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On the Rising Sun Stereographic l –Axial Reflected Log - Logistic Model

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ABSTRACT
The circular models based on the Rising Sun function are motivated by purely mathematical considerations as a smoothing function and possible application. This work takes a further step in this direction using several mathematical tools such as Real Analysis along with MATLAB and are applied to enlarge the horizon of Mathematical Statistics. Here an attempt is made to construct new circular model using the Rising Sun function on the Stereographic Reflected Loglogistic model and both linear and circular representations of graphs of pdf are plotted using MATLAB.

KEYWORDS: Circular model, trigonometric moments, Rising Sun function, characteristic function, Stereographic Reflected Loglogistic model.

1. INTRODUCTION

The accessible techniques for constructing circular models are wrapping a linear model, offsetting a bivariate linear model and applying stereographic projection on a linear model. The Rising Sun function [Van Rooij and Schikoff (1982), p.10] smoothens the existing curve and many bumps disappear. This may lead to the effect of increasing the smoothing the curve in density estimation. Girija (2010) proposed a new method of generating circular model by using the Rising Sun function (**RSF)** and Radhika.et al (2013) derived circular models based on the Rising Sun function motivated by the mathematical significance of the Rising Sun function behind the construction of circular models, here an attempt is made to construct a new circular model named The Rising Sun Stereographic Reflected Loglogistic model and various properties are discussed.

2. CONSTRUCTION OF CIRCULAR MODELS USING THE RISING SUN FUNCTION

The Circular Distribution [Jammalamadaka and Sengupta (2001)] is defined as under

In the continuous case $g:[0,\ 2\pi)\rightarrow\mathbb{R}$ is the probability density function of a circular distribution if and only if *g* has the following basic properties

- $g(\theta)\geq 0,\ \ \forall\theta$ (2.1)
- $\int_0^{2\pi} g(\theta)\,d\theta=1$ (2.2)
- $g(\theta)=g(\theta+2k\pi)$ (2.3)

for any integer k (i.e., g is periodic) (Mardia,1972)

It may be noted that the circular distribution is a probability distribution whose total probability is concentrated on the unit circle $\{(\cos\theta,\sin\theta)/0\leq\theta<2\pi\}$ in the plane which satisfies the properties (2.1) through (2.3).

If $G(\theta)$ denotes the cdf of the r.v., the characteristic function of the circular model is given by

$$\varphi_\theta(t)=E\left(e^{it\theta}\right)=\int_0^{2\pi}e^{it\theta}dG(\theta)=\rho_t e^{i\mu_t}\qquad t\in\mathbb{R} \tag{2.4}$$

It is known that whenever $\varphi(t) \neq 0$, $e^{2\pi i t} = 1$ (Mardia 1972 p. 41). This suggests that the function $\varphi(t)$ should only be defined for integer values of *t*. Accordingly the characteristic function $\varphi(p) = \varphi_p$ is defined by

$$\varphi_\theta(p) = E\left(e^{ip\theta}\right) = \int_0^{2\pi} e^{ip\theta} dF(\theta) = \rho_p e^{i\mu_p} \qquad p \in \mathbb{Z} \tag{2.5}$$

Clearly, $\varphi_0 = 1$, $\bar{\varphi}_p = \varphi_{-p}$.

Trigonometric moments [Jammalamadaka and Sengupta (2001)]

The value of the characteristic function φ_p at an integer *p* is called the p^{th} trigonometric moment of θ. The real and the imaginary parts of φ_p are denoted by α_p and β_p respectively. We can also view these trigonometric moments in terms of

$$\alpha_p = E(\cos p\theta), \quad \beta_p = E(\sin p\theta), \quad p \in \mathbb{Z}$$

The Rising Sun function (RSF) of a bounded function $f : [a,b] \to \mathbb{R}$ is defined by

$$f_\Theta(x) = \sup\{f(t) : x \le t \le b\} \tag{2.6}$$

It is easy to show that

- when f is nonnegative then f_Θ is nonnegative
- when f is continuous then f_Θ is continuous
- f_Θ is monotonically decreasing , hence $f_\Theta = f$ when f is decreasing and f_Θ is the smallest monotonically decreasing function such that $f_\Theta = f$.

Imagine the Rising Sun on x – axis. Then $\left\{(x, y) \in \mathbb{R}^2 : y \ge f_\Theta(x)\right\}$ is illuminated by the sun whereas $\left\{(x, y) \in \mathbb{R}^2 : y < f_\Theta(x)\right\}$ is covered by darkness. The set $\left\{(x, f(x)) : f(x) = f_\Theta(x)\right\}$ is the collection of those points of the graph of f that receive light from the sun.

If f is continuous on [*a*,*b*] then for any *k* in the range of f_Θ, $S = \left\{x \in [a,b] : f_\Theta(x) = k\right\}$ is a closed and bounded interval **[**Van Rooij and Schikoff (1982)].

The Rising Sun Lemma 2.1: [Van Rooij and Schikoff (1982)]: Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then $E = \left\{x \in (a,b) : f_\Theta(x) > f(x)\right\}$ is open. If (α, β) is a component of E then $f_\Theta(\beta) = f(\beta)$ and $f_\Theta(\alpha) = f(\alpha)$ for $\alpha \neq a$, and f_Θ is constant in that interval.

The well known Lebesgue theorem is also proved using the Rising Sun function.

Another method of constructing a class of circular Models utilizing RSF is observed in the accompanying hypothesis and a representation is likewise remembered for this section. These models are named as 'Rising Sun Circular models'

Theorem 2.2: [Girija (2010)]: If g is the pdf and *G* is the cdf of a random variable of a circular distribution then the Rising Sun function g_Θ, gives rise to the pdf g_c of a circular model. The distribution function of g_c is given by

$$G_c = \begin{cases} \frac{1}{K}\left[\theta_1\, g(\theta_1) + G(\theta) - G(\theta_1)\right] & \text{for } \theta_1 < \theta \\ \frac{1}{K}\left[\theta\, g(\theta_1)\right] & \text{for } \theta_1 \geq \theta \end{cases} \tag{2.7}$$

Analogous to Theorem 2.1 the Rising Sun lemma for circular data is presented as follows

The Circular Rising Sun Lemma 2.3 [Radhika et al (2013)]: Let $g:[0,2\pi]\to\mathbb{R}$ be a continuous function. Then $E=\{\theta\in(0,2\pi): g_\Theta(\theta)>g(\theta)\}$ is open. If (α,β) is a component of E then $g_\Theta(\beta)=g(\beta)$ and $g_\Theta(\alpha)=g(\alpha)$ for $\alpha\neq 0$, and g_Θ is constant in that interval. g_c is the normalized function of g_Θ, hence β represents the mode of the circular model at which both g_Θ and g_c have maximum value. Hence component of g_c as well as g_Θ is (0, β) and g_c is the pdf of circular model known as Rising Sun circular model.

3. THE RISING SUN STEREOGRAPHIC REFLECTED LOGLOGISTIC MODEL

The Stereographic Reflected Loglogistic SRLLG(α,σ) distribution is derived by Sreekanth et al (2018).

The pdf $g(\theta)$ and cdf $G(\theta)$ of the Stereographic Reflected Loglogistic model are respectively given by

$$g(\theta)=\frac{\alpha\sigma\sec^2\left(\frac{\theta}{2}\right)\left(\sigma\left|\tan\left(\frac{\theta}{2}\right)\right|\right)^{\alpha-1}}{4\left(1+\left(\sigma\left|\tan\left(\frac{\theta}{2}\right)\right|\right)^{\alpha}\right)^2}\text{, where } \alpha,\sigma>0\text{ }, -\pi<\theta<\pi \tag{3.1}$$

and

$$G(\theta)=\begin{cases} 1-\frac{1}{2}\left[\dfrac{2+\left(\sigma\left|\tan\left(\frac{\theta}{2}\right)\right|\right)^{-\alpha}}{1+\left(\sigma\left|\tan\left(\frac{\theta}{2}\right)\right|\right)^{-\alpha}}\right], & \alpha,\sigma>0,\ -\pi<\theta<0 \\ \frac{1}{2}\left[\dfrac{2+\left(\sigma\left|\tan\left(\frac{\theta}{2}\right)\right|\right)^{-\alpha}}{1+\left(\sigma\left|\tan\left(\frac{\theta}{2}\right)\right|\right)^{-\alpha}}\right], & \alpha,\sigma>0,\ 0<\theta<\pi \end{cases} \tag{3.2}$$

respectively.

The Rising Sun function of the Stereographic Reflected Loglogistic distribution is defined as

$$g_\Theta(\theta)=\operatorname{Sup}\left(g(t):\theta\leq t<2\pi\right) \tag{3.3}$$

$$=\operatorname{Sup}\left(\frac{\alpha\sigma\sec^2\left(\frac{t}{2}\right)\left(\sigma\left|\tan\left(\frac{t}{2}\right)\right|\right)^{\alpha-1}}{4\left(1+\left(\sigma\left|\tan\left(\frac{t}{2}\right)\right|\right)^{\alpha}\right)^2}\text{, where } \alpha,\sigma>0\text{ }, -\pi<\theta<\pi\ :\theta\leq t<\pi\right)$$

Normalizing this function with the constant $K_1 = \int_0^{2\pi} g_\Theta(\theta)\ d\theta$ the pdf of the Rising Sun Stereographic Reflected Loglogistic distribution (RSRLLG) is obtained.

$$g_c(\theta) = \frac{\operatorname{Sup}\left(\dfrac{\alpha\sigma \sec^2\left(\frac{t}{2}\right)\left(\sigma\left|\tan\left(\frac{t}{2}\right)\right|\right)^{\alpha-1}}{4\left(1+\left(\sigma\left|\tan\left(\frac{t}{2}\right)\right|\right)^{\alpha}\right)^2} \text{, where } \alpha,\sigma>0\ , -\pi<\theta<\pi\ \ :\theta\le t<\pi \right)}{\int_0^{2\pi} g_\Theta(\theta)\ d\theta}$$

(3.4)

The graphs of pdf of the Rising Sun Stereographic Reflected Loglogistic distribution (RSRLLG) are plotted and population characteristics are studied using MATLAB.

Figure 3.1 Graph of pdf of (RSRLLG) (Linear Representation)

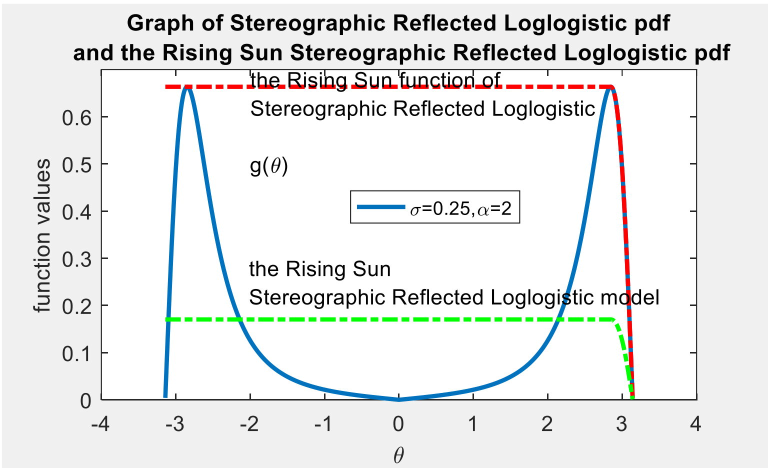
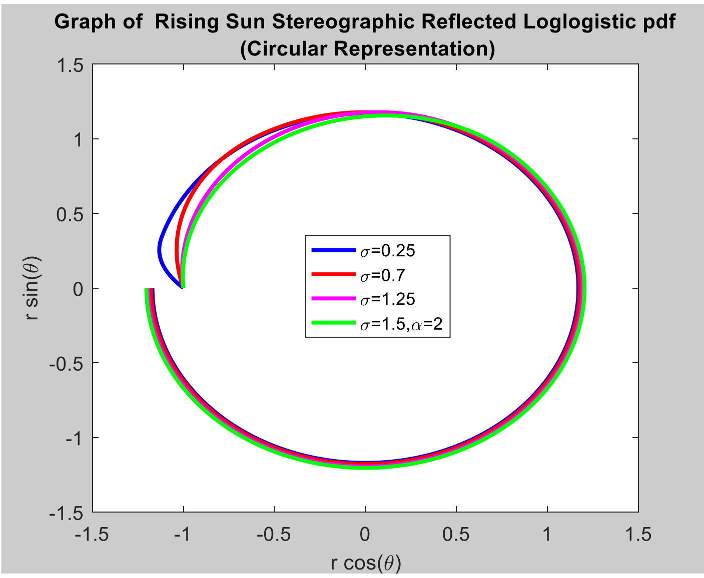


Figure 3.2 Graph of pdf of RSRLLG (Circular Representation)



4. THE POPULATION CHARACTERISTICS OF NEW RISING SUN CIRCULAR MODEL

The characteristic function of a circular model with probability density function $g(\theta)$ is defined as $\varphi_p = \int_0^{2\pi} e^{ip\theta} g(\theta)\, d\theta$, $p \in$.

The Characteristic function of the Rising Sun circular model is derived here under

Theorem 4.1 [Radhika et al (2013)]: Let g and φ_p be the pdf and the characteristic function of a circular model and θ_1 be the mode of g. Then the characteristic function of the corresponding Rising Sun circular model with pdf g_c is

$$\varphi_\Theta(p) = \begin{cases} 1 \text{ for } p = 0 \\ \dfrac{\varphi_p(\theta)}{k} + \dfrac{g(\theta_1)}{kp} i\left(1 - e^{ip\theta_1}\right) - \dfrac{1}{k}\displaystyle\int_0^{\theta_1} e^{ip\theta} g(\theta)\, d\theta \text{ for } p \neq 0 \end{cases} \quad (4.1)$$

where g_Θ is the Rising Sun function of g, and $k = \int_0^{2\pi} g_\Theta(\theta)\ d\theta$

Proof: The pdf of the Rising Sun circular Model is

$$g_c(\theta) = \frac{1}{k} g_\Theta(\theta) \quad (4.2)$$

For $p \neq 0$

The characteristic function of the Rising Sun circular model is

$$\varphi_\Theta(p) = \int_0^{2\pi} e^{ip\theta} g_c(\theta)\, d\theta$$

$$= \frac{1}{k}\int_0^{2\pi} e^{ip\theta} g_\Theta(\theta)\, d\theta$$

$$\varphi_\Theta(p) = \frac{1}{k}\left[\int_0^{\theta_1} e^{ip\theta} g_\Theta(\theta)\, d\theta + \int_{\theta_1}^{2\pi} e^{ip\theta} g_\Theta(\theta)\, d\theta\right]$$

$$= \frac{1}{k}\left[\int_0^{\theta_1} e^{ip\theta} g(\theta_1)\, d\theta + \int_{\theta_1}^{2\pi} e^{ip\theta} g(\theta)\, d\theta\right]$$

$$= \frac{g(\theta_1)}{k}\left(\frac{-i\, e^{ip\theta}}{p}\right)_0^{\theta_1} + \frac{1}{k}\phi_p(\theta) - \frac{1}{k}\int_0^{\theta_1} e^{ip\theta} g(\theta)\, d\theta$$

by known theorem $g_\Theta(\theta) = g(\theta_1)$ for $\theta \le \theta_1$

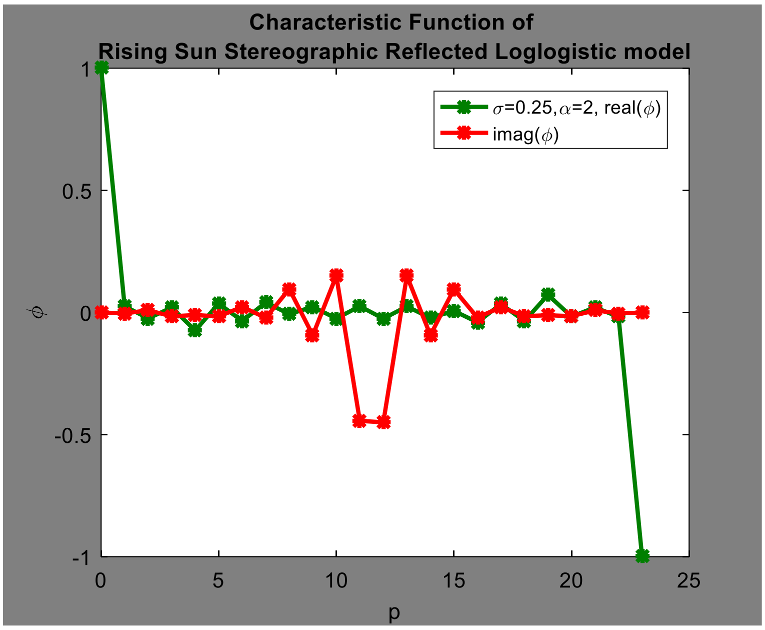
$= g(\theta)$ for $\theta_1 < \theta$

$$\text{for } p = 0,\ \varphi_\Theta(0) = \frac{1}{k}\int_0^{2\pi} g_\Theta(\theta)\ d\theta = 1$$

Hence $$\varphi_{\Theta}(p)=\begin{cases} 1 \text{ for } p=0 \\ \dfrac{\phi_p(\theta)}{k}+\dfrac{g(\theta_1)}{kp}i\left(1-e^{ip\theta_1}\right)-\dfrac{1}{k}\displaystyle\int_0^{\theta_1} e^{ip\theta}g(\theta)d\theta \text{ for } p\neq 0 \end{cases} \quad (4.3)$$

The graph of the characteristic function of new Rising Sun circular model is presented here.

Figure 3.3 Graph of the Characteristic Function of the Rising Sun Stereographic Reflected Loglogistic distribution for $\sigma=0.25,\alpha=2$



To study the characteristics of the circular models, the trigonometric moments which are the real and imaginary parts of the characteristic function are required. As the pdf of the newly constructed Rising Sun circular model is not in closed form, the values of the characteristic functions can be evaluated using numerical methods in MATLAB. The population characteristics for the Rising Sun Offset Pearson Type II model are also based on their respective trigonometric moments. These can be expressed in terms of trigonometric moments α_p and β_p [Mardia and Jupp (2000)] and are presented in table 3.1.

From the population characteristics, it can be observed that with increasing value of scale parameter σ and fixed value of $\alpha=2$, the circular variance gradually decreases, the distribution remains platykurtic.

Table 3.1: The Characteristics of the Rising Sun Stereographic Reflected Loglogistic distribution

	$\sigma=0.75,\ \alpha=2$	$\sigma=1.25,\ \alpha=2$	$\sigma=1.5,\ \alpha=2$	$\sigma=1.75,\ \alpha=2$
Trigonometric Moments				
α_1	0.0521	0.1126	0.1375	0.1531
α_2	-0.0416	-0.0444	-0.0267	-0.0058
β_1	-0.0176	-0.0776	-0.1211	-0.1644
β_2	0.0321	0.1022	0.1323	0.1496
Resultant Length				
ρ_1	0.0550	0.1368	0.1832	0.2247
ρ_2	0.0526	0.1114	0.1350	0.1497

Mean Direction μ_0	-0.3254	-0.6036	-0.7220	-0.8210
Circular Variance v_0	0.9450	0.8632	0.8168	0.7753
Circular Standard Deviation				
σ_0	2.4088	1.9946	1.8424	1.7280
	2.4273	2.0950	2.0012	1.9489
	0.0550	0.1368	0.1832	0.2247
	-0.0526	-0.1113	-0.1347	-0.1488
	0	0	0	0
	1.0e-03 0.3060	-0.0052	-0.0097	-0.0164
	3.3312e-04	-0.0064	-0.0132	-0.0240
	-0.0589	-0.1498	-0.2035	-0.2518

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