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Sum of Divisors Function

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ABSTRACT
We show that the recurrence relation deduced by Robbins and Osler et al for the sum of divisors function $\sigma(n)$can be solved in terms of the complete Bell polynomials. Besides, the connection between $\sigma(2n+1)$ and the number of representations of n as the sum of four triangular numbers allows obtain arecurrence relation where only participate the values of $\sigma(m)$ with m odd.

KEYWORDS: Sum of divisors function, Bell polynomials, Sums of triangular numbers, Recurrence relations, Osler et al - Robbins identity, Colour partitions, Pentagonal numbers, Partition function.

1.- INTRODUCTION

In [1] were considered recurrence relations with the structure of a Cauchy convolution [2]:

$$n\, f_k(n) = \sum_{j=1}^{n} h(j)\, f_k(n-j), \qquad k \geq 1, \quad n \geq 0, \tag{1}$$

verifying the properties $f_k(0) = 1 \ \forall\, k$ and $h(0) = 0$, where it was used the Z- transform to obtain the following solution:

$$f_k(n) = \frac{1}{n!} B_n\big(0!\,h(1), 1!\,h(2), 2!\,h(3), \dots, (n-1)!\,h(n)\big), \tag{2}$$

in terms of the complete Bell polynomials [3-9].

In Sec. 2 we observe that the Robbins [10] – Osler et al [11-13] identity for the sum of divisors function $\sigma(n)$ [2] has the structure (1), hence it is applicable the result (2). In Sec. 3 it is used the connection between$\sigma(m)$ and $t_4(N)$, that is, the number of representations of N as a sum of 4 triangular numbers [14, 15], to deduce an interesting recurrence relation involving only the values of $\sigma(m)$ with m odd.

2.- OSLER et al – ROBBINS IDENTITY

We know the following recurrence relation [10-13] with the structure (1):

$$n\, a_n = -\sum_{j=1}^{n} \sigma(j)\, a_{n-j}\,, \qquad a_0 = 1, \qquad \sigma(0) = 0, \qquad n \geq 1, \tag{3}$$

where [16]:

$$a_j = \begin{cases} 0\,, & j \neq \frac{m}{2}(3m+1), \\ (-1)^m, & j = \frac{m}{2}(3m+1), \end{cases} \qquad m = 0,\ \pm 1,\ \pm 2, \dots \tag{4}$$

that is:

$$a_j = \begin{cases} 1, & j = 0, 5, 7, 22, 26, 51, 57, 92, 100, 145, 155, \dots \\ -1, & j = 1, 2, 12, 15, 35, 40, 70, 77, 117, 126, 176, \dots \\ 0 & otherwise \end{cases} \tag{5}$$

hence (2) implies the closed expression:

$$a_n = \frac{1}{n!} B_n\big(-0!\ \sigma(1), -1!\ \sigma(2), -2!\ \sigma(3), \dots, -(n-1)!\ \sigma(n)\big), \qquad n \geq 0, \tag{6}$$

and its corresponding inversion is given by:

$$\sigma(n) = \frac{1}{(n-1)!} \sum_{j=1}^{n} (-1)^k\, (k-1)!\ B_{n,k}(1!\ a_1, 2!\ a_2, \dots, (n-k+1)!\ a_{n-k+1}), \qquad n \geq 0, \tag{7}$$

in terms of the partial Bell polynomials.

3.- RECURRENCE RELATION FOR $\sigma(m)$ WITH m ODD

In [12] it was obtained the following recurrence relation:

$$n\, t_k(n) = -k \sum_{j=1}^{n} j\ T(j) t_k(n-j), \tag{8}$$

where $t_k(n)$ is the number of representations of n as a sum of k triangular numbers, and:

$$T(j) = \sum_{d/j} \frac{1+2\,(-1)^d}{d} = \frac{1}{j}\sum_{d/j} (-1)^d\, d\,. \quad (9)$$

On the other hand, we have the connection [14, 15]:

$$t_4(n) = \sigma(2n+1), \qquad (10)$$

then (10) and (8) with $k = 4$ imply the relation:

$$n\,\sigma(2n+1) = -\,4\sum_{J=1}^{n} j\;T(j)\,\sigma(2n+1-2j), \qquad n \geq 0, \qquad (11)$$

as an alternative recurrence to (3); we note that into (11) only participate the values of $\sigma(m)$with m odd. We know that any positive integer can be written in the form $n = 2^k\, m,\ k \geq 0$ such that m is odd, therefore:

$$\sigma(n) = (2^{k+1} - 1)\,\sigma(m), \qquad (12)$$

hence all values of the sum of divisors function are generated by the quantities $\sigma(m)$, where m is odd, which are determined using (11).

Similarly, we have the result of Ewell [17, 18]:

$$t_2(n) = \frac{1}{4} r_2(4n+1), \qquad (13)$$

where $r_2(n)$ is the number of representations of n as a sum of two squares [19-21], thus (13) and (8) with $k = 2$ imply the recurrence relation:

$$n\,r_2(4n+1) = -2\,\sum_{j=1}^{n} j\;T(j) r_2(4n+1-4j), \qquad n \geq 0, \qquad (14)$$

which is a companion expression for the following formula obtained in [12]:

$$n\,r_2(n) = -4\,\sum_{j=1}^{n} (-1)^j\; j\,D(j) r_2(n-j), \qquad D(j) = \sum_{odd\ d/j} \frac{1}{d}\,, \qquad n \geq 0. \qquad (15)$$

Remark 1: The recurrence relation (3) can be written in the form [10]:

$$\sigma(n) + \sum_{k\geq 1} (-1)^k\,[\sigma(n-\omega(k)) + \sigma(n-\omega(-k))] = \begin{cases}(-1)^{m-1}\,n, & if\ \ n = \omega(\pm m),\\ 0, & otherwise,\end{cases} \qquad (16)$$

where $\omega(k) = \frac{k}{2}(3k-1)$ are the pentagonal numbers.

Remark2: Gandhi [12, 22, 23] deduced the following recurrence relation for the colour partitions$p_k(n)$ [13, 24, 25]:

$$n\,p_r(n) = -r\,\sum_{r=1}^{n} \sigma(r)\,p_r(n-r), \qquad r, n \geq 1, \qquad (17)$$

where we can employ $r = 1$ to obtain (3) because $p_1(n) = a_n$; furthermore, letting $r = -1$ in (17) gives the well-known expression [13]:

$$n\,p(n) = \sum_{j=1}^{n} \sigma(j)\,p(n-j), \qquad (18)$$

involving the partition function.

*Remark 3:*The property (7) implies the following determinant [4, 25]:

$$\sigma(n) = -\begin{vmatrix} na_n & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\ (n-1)a_{n-1} & 1 & a_1 & a_2 & \cdots & a_{n-2} \\ (n-2)a_{n-2} & 0 & 1 & a_1 & \cdots & a_{n-3} \\ \vdots & 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2a_2 & 0 & 0 & 0 & \ddots & a_1 \\ a_1 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}, \qquad n \geq 1. \qquad (19)$$

REFERENCES

1. R. Sivaraman, J. D. Bulnes, J. López-Bonilla, *Complete Bell polynomials and recurrence relations for arithmetic functions*, European of Theoretical and Applied Sci. **1**, No. 3 (2023) 51-55.
2. R. Sivaramakrishnan, *Classical theory of arithmetic functions*, Marcel Dekker, New York (1989).
3. K. S. Kölbig, *The complete Bell polynomials for certain arguments in terms of Stirling numbers of the first kind*, J. Comput. Appl. Math. **51** (1994) 113-116.
4. W. P. Johnson, *The curious history of Faä di Bruno's formula,* The Math. Assoc. of America **109**(2002) 217-234.
5. D. F. Connon, *Various applications of the (exponential) complete Bell polynomials*, http://arxiv.org/ftp/arkiv/papers/1001/1001.2835.pdf , 16 Jan 2010.
6. J. López-Bonilla, S. Vidal-Beltrán, A. Zúñiga-Segundo, *Some applications of complete Bell polynomials*, World Eng. & Appl. Sci. J. **9**, No. 3 (2018) 89-92.
7. J. López-Bonilla, R. López-Vázquez, S. Vidal-Beltrán, *Bell polynomials*, Prespacetime J. **9**, No. 5 (2018) 451-453.
8. J. López-Bonilla, S. Vidal-Beltrán, A. Zúñiga-Segundo, *Characteristic equation of a matrix via Bell polynomials*, Asia Mathematika **2**, No. 2 (2018) 49-51.

9. https://en.wikipedia.org/wiki/Bell_polynomials#Definitions
10. N. Robbins, *Some identities connecting partition functions to other number theoretic functions*, Rocky Mountain J. Math. **29**, No. 1 (1999) 335-345.
11. T. J. Osler, A. Hassen, T. R. Chandrupatla, *Surprising connections between partitions and divisors*, The
12. College Maths. J. **38**, No. 4 (2007) 278-287.12.- G. E. Andrews, S. Kumar Jha, J. López-Bonilla, *Sums of squares, triangular numbers, and divisor sums*, J. of Integer Sequences **26** (2023) Article 23.2.5
13. J. D. Bulnes, T. Kim, J. López-Bonilla, S. Vidal-Beltrán, *On the partition function*, Proc. of the Jangjeon Math. Soc. **26**, No. 1 (2023) 1-9.
14. K. Ono, S. Robins, P. T. Wahl, *On the representation of integers as sums of triangular numbers*, Aequationes Mathematicae **50**, No. 1 (1995) 73-94.
15. 15.- S. Cooper, *On sums of an even number of squares, and an even number of triangular numbers: an elementary approach based on Ramanujan's* ${}_1\psi_1$ *summation formula*, Contemp. Math. **291**, Am. Math. Soc., Providence, RI, USA (2001).
16. J. López-Bonilla, S. Vidal-Beltrán, *On the Jha and Malenfant formulae for the partition function*, Comput. Appl. Math. Sci. **6**, No. 1 (2021) 21-22.
17. 17.- J. A. Ewell, *On sums of triangular numbers and sums of squares*, Amer. Math. Monthly **99**, No. 8 (1992) 752-757.
18. A. Moreno-Cañadas, *On sums of three squares*, arXiv: 0812.0540v2 [math. NT] 4 Dec 2008.
19. E. Grosswald, *Representations of integers as sums of squares*, Springer-Verlag, New York (1985).
20. Goro Shimura, *The representation of integers as sums of squares*, Amer. J. Math. **124**, No. 5 (2002) 1059-1081.
21. C.Moreno, S. Wagstaff, *Sums of squares of integers*, Chapman & Hall/CRC, Boca Raton, Fl (2006).
22. J. M. Gandhi, *Congruences for* $p_r(n)$ *and Ramanujan's* τ*-function*, Amer. Math. Monthly **70**, No. 3 (1963) 265-274.
23. O. Lazarev, M. Mizuhara, B. Reid, *Some results in partitions, plane partitions, and multipartitions*, Summer 2010 REU Program in Maths. at Oregon State University, Aug. 13 (2010).
24. A. Lucas-Bravo, J. López-Bonilla, O. Marín-Martínez, *On the colour partitions* $p_r(n)$,Comput. Appl. Math. Sci. **6**, No. 2 (2021) 35-37.
25. J. López-Bonilla, M. Morales-García, *Sum of divisors functions in terms of colour partitions*, Studies in Nonlinear Sci. **7**, No. 1 (2022) 4-5.