$. The sprays G of F and

$\hat{G}^i = G^i_\alpha + T^i$ are projectively related and thus they have the same Douglas tensor, hence $D^i_{jkl} = \hat{D}^i_{jkl} = \frac{\partial^3}{\partial y^i \partial y^j \partial y^l}\{T^i - \frac{1}{n+1}T^m_m y^i\}$ where,

$$T^i = \alpha Q s^i_0 + \Psi\{-2Q\alpha s_0 + r_{00}\}b^i$$
$$Q = \frac{\varphi'}{\varphi - s\varphi'} = \frac{1}{1+2s}$$
$$\Psi = \frac{1}{2}\frac{\varphi''}{(\varphi - s\varphi') + (b^2 - s^2)\varphi''} = \frac{1}{1-3s+2b^2}$$
$$T^i = \frac{\alpha}{1-2s}\, s^i_0 - \frac{1}{1-3s+2b^2}\left\{\left(\frac{2}{1-2s}\right)2s^i_0 \; - r_{00}\right\}b^i$$
$$T^m_m = \varphi' s_0 + \Psi'\alpha^{-1}(b^2 - s^2)(r_{00} - 2\varphi\alpha s_0)$$
$$+2\Psi[r_0 - \varphi^1(b^2 - s^2)s_0 - \varphi s s_0] \qquad (1)$$

From [4] $T^m_m = 0$, and take $T^i = \frac{\alpha}{1-2s}s^i_0$, we have $L_{\hat{v}}\, D^i_{jkl} = L_{\hat{v}}\; T^i_{.j.k.l}\{\frac{\alpha}{1-2s}s^i_0\;\}_{.j.k.l} = 0$.

Therefore, we have the functions $H^i(x, y)$, $(i = 1,2, \dots n)$ quadratic in y such that

$$L_{\hat{v}}\left\{\frac{\alpha}{1-2s}s^i_0\right\} = H^i. \qquad (2)$$

Now, let us put $t_{ij} = L_{\hat{v}}\, a_{ij}$.

Here, observe that $L_{\hat{v}}\left\{\frac{\alpha}{1-2s}s^i_0\right\} = \frac{t_{00}}{2\alpha}s^i_0 + \frac{\alpha}{1-2s}L_{\hat{v}}\, s^i_0 \qquad (3)$

Using the (3), equation (2) is as follows:

$$t_{00}\, s^i_0 + 2\alpha^2 L_{\hat{v}}\, s^i_0 = (1-2s)H^i \qquad (4)$$

Since we have $\alpha^2 = \; a_{ij}(x)y^i y^j$, $t_{00}\, s^i_0 = \; t_{ij}(x)s^i_k(x)y^i y^j y^k$ and $L_{\hat{v}}\, s^i_0 = L_v s^i_k(x)y^k$
are polynomials.

Hence, the left hand side of (4) is a polynomial in $y^1, y^2 \dots . y^n$ for every i, while the right hand side is not a polynomial. Which implies that $H^i \;=\; 0$ and (3) leads as $L_{\hat{v}}\left\{\frac{\alpha}{1-2s}s^i_0\right\} = 0$.

$G^i_\alpha = \; G{^a}{^i}_\alpha - \frac{\alpha}{1-2s}s^i_0 - \left(\frac{2\alpha}{1-2s}s_0 - r_{00}\right)\frac{1}{1-3s+2b^2}\{b^2 + \alpha^{-1}\frac{1-4s}{2}\; y^i\;\}$.

$G_\alpha^i = G{a^i}_\alpha - \left(\frac{2\alpha}{1-2s}\alpha s_0 - r_{00}\right)\left(\frac{1-4s}{1-3s+2b^2}\right)\alpha^{-1} y^i + \frac{\alpha}{1-2s} s_0^i$ (5)

Since $L_{\hat{v}}\left\{\frac{\alpha}{1-2s} s_0^i\right\} = 0$ and $L_{\hat{v}}\, G^i = py^i$, we have

$$L_{\hat{v}}\, G_\alpha^i = L_{\hat{v}}\left\{G_\alpha^i\left(\frac{2\alpha}{1-2s} s_0 - r_{00}\right)\left(\frac{1-4s}{1-3s+2b^2}\right)\alpha^{-1} y^i\right\} = py^i$$

$$L_{\hat{v}}\, G_\alpha^i = \left\{p + L_{\hat{v}}\left(\frac{2\alpha}{1-2s} s_0 + r_{00}\right)\left(\frac{1-4s}{1-3s+2b^2}\right)\alpha^{-1}\right\} y^i$$

Which implies that V is a $\alpha -$projective.

Conversely, suppose V is a $\alpha -$ projective, $L_{\hat{v}}\, G_\alpha^i = w_0 y^i$, where $w_0 = w_k(x)y^k$ on M and $L_{\hat{v}}\left\{\frac{\alpha}{1-2s} s_0^i\right\} = 0$.

From (5) it follows $L_{\hat{v}}\, G^i = \left\{G_\alpha^i\left(\frac{2\alpha}{1-2s} s_0 - r_{00}\right)\left(\frac{(1-4s)\alpha^{-1}}{1-3s+2b^2}\right) y^i + \frac{\alpha}{1-2s} s_0^i\right\}$,

$$= L_{\hat{v}}\, G_\alpha^i - L_{\hat{v}}\left\{\left(\frac{2\alpha}{1-2s} s_0 - r_{00}\right)\left(\frac{(1-4s)\alpha^{-1}}{1-3s+2b^2}\right) y^i\right\},$$

$$= \left\{w_0 - L_{\hat{v}}\left(\frac{2\alpha}{1-2s} s_0 - r_{00}\right)\left(\frac{(1-4s)\alpha^{-1}}{1-3s+2b^2}\right) y^i\right\}.$$

Which proves that V is $F -$ projective vector field.

Lemma 1. Let $\left(M, F = \frac{\alpha^2}{\alpha-\beta}\right)$ be an $n -$ dimensional Matsumoto space. If $s_j^i \neq 0$ then V is $F -$ projective vector field if and only if it is a $\alpha -$ homothety and $L_{\hat{v}} d\beta = \mu d\beta$ where $L_{\hat{v}}\, a_{ij} = t_{ij} = 2\mu a_{ij}$.

Proof: By theorem (1) says that V is $F -$ projective vector field if and only if it is $\alpha -$ Projective and $L_{\hat{v}}\left\{\frac{\alpha}{1-2s} s_0^i\right\} = 0$ Let $t_{ij} = L_{\hat{v}}\, a_{ij}$ and since $L_{\hat{v}}\left\{\frac{\alpha}{1-2s} s_0^i\right\} = 0$ and suppose that $s_j^i \neq 0$.

Now, let us suppose $t_{ij} = L_{\hat{v}}\, a_{ij}$ and $L_{\hat{v}}\left\{\frac{\alpha}{1-2s} s_0^i\right\} = 0$.

Therefore (4) becomes $\{t_{00} s_0^i + 2\alpha^2 L_{\hat{v}}\, s_0^i = 0\}$. (6)

It follows that α^2 divides $t_{00} s_0^i$ for every index i. This equivalent to that $s_j^i = 0$ or $\alpha^2 =$ t00 divides. Which contradicts that $s_j^i \neq 0$.

Therefore V is conformal vector field on (M, α), i.e., α - projective then there is a constant μ such that $L_{\hat{v}}\, a_{ij} = 2\mu a_{ij}$.

From (6) we obtain $L_v s_j^i = -\mu\, s_j^i$.

Here, observe that $L_v s_j^i = (L_v\, a_{ik}) s_j^k$

$$= (L_v\, a_{ik}) s_j^k + a_{ik} L_v s_j^k$$

$$= 2\mu s_{ij} - \mu s_{ij}.$$

It shows that $L_{\hat{v}} d\beta = \mu d\beta$.

Remark 1. From lemma (1), since $s_j^i = 0$, then V is $F -$ projective vector field, but it is not a $\alpha -$ homothety.

Theorem 2. Let $\left(M, F = \frac{\alpha^2}{\alpha-\beta}\right)$ be a Matsumoto space of vanishing $S -$curvature and admits a $\alpha -$ projective vector field then Matsumoto space is a Berwald space.

Proof: Since F is of vanishing $S -$ curvature and $\alpha -$ projective vector field. From [1] we have $\alpha^\Psi r_{00} = 2\sigma(x)[\beta^2(1 - \alpha^\Psi)\alpha(2\beta - 1)]$.

If Ψ and $\Psi(x, y)$ is linear w.r.t. to y s. t. $L_{\hat{v}}\, G^i = \Psi\, y^i$.

From theorem (1) we get, $L_{\hat{v}}\, G^i = V_{\hat{v}}\widetilde{G^i} + L_{\hat{v}}\left(\left(\frac{(\sigma(\beta^2(1-\alpha^\Psi)\,\alpha(1-2\beta))}{\alpha^\Psi y^i}\right)\right) - L_{\hat{v}}\, s_0 y^i = \Psi\, y^i$.

Since $L_{\hat{v}}\; y^i = 0$, it follows that $t00 = L_{\hat{v}}\, \alpha^2$ and

$L_{\hat{v}}\, G^i + \left(\frac{(\sigma(\beta^2(1-\alpha^\Psi)\,\alpha(1-2\beta))}{\alpha^\Psi}\right) L_{\hat{v}}\, \sigma y^i + \frac{t_{00}}{2\alpha} c\, y^i - L_{\hat{v}}\, s_0 y^i = \Psi\, y^i$ (7)

Recall that the natural coordinates $(x^i, y^i)\varphi^{-1}(U)$ and $x \in U$ (7) is a polynomial equation. On Multiplying (7) by α. Then we obtain, $R^i + \alpha\, I^i = 0, i = 1,2, \dots n$.

where, R^i and $I^{\,i}$ are polynomials.

$$R^i = \alpha^2\; L_{\hat{v}}\, \sigma\, y^i + \frac{1}{2}\mu a_{ij}\sigma\, y^i.$$

$$I^i = V_{\hat{v}}\widetilde{G^i}\left(\left[\frac{\beta^2(1-\alpha^\Psi)\,\alpha(1-2\beta)}{\alpha^\Psi}\right] L_{\hat{v}}\, \sigma - L_{\hat{v}}\, s_0 - \Psi\right) y^i$$

Now, assume that $s = 0$. By lemma (1), Matsumoto space must be locally projectively at, otherwise vector field V is a $\alpha -$ homothety. Which is a contradiction to that V is non -$\alpha -$ homothetic.

It implies $s_{ij} = 0$ and $e_{ij} = r_{ij} + b_i s_j + b_j s_i$ and$r_{ij} = 0$. Which is equivalent to $\nabla_i b_j = 0$ and Matsumoto space is a Berwald space.

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