Domination Parameters on Bipartite graphs of the Commutative ring Z_n

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ABSTRACT
In 1977,E.J.Cockayne and S.T. Hedetniemi conducted a commendable and broad survey on the outcomes of the existing concepts of dominating set in graphs. It's basic concept is the dominating set and the domination number. In this paper our main aim is to find out the dominating parameters of Bipartite graph obtained on the Commutative ring of type Z_n. A subset D of V is said to be a dominating set of G if every vertex in V-D is adjacent to a vertex in D. A bipartite graph is a graph G whose vertex set is partitioned into two disjoint subsets X and Y such that each edge in G has one end in X and the other end in Y. We determine the domination parameters i.e., domination number, dominating set and Minimum domination number of Bipartite graphs of the commutative ring Z_n.

KEYWORDS: Commutative ring Z_n, Dominating set, Minimum dominating number, Bipartite graph.

I. INTRODUCTION

Graph theory is an important branch of Mathematics. Domination is an area in graph theory with an extensive research activity. The theory of domination in graphs introduced by Ore [5] and Berge [6] is an emerging area of research in graph theory today. For the first time in 1962, the concepts were entitled 'dominating set and dominating number by Ore. In 1977,E.J.Cockayne and S.T. Hedetniemi [8] conducted a commendable and broad survey on the outcomes of the existing concepts of dominating set in graphs at that time. The notation γ (G) for the domination number of a graph was applied by the pair and was accepted widely since then.

II. BASIC CONCEPTS AND PRELIMINARIES:

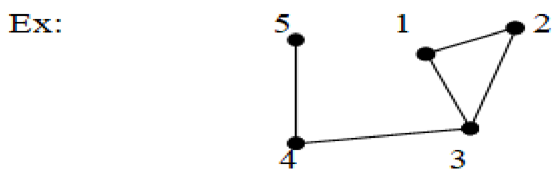
Ring: A non-empty set R together with two binary operations + and · is called a ring if the following conditions are satisfied.

(i) (R,+) is an abelian group
(ii) (R,·) is a semi group.
(iii) The operation · is distributive over +, i.e., a · (b+c) = a.b + a.c and (a+b).c = a.c + b.c.

Commutative ring: The ring (R, +, ·) is called a commutative ring if it satisfies commutative property i.e., a.b = b.a ∀ a,b∈ R.

Connected dominating set: A dominating set D of a graph G is a connected dominating set if the induced sub graph <V-D> is connected domination number γ_c (G) of the graph G is the minimum cardinality of the connected dominating set.

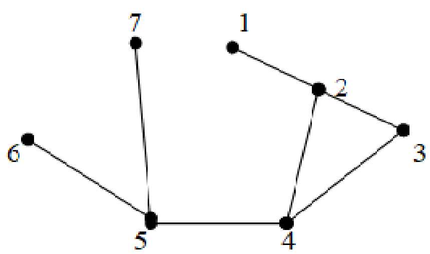
Ex:



Graph G

Minimum dominating set: A dominating set D of a Graph G is called a minimum dominating set if no proper subset of D is a dominating set.

Ex:



Graph G

In this graph G, the dominating sets are {2,5},{1,4,5},{1,4,6,7}

The minimum dominating set={2,5}.

Bipartite Graph: A bipartite graph is a graph G whose vertex set is partitioned into two disjoint subsets X and Y such that each edge in G has one end in X and the other end in Y. Such a partition (X, Y) is called a bipartition of the graph.

III. MAIN RESULT

Result 1: Let R= Z_n be a commutative ring of integers modulo n and let G(R)=(V(R),E(R)) be the graph of R, where V(R) is vertex set of G(R) and E(R) is edge set of G(R) and E(R) is defined as E(R)= {x,y ∈R / x, y are adjacent to each other iff $x+_ny=1, x\neq y$}.Then the domination number of G(R) is either n/2 or (n-1)/2 (n >1) .

Proof: Let R= Z_n be a commutative ring of integers modulo n and let G(R)= (V(R), E(R)) denote the graph of R with vertex set V(R) and edge set E(R) ,where E(R) is define by E (R)= {x,y ∈R / x, and y are adjacent to each other iff $x+_ny=1, x\neq y$}, then the graph G(R) is a **Bipartite graph**.

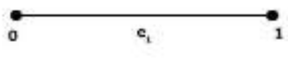
Now we want to find the domination number γ (G) of the graph G(R).

First we construct the dominating set for the bipartite graph where V be the vertex set and let D be the sub set of the vertex set V of G(R). Then D will become the dominating set if every vertex in V-D is adjacent to some vertex in D.

If n=1 then the graph of G(R) is null graph .So there does not exists dominating set for the **null graph.** If n(≥2) is even, then the graph G(R) is an **(n/2,n/2)- bipartite graph.** For the (n/2,n/2)- bipartite graph, in the bipartition each partition contains n/2 vertices and each vertex in the partition is adjacent to only one vertex in the other partition of the graph. Hence the dominating set for this graph contains at least n/2 vertices. $|D| \geq$ n/2.Therefore the minimum cardinality of D is n/2.Hence the domination number is n/2. Similarly if n(≥3) is odd then the graph G(R) is a **((n-1)/2,(n-1)/2))-bipartite graph.** So for this graph the dominating set consist of at least (n-1)/2 vertices .i.e,. $|D| \geq$ (n-1)/2. Therefore the minimum cardinality of D is (n-1)/2Hence the domination number for this graph is (n-1)/2.

Illustration 1: Let R be a commutative ring of integers modulo n.

Case(i) : Let n=2,then R={0,1} and V(R)={0,1} E(R)={e_1} where e_1 =(0,1) the graph of G(R) is a **(1,1)- bipartite graph**.



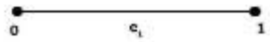
(1,1) - Bipartite graph

Let D={ 0 } then V-D={1} where V={0,1} and every vertex of V-D is adjacent to some vertex in D

Therefore D={0} is the dominating set. Similarly we can see that D={1} is dominating set for G(R). Hence the minimum dominating set of G(R) is one of the set {1} and {0}.

Therefore domination number γ (G)=1.

Similarly let n=3,then R={0,1,2} and V(R)={0,1}, E(R)={e_1} where e_1 =(0,1), the graph of G(R) is a **(1,1)- bipartite graph**.



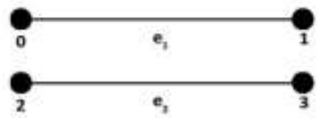
(1,1) - Bipartite graph

Let D={ 0 } then V-D={1} where V={0,1} and every vertex of V-D is adjacent to some vertex in D

Therefore D={0} is the dominating set .Similarly we can see that D={1} is dominating set for G(R). Hence the minimum dominating set of G(R) is one of the set {0} and {1}.

Therefore domination number γ (G)=1.

Case(ii): Let n=4 then R={0,1,2,3} and V(R)={0,1,2,3} E(R)={ e_1, e_2} where e_1 =(0,1)
e_2=(2,3) the graph G(R) is a **(2,2)- bipartite graph**.



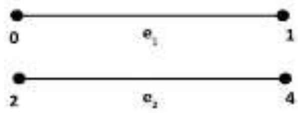
(2,2) - Bipartite graph

Let D={ 0,2 } then V-D={1,3} where V={0,1,2,3} and every vertex of V-D is adjacent to some vertex in D

Therefore D={0,2} is the dominating set. Similarly we can see that D={1,3} is dominating set for G(R). Hence the minimum dominating set of G(R) is one of the set {0,2}and {1,3}.

Therefore domination number γ (G)=1.

Similarly let n=5 then R={0,1,2,3,4} and V(R)={0,1,2,4}, E(R)={ e_1, e_2} where e_1 =(0,1) e_2=(2,4) the graph G(R) is a **(2,2)- Bipartite graph**.



(2,2) - Bipartite graph

Let D={ 0,2 } then V-D={1,4} where V={0,2} and every vertex of V-D is adjacent to some vertex in D

Therefore D={0,2} is the dominating set .Similarly we can see that D={1,4} is dominating set for G(R).Hence the minimum dominating set of G(R) is one of the set {0,2} and {1,4}.

Therefore domination number γ (G)=2.

Conclusion : From the above Illustration we observe that the domination number for the bipartite graph G(R) is either **(n/2) or (n-1)/2 (n >1).**

Result 2: Let R = Z_n be a commutative ring of integers modulo n and let G(R)=(V(R),E(R)) be the graph of R, where V(R) is vertex set of G(R) and E(R) is edge set of G(R),where E(R) is defined by E (R)={x,y∈R / x,y and z are adjacent to each other iff $x+_ny=1, x\neq y$ and x , y ≠0 }.Then the domination number of G(R) is either **(n-2)/2,** or **(n-3)/2** (n >3).

Proof: Let R= Z_n be a commutative ring of integers modulo n and let G(R)= (V(R), E(R)) denote the graph of R with vertex set V(R) and edge set E(R) ,where E(R) is define by E (R)= {x,y ∈R / x, and y are adjacent to each other iff $x+_n y=1$, x≠y and exactly one of x and y does not be zero}, then the graph G(R) is a **bipartite graph.**

Now we want to find the domination number γ (G) of the graph G(R).

First we construct the dominating set for the bipartite graph where V be the vertex set and let D be the subset of the vertex set V of G(R). Then D will become the dominating set if every vertex in V-D is adjacent to some vertex in D.

If n>3 then the graph of G(R) is null graph .So there does not exists dominating set for the **null graph.** If n(≥4) is even, then the graph G(R) is an **((n-2)/2,(n-2)/2)- bipartite graph.** For the (n-2/2,n-2/2)- bipartite graph, in the bipartition each partition contains (n-2)/2 vertices and each vertex in the partition is adjacent to only one vertex in the other partition of the graph. Hence the dominating set for this graph contains at least (n-2)/2 elements. $|D| \geq$ (n-2)/2.Therefore the minimum cardinality of D is (n-2)/2. Hence the domination number is (n-2)/2. Similarly if n(≥5) is odd then the graph G(R) is a **((n-3)/2,(n-3)/2))-bipartite graph.** So for this graph the dominating set consist of at least (n-3)/2 vertices. i.e,. $|D| \geq$ (n-3)/2. the minimum cardinality of D is (n-3)/2.Hence the domination number for this graph is (n-3)/2.

Illustration 2: Let R be a commutative ring of integers modulo n, .

Case(i): Let n=4 then R={0,1,2,3} and V(R)={2,3} E(R)={e_1} where e_1 =(2,3), the graph G(R) is a **(1,1)- bipartite graph**.

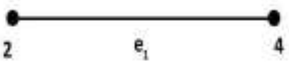
(1,1) - Bipartite graph

Let D={ 2 } then V-D={3} where V={2,3} E(R)={ e_1} where e_1 =(2,3) and every vertex of V-D is adjacent to some vertex in D

Therefore D={2} is the dominating set. Similarly we can see that D={3} is dominating set for G(R). Hence the minimum dominating set of G(R) is one of the set {2}and {3}.

Therefore domination number γ (G)=1.

Similarly let n=5 then R={0,1,2,3,4} and V(R)={2,4} E(R)={ e_1} where e_1 =(2,4) the graph G(R) is a **(1,1)- bipartite graph**.



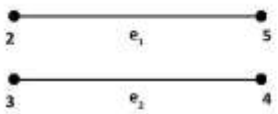
(1,1) - Bipartite graph

Let D={ 2 } then V-D={4} where V={2,4} and every vertex of V-D is adjacent to some vertex in D

Therefore D={2} is the dominating set .Similarly we can see that D={4} is dominating set for G(R). Hence the minimum dominating set of G(R) is one of the set {2} and {4}.

Therefore domination number γ (G)=1.

Case(ii): Let n=6 then R={0,1,2,3,4,5} and V(R)={2,3,4,5} E(R)={ e_1, e_2} where e_1 =(2,5) e_2 = (3,4) the graph G(R) is a **(2,2)- bipartite graph**.



(2,2) - Bipartite graph

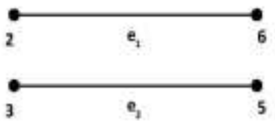
Let D={2,3 } then V-D={5,4} where V={2,3,4,5} and every vertex of V-D is adjacent to some vertex in D

Therefore D={2,3} is the dominating set .Similarly we can see that D={5,4} is dominating set for G(R).

Hence the minimum dominating set of G(R) is one of the set {2,3} and {5,4}.

Therefore domination number γ (G)=2**.**

Similarly let n=7.Then R={0,1,2,3,4,5,6} and V(R)={0,1,2,4} E(R)={ e_1, e_2} where e_1 =(2,6) e_2 = (3,5) the graph G(R) is a **(2,2)- Bipartite graph**.



(2,2) - Bipartite graph

Let D={2,6 } then V-D={6,5} where V={2,3,5,6} and every vertex of V-D is adjacent to some vertex in D

Therefore D={2,6} is the dominating set .Similarly we can see that D={5,6} is dominating set for G(R). Hence the minimum dominating set of G(R) is one of the set {2,6} and{5,6}.

Therefore domination number γ (G)=2.

Conclusion: From the above illustration we observe that, the domination number for the bipartite graph G(R) is either **(n-2)/2 or (n-3)/2 (n >3).**

Result 3: Let R= Z_n be a commutative ring of integers modulo n and let G(R)=(V(R),E(R)) be the graph of R, where V(R) is vertex set of G(R) and E(R) is edge set of G(R) and E(R) is defined as E (R)= {x,y ∈R / x, y are adjacent to each other iff $x+_n y=0$,x≠y}.Then the domination number of G(R) is either (n-1)/2 or (n-2)/2 (n >3) .

Proof: Let R= Z_n be a commutative ring of integers modulo n and let G(R)= (V(R), E(R)) denote the graph of R with vertex set V(R) and edge set E(R) ,where E(R) is define by E (R)= {x,y ∈R / x, and y are adjacent to each other iff $x+_n y=0$, x≠y and x,y≠0 } then the graph G(R) is a bipartite graph**.**

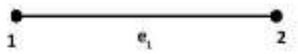
Now we want to find the domination number γ (G) of the graph G(R).

First we construct the dominating set for the bipartite graph where V be the vertex set and let D be the sub set of the vertex set V of G(R). Then D will become the dominating set if every vertex in V-D is adjacent to some vertex in D.

If n=1 then the graph of G(R) is null graph .So there does not exists dominating set for the **null graph.** If n(≥3) is odd, then the graph G(R) is an **((n-1)/2, (n-1)/2))- bipartite graph.** For the ((n-1)/2,(n-1)/2))- bipartite graph, in the bipartition each partition contains (n-1)/2 vertices and each vertex in the partition is adjacent to only one vertex in the other partition of the graph. Hence the dominating set for this graph contains at least (n-1)/2 vertices. $|D| \geq$(n-1)/2.Therefore the minimum cardinality of D is (n-1)/2.Hence the domination number is (n-1)/2. Similarly if n(≥4) is even then the graph G(R) is a **((n-2)/2,(n-2)/2))-bipartite graph.** So for this graph the dominating set consist of at least (n-2)/2 vertices i.e. $|D| \geq$ (n-2)/2. the minimum cardinality of D is (n-2)/2.Hence the domination number for this graph is (n-2)/2.

Illustration 3: Let R be a commutative ring of integers modulo n.

Case(i): Let n=3 then R={0,1,2} and V(R)={1,2}, E(R)={e_1} where e_1 =(1,2), the graph G(R) is a **(1,1)- bipartite graph**.



(1,1) - Bipartite graph

Let D={ 2 } then V-D={1} where V={1,2} and every vertex of V-D is adjacent to some vertex in D

Therefore D={2} is the dominating set .Similarly we can see that D={1} is dominating set for G(R). Hence the minimum dominating set of G(R) is one of the set {2} and {1}.

Therefore domination number γ (G)=1.

Similarly let n=4 then R={0,1,2,3} and V(R)={2,3} E(R)={e_1} where e_1 =(2,3), the graph of G(R) is a **1- bipartite graph**.

(1,1) - Bipartite graph

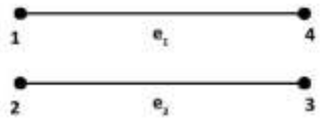
Let D={ 2 } then V-D={3} where V={2,3} and every vertex of V-D is adjacent to some vertex in D

Therefore D={2} is the dominating set .Similarly we can see that D={3} is dominating set for G(R).

Hence the minimum dominating set of G(R) is one of the set {3} and{1}.

Therefore domination number γ (G)=1.

Case(ii): Let n=5 then R={0,1,2,3,4} and V(R)={1,2,3,4} E(R)={ e_1, e_2} where e_1 =(1,4) e_2=(2,3), the graph G(R) is a **(2,2)- bipartite graph** .



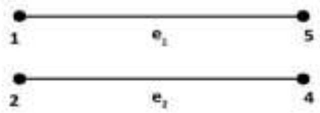
(2,2) - Bipartite graph

Let D={ 1,2 } then V-D={3,4} where V={1,2,3,4} and every vertex of V-D is adjacent to some vertex in D

Therefore D={1,2} is the dominating set .Similarly we can see that D={3,4} is dominating set for G(R).Hence the minimum dominating set of G(R) is one of the set {3,4} and{1,2}

Therefore domination number γ (G)=2.

Similarly let n=6 then R={0,1,2,3,4,5} and V(R)={1,2,4,5} E(R)={ e_1, e_2} where e_1 =(1,5) e_2=(2,4) the graph of G(R) is a **(2,2)- Bipartite graph**.



(2,2) - Bipartite graph

Let D={ 1,2 } then V-D={4,5} where V={1,2,4,5} and every vertex of V-D is adjacent to some vertex in D.

Therefore the D={1,2} is the dominating set .Similarly we can see that D={4,5} is dominating set for G(R). Hence the minimum dominating set of G(R) is one of the set {3,4} and {1,2}.

Therefore domination number γ (G)=2.

Conclusion : From the above Illustration we observe that the domination number for the bipartite graph G(R) is either **(n-3)/2 or (n-2)/2 (n >3) .**

Result 4: Let R= $Z_n \times Z_n$ be a commutative ring of integer modulo n and let G(R)=(V(R), E(R)) be the graph of R where V(R) is vertex set of G(R) and E(R) is edge set of G(R). If E (R)= {x,y∈R / x and y are adjacent to each other iff $x+_n y=0$, where x=(x_1,y_1), y=(x_2,y_2), x≠y and x,y≠0}. If n=2r+1, where r=1,2,3- - - .Then the domination number of G(R) is 2r(r+1) (r≥1).

Proof : Let R= Z_n be a commutative ring of integers modulo n and let G(R)= (V(R), E(R)) denote the graph of R with vertex set V(R) and edge set E(R) ,where E(R) is define by E (R)= {x,y ∈R / x, and y are adjacent to each other iff $x+_n y=0$, where x=(x_1,y_1), y=(x_2,y_2), x≠y and x,y≠0} then the graph G(R) is a bipartite graph.

Now we want to find the domination number γ (G) of the graph G(R).

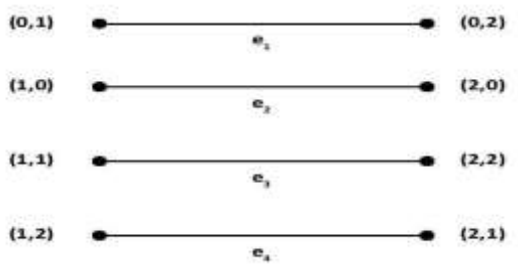
First we construct the dominating set for the bipartite graph where V be the vertex set and let D be the sub set of the vertex set V of G(R). Then D will become the dominating set if every vertex in V-D is adjacent to some vertex in D.

If n is even then the graph G(R) is a null graph therefore there does not exist any dominating set for the null graph. If n is odd ($n\geq 3$) i.e., n=2r+1,r=1,2,3 - - -.Then the graph G(R) is (2r(r+1),2r(r+1))-bipartite graph .For this bipartite graph in the bipartition each partition contain 2r(r+1) vertices and each vertex in one partition is adjacent to exactly one vertex in the other partition and no two vertices in the partition are adjacent .Hence the dominating set for this graph contains at least 2r(r+1) vertices $|D|\geq 2r(r+1)$.Therefore the minimum cardinally of D is 2r(r+1).Hence the domination number is 2r(r+1).

Therefore D is the domination number for the above bipartite graphs and D is the domination number of the bipartite graph of G(R) is 2r(r+1) ($r\geq 1$).

Illustration 4 : Let R be a commutative ring of integers modulo n.

Case(i): Let n=3 then R which consists of 3^2 elements, R={(0,0) (0,1) (0,2) (1,0) (1,1) (1,2) (2,0) (2,1) (2,2)} where V(R)={(0,1), (0,2), (1,0), (2,0), (1,1), (2,2), (1,2),(2,1) } E(R) ={ e_1,e_2,e_3,e_4} where e_1=[(0,1) (0,2)] e_2=[(1,0) (2,0)] e_3=[(1,1) (2,2)] e_4=[(1,2) (2,1)] the graph of G(R) is a **(4,4)-Bipartite Graph**.



(4,4) - Bipartite graph

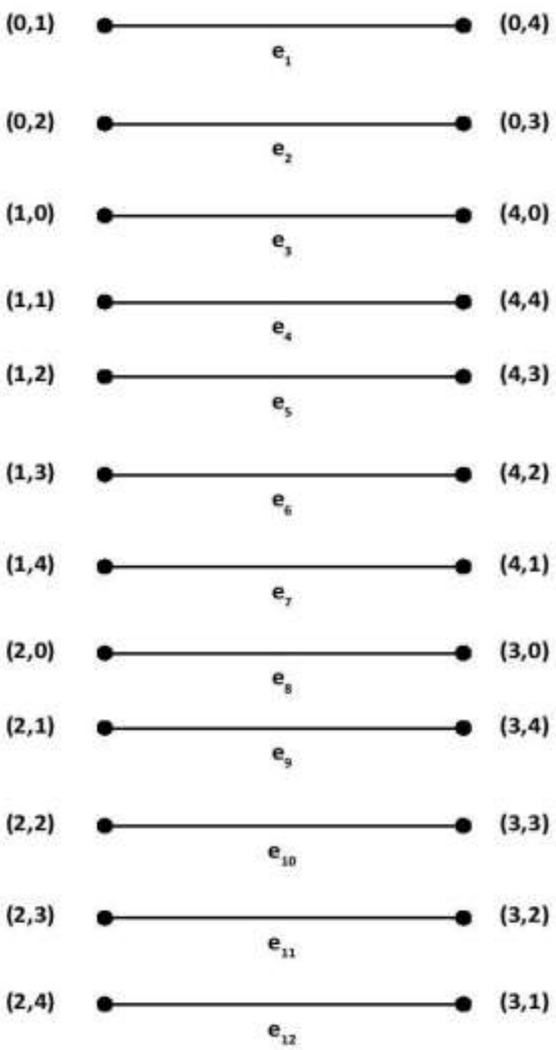
Let D={(0,1),(1.0),(1,1),(1,2)} then V-D={(0,2),(2,0),(2,2),(2,1)}

where V={ (0,1),(0,2),(1,0), (2,0),((1,1)(2,2),(1,2),(2,1)} and every vertex of V-D is adjacent to some vertex in D

Therefore D={(0,1),(1.0),(1,1),(1,2)} is the dominating set. Similarly we can see that D={(0,2),(2,0),(2,2),(2,1)} is dominating set for G(R). Hence the minimum dominating set of G(R) is one of the set {(0,1),(1.0),(1,1),(1,2)} and {(0,2),(2,0),(2,2),(2,1)}

Therefore domination number γ (G)=4.

Case (ii): Let n=5 then R which consists of 5^2 elements
R={(0,0)(0,1)(0,2)(0,3)(0,4)(1,0)(1,2)(1,3)(1,4)(2,0)(2,1)(2,2)(2,3)(2,4)(3,0) (3,1)(3,2)(3,3)(3,4)(4,0)(4,1)(4,2)(4,3)(4,4)}
WhereV(R)={(0,1),(0,2),(0,3),(0,4),(1,0),(1,2),(1,3),(1,4),(2,0),(2,1),(2,2),(2,3),(2,4)(3,1),(3,2),(3,3),(3,4),(4,0)(4,1), (4,2), (4,3),(4,4)}
E(R)={$e_1,e_2,e_3,e_4,e_5,e_6,e_7,e_8,e_9,e_{10},e_{11},e_{12}$}
where e_1=[(0,1),(0,4)], e_2=[(0,2),(0,3)], e_3=[(1,0),(4,0)] e_4=[(1,1),(4,4)] e_5=[(1,2), (4,3)] e_6=[(1,3),(4,2)] e_7=[(1,4),(4,1)] e_8=[(2,0),(3,0)] e_9 =[(2,1),(3,4)] e_{10}=[(2,2),(3,3)]
e_{11}=[(2,3) ,(3,2)] e_{12}=[(2,4) ,(3,1)].



(12,12) - Bipartite graph

Let D={(0,1),(0,2),(1,0),(1,1),(1,2),(1,3),(1,4),(2,0),(2,1),(2,2),(2,3)(2,4)}.

Then V-D={(0,4),(0,3),(4,0),(4,4),(4,3),(4,2), (4,1),(3,0),(3,4),(3,3),(3,2),(3,1)}.

Where V={(0,1), (0,2), (0,3) (0,4), (1,0), (1,2), (1,3), (1,4), (2,0), (2,1), (2,2), (2,3), (2,4)(3,1),(3,2),(3,3),(3,4),(4,0)(4,1),(4,2),(4,3),(4,4)}and every vertex of V-D is adjacent to some vertex in D.

Therefore D={(0,1),(0,2),(1,0),(1,1),(1,2),(1,3),(1,4),(2,0),(2,1), (2,2), (2,3), (2,4)} is the dominating set.

Similarly we can see that D={(0,4),(0,3),(4,0),(4,4),(4,3),(4,2),(4,1),(3,0),(3,4),(3,3),(3,2),(3,1)}.is dominating set for G(R). Hence the minimum dominating set of G(R) is one of the set
{(0,1),(0,2),(1,0),(1,1),(1,2),(1,3),(1,4),(2,0),(2,1),(2,2),(2,3)(2,4)}and
{(0,4),(0,3),(4,0),(4,4),(4,3),(4,2),(4,1),(3,0),(3,4),(3,3),(3,2),(3,1)}.

Therefore domination number γ (G)=12.

Conclusion: From the above illustration we observe that, the domination number for the bipartite graph G(R) is **2r(r+1). (r>1).**

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