International Journal of Mathematics and Computer Research
ISSN: 2320-7167
Volume 11 Issue 09 September 2023, Page no. – 3782-3788
Index Copernicus ICV: 57.55, Impact Factor: 7.362
DOI: 10.47191/ijmcr/v11i9.12

Common Fixed Point Theorem for Compatible Mappings of Type (P) in Multiplicative Cone Metric Space

A. S. Saluja[1], Jyoti Jhade[2]
[1,2] Department of Mathematics, Institute for Excellence in Higher Education, Bhopal – 462016, Madhya Pradesh, India.

ARTICLE INFO	ABSTRACT
Published Online: 30 September 2023	In this paper, we introduce the notion of compatible mappings of type (P) in multiplicative cone metric space and prove a common fixed point theorem for two pairs of compatible mappings of type (P) with multiplicative normal cone setting. Also, we give an example to show the validity of our result.
Corresponding Author: **A. S. Saluja**	**2020 Mathematical Sciences Classification:** 54H25, 47H10.

KEYWORDS: multiplicative cone metric space, compatible mappings of type (P), common fixed point.

1. INTRODUCTION

There exist numerous generalizations of metric space in fixed point theory. One of them is cone metric space, which is introduced by Huang and Zhang [10] in 2007. They analysed convergence and substituted real numbers by ordered Banach space. After that, various authors proved and extend many fixed point and *CFP* (common fixed point) results to this space with normal and non-normal cone conditions (see, e.g., [3, 6, 8, 9, 11, 12, 13, 14, 16, 17, 18, 19]).

Recently in 2017, C. Boateng Ampadu [1] introduced the notion of multiplicative cone metric in which he replaced triangle inequality property in cone metric space by multiplicative triangle inequality property and established a coupled version of higher-order "Banach contraction principle" with multiplicative normal cone condition, also he [2] proved a Hardy-Rogers fixed point theorem in this space uses the c- class multiplicative cone functions.

On the other hand, in 1976 it was the turning point in the theory of the existence of *CFP* for mappings when Jungck [4] introduced the idea of commutative mappings by generalizing the Banach contraction theorem and proved some *CFP* theorems by using these mappings. This opens a new and interesting area of research for researchers. Then in the sequel, in 1982, a less restrictive concept was introduced by Sessa [15] called weakly commutativity in order to generalize the commutativity concept. Thereafter, many authors prove and extend a variety of common fixed point theorems by substituting commutativity to weakly commutativity. Further, in 1986, Jungck [5] define a new notion of compatible mappings. These mappings are more general in nature than commutative and weakly commutative mappings that commutative mappings are weakly commutative and weakly commutative mappings are compatible but the converse may not be true. Again, in 1996, H. K. Pathak et al. [7] defined a different type of compatibility called compatible of type (P) and compared with other concepts of compatibility.

In this paper, firstly the notion of compatible mappings of type (P) is introduced to multiplicative cone metric space and then next we prove a *CFP* theorem for these mappings. Also, in the last, we show the validity of our proven result by an example.

2. PRELIMINARIES

In 2017, C. Boateng Ampadu gave the perception of multiplicative cone metric space as follows:

Definition 2.1 ([1]). *"Let K be a real Banach space. A subset L of K is called a multiplicative cone if and only if,*

$(L1)$ *L is closed, nonempty, and $L \neq \{1\}$,*

$(L2)$ *$u^m \cdot v^n \in L$, for all $u, v \in L$ and $m, n \geq 0$,*

$(L3)$ *$u \in L$ and $\frac{1}{u} \in L$ imply $u = 1$ i.e., $L \cap \frac{1}{L} = 1$."*

Definition 2.2 ([1]). *Let $L \subseteq K$ be a multiplicative cone, then partial ordering $\leq$ is defined on L by $u \leq v$ iff $\frac{u}{v} \in L$. Here $u < v$ indicates $u \leq v$ but $u \neq v$ and $u \ll v$ will stand for $\frac{u}{v} \in$ int (L) (interior of L).*

Definition 2.3 ([1]). *"Let $L \subseteq K$ is a multiplicative cone then it is called multiplicative normal if,*

$$\exists\ \Psi > 0 \text{ s.t., } \forall\ u, v \in K,\ 1 \leq u \leq v \text{ implies that, } \|u\| \leq \|v\|^{\Psi}.$$

The least positive number which satisfies the above condition is called the multiplicative constant of L. Here $\|\cdot\|$ denotes a multiplicative norm."

Definition 2.4 ([1]). *"Let K be a real Banach space and $L \subseteq K$ be a multiplicative cone. Let M be any non-empty set, then if the mapping $d: M \times M \to K$ satisfies the following:*

$(d1)$ $1< d(u, v)$, $\forall\ u, v \in M$ and $d(u, v) = 1$ iff $u \neq v$,

$(d2)$ $d(u, v) = d(v, u)$, $\forall\ u, v \in M$,

$(d3)$ $d(u, v) \leq d(u, w) \cdot d(w, v)$ $\forall\ u, w, v \in M$ (*multiplicative triangle inequality*).

Then pair (M, d) represents a multiplicative cone metric space (for short, MCM-space) and d is called a multiplicative cone metric on M."

Example 2.5. Let $K = R^2$, $L = \{(u, v) \in K: u, v \geq 1\} \subseteq R^2$, $M = R$ and mapping $d: M \times M \to K$ be such that, $d(u, v) = (\omega^{|u-v|}, \omega^{\sigma|u-v|})$, where $\omega > 1$ and $\sigma \geq 0$ is a constant. Then pair (M, d) is a MCM-space.

Example 2.6. Let $K = R_+^2$, $L = \{(u, v) \in K: u, v \geq 1\} \subseteq R^2$, $M = R$ and mapping $d: M \times M \to K$ be such that, $d(u, v) = (\left|\frac{u}{v}\right|, \left|\frac{u}{v}\right|^{\sigma})$, where $\sigma \geq 0$ is a constant. Then pair (M, d) is a MCM-space.

Definition 2.7 ([1]). *"Let (M, d) is multiplicative cone metric space, and $\{u_n\} \subset M$ be a sequence, then we say that sequence $\{u_n\}$ is;*

(i) *Multiplicative convergent and multiplicative converges to a point $u \in M$, if for every $\mu \in K$ with $1 << \mu$, there is N s. t., $\forall\ n > N$, $d(u_n, u) << \mu$, i.e. $\lim_{n\to\infty} u_n = u$.*

(ii) *Cauchy sequence, if for any $\mu \in K$ with $1 << \mu$, $\exists$ N s.t., $\forall\ n, m > N$, $d(u_n, u_m) << \mu$."*

Definition 2.8 ([1]). *"A multiplicative cone metric space is said to be complete if for every multiplicative Cauchy sequence is multiplicative convergent in M."*

Definition 2.9. *Let E, F: $M \to M$ are two self-mappings of MCM-space (M, d). Then E and F are said to be compatible mappings of type (P) if,*

$$\lim_{n\to\infty} d(EEu_{n,}\ FFu_n) = 1,$$

Whenever, sequence $\{u_n\} \subset M$ be such that $\lim_{n\to\infty} Eu_n = \lim_{n\to\infty} Fu_n = \lambda$, for some $\lambda \in M$.

Preposition 2.10. *Let E and F be two self-mappings of compatible of type (P) of a MCM-space (M, d). If $E\lambda = F\lambda$ for some $\lambda \in M$. Then $EF\lambda = EE\lambda = FF\lambda = FE\lambda$.*

Proof. Let $\{u_n\} \subseteq M$ s.t., $u_n = \lambda$, where $\lambda \in M$ and n = 1, 2, 3… and $E\lambda = F\lambda$. Then we have,

$$\lim_{n\to\infty} Eu_n = \lim_{n\to\infty} Fu_n = E\lambda$$

Since mappings E and F are compatible of type (P), i.e.,

$$d(EE\lambda, FF\lambda) = \lim_{n\to\infty} d(EEu_{n,} FFu_n) = 1$$

Therefore, $EE\lambda = FF\lambda$. Since $E\lambda = F\lambda$, then finally we get, $EF\lambda = EE\lambda = FF\lambda = FE\lambda$.

Preposition 2.11. *Let E and F be two self-mappings of compatible of type (P) of a MCM-space (M, d) and $\lim_{n\to\infty} Eu_n = \lim_{n\to\infty} Fu_n = \lambda$, for some $\lambda \in M$. Then,*

(i) $\lim_{n\to\infty} FFu_n = E\lambda$ *if E is continuous at λ.*

(ii) $\lim_{n\to\infty} EEu_n = F\lambda$ *if F is continuous at λ.*

(iii) $EF\lambda = FE\lambda$ *and* $E\lambda = F\lambda$ *if E and F are continuous at λ.*

Proof. (i). Let E is continuous at λ. Since $\lim_{n\to\infty} Eu_n = \lim_{n\to\infty} Fu_n = \lambda$, for some $\lambda \in M$, so we have

$$\lim_{n\to\infty} EEu_n = \lim_{n\to\infty} EFu_n = E\lambda.$$

Since mappings E and F are compatible of type (P), i.e.

$$\lim_{n\to\infty} d(E\lambda, FFu_n) = \lim_{n\to\infty} d(EEu_n, FFu_n) = d(E\lambda, E\lambda) = 1.$$

Therefore, $\lim_{n\to\infty} FFu_n = E\lambda$.

(ii). This can be proven by a similar argument to (i).

(iii). Let E and F be continuous at λ. Since $\lim_{n\to\infty} Fu_n = \lambda$ and E is continuous at λ, then by (i) we have $\lim_{n\to\infty} FFu_n = E\lambda$, also F is continuous at λ. So, $\lim_{n\to\infty} FFu_n = F\lambda$. Thus we get $E\lambda = F\lambda$ and by preposition 2.10, and using the uniqueness of the limit we obtain, $EF\lambda = FE\lambda$.

3. MAIN RESULT

We now prove the following *CFP* theorem for compatible mappings of type (P), which satisfy a contractive condition in the context of multiplicative cone metric space.

Theorem 3.1. *Let* (M,d) *be a complete multiplicative cone metric space and L be a multiplicative normal cone with multiplicative constant* Ψ. *Let E, F, H, I:* $M\to M$ *be four self-mappings of* (M, d), *which satisfy the following conditions:*

(1) $E(M) \subset I(M)$ *and* $F(M) \subset H(M)$

(2) $d(Eu, Fv) \le [\max\{d(Hu, Iv), d(Hu, Eu), d(Iv, Fv), d(Eu, Iv), d(Hu, Fv)\}]^{\eta}$,
$\forall\, u, v \in M$ *and* $\eta \in (0, \frac{1}{2})$

(3) *One of the mappings E, F, H, and I is continuous*

(4) (H, E) *and* (I, F) *are pairs of compatible mappings of type* (*P*).

Then mappings E, F, H, and I have a unique CFP.

Proof. Since $E(M) \subset I(M)$, consider a point $u_0 \in M$, there exists $u_1 \in M$ such that $Eu_0 = Iu_1 = v_0$. Now for this point u_1, there exists $u_2 \in M$ such that $Fu_1 = Hu_2 = v_1$. This continues to form sequences such that,

$$v_{2n} = Eu_{2n} = Iu_{2n+1}, \text{ and } v_{2n+1} = Fu_{2n+1} = Hu_{2n+1} \tag{3.1}$$

Now, put $u = u_{2n}$ and $v = u_{2n+1}$ in (2) we obtain,

$$\begin{aligned} d(v_{2n}, v_{2n+1}) &= d(Eu_{2n}, Fu_{2n+1}) \\ &\le [\max\{ d(Hu_{2n}, Iu_{2n+1}), d(Hu_{2n}, Eu_{2n}), d(Iu_{2n+1}, Fu_{2n+1}), d(Eu_{2n}, Iu_{2n+1}), \\ &\quad d(Hu_{2n}, Fu_{2n+1})\}]^{\eta} \\ &\le [\max\{ d(v_{2n-1}, v_{2n}), d(v_{2n-1}, v_{2n}), d(v_{2n}, v_{2n+1}), d(v_{2n}, v_{2n}), d(v_{2n-1}, v_{2n+1})\}]^{\eta} \\ &\le [\max\{ d(v_{2n-1}, v_{2n})\cdot d(v_{2n}, v_{2n+1}), d(v_{2n-1}, v_{2n})\cdot d(v_{2n}, v_{2n+1}), d(v_{2n-1}, v_{2n})\cdot \\ &\quad d(v_{2n}, v_{2n+1}), 1, d(v_{2n-1}, v_{2n})\cdot d(v_{2n}, v_{2n+1})\}]^{\eta} \\ &= \{d(v_{2n-1}, v_{2n})\}^{\eta} \cdot \{d(v_{2n}, v_{2n+1})\}^{\eta} \end{aligned}$$

This implies that,

$$d(v_{2n}, v_{2n+1}) \le \{d(v_{2n-1}, v_{2n})\}^{\frac{\eta}{1-\eta}}$$

$$d(v_{2n}, v_{2n+1}) \le \{d(v_{2n-1}, v_{2n})\}^{h} \tag{3.2}$$

Here $h = \frac{\eta}{1-\eta} \in (0, \frac{1}{2})$. Similarly, by using (2) we obtain,

$$\begin{aligned} d(v_{2n}, v_{2n+1}) &= d(Eu_{2n}, Fu_{2n+1}) \\ &\le [\max\{d(Hu_{2n}, Iu_{2n+1}), d(Hu_{2n}, Eu_{2n}), d(Iu_{2n+1}, Fu_{2n+1}), d(Eu_{2n}, Iu_{2n+1}), \\ &\quad d(Hu_{2n}, Fu_{2n+1})\}]^{\eta} \\ &\le [\max\{d(v_{2n-1}, v_{2n}), d(v_{2n-1}, v_{2n}), d(v_{2n}, v_{2n+1}), d(v_{2n}, v_{2n}), d(v_{2n-1},v_{2n+1})\}]^{\eta} \\ &= \{d(v_{2n}, v_{2n+1})\}^{\eta} \cdot \{d(v_{2n+1}, v_{2n+2})\}^{\eta} \end{aligned}$$

This implies that,

$$d(v_{2n+1}, v_{2n+2}) \le \{d(v_{2n}, v_{2n+1})\}^{\frac{\eta}{1-\eta}}$$

$$d(v_{2n+1}, v_{2n+2}) \le \{d(v_{2n}, v_{2n+1})\}^{h}, \quad h = \frac{\eta}{1-\eta} \in (0, \tfrac{1}{2}) \tag{3.3}$$

So, from (3.2) and (3.3), $\forall\, n \in N$ we get,

$$d(v_n, v_{n+1}) \le d(v_{n-1}, v_n)^{h} \le d(v_{n-2}, v_{n-1})^{h^2} \le \ldots\ldots\ldots d(v_0, v_1)^{h^n}.$$

Therefore, by using multiplicative triangle inequality, we obtain $\forall\, n, m \in N$ such that $n < m$,

$$\begin{aligned} d(v_n, v_m) &\le d(v_n, v_{n+1}) \cdot d(v_{n+1}, v_{n+2}) \ldots\ldots\ldots d(v_{m-1}, v_m) \\ &\le d(v_0, v_1)^{h^n} \cdot d(v_0, v_1)^{h^{n-1}} \cdot \ldots\ldots\ldots d(v_0, v_1)^{h^{m-1}}. \end{aligned}$$

$$\leq d(v_0, v_1)^{\frac{h^n}{1-h}}$$

Now, by using the condition of multiplicative normality of the cone, we get;

$$\|d(v_{n,} \ v_m)\| \leq \|d(v_0 \ v_1)\|^{\Psi\frac{h^n}{1-h}}.$$

Since $h < 1$ it follows that,

$\lim_{n,m\to\infty} d(v_{n,} \ v_m) = 1$. Hence $\{v_n\}$ is a multiplicative Cauchy sequence in M. Now since M is multiplicative complete so, there is a point $s \in M$ s.t. $\lim_{n\to\infty} v_n = s$. Consequently, we have,

$$\lim_{n\to\infty} Eu_{2n} = \lim_{n\to\infty} Iu_{2n+1} = \lim_{n\to\infty} Fu_{2n+1} = \lim_{n\to\infty} Hu_{2n+2} = s \quad (3.4)$$

Because, $\{v_{2n}\} = \{Eu_{2n}\} = \{Iu_{2n+1}\}$, $\{v_{2n+1}\} = \{Fu_{2n+1}\} = \{Hu_{2n+1}\}$ are sub sequences of $\{v_n\}$.

Case (i). First, suppose that E is continuous then,

$$\lim_{n\to\infty} EHu_{2n} = \lim_{n\to\infty} E^2u_{2n} = Es \quad (3.5)$$

Since, (H, E) is a pair of Compatible of type (P) mappings, so it follows from the preposition 2.11, we have,

$$\lim_{n\to\infty} H^2u_{2n} = Es \quad (3.6)$$

On putting $u = Hu_{2n}$ and $v = u_{2n+1}$ in inequality (2) we get,

$$d(EHu_{2n}, Fu_{2n+1}) \leq [\max\{d(H^2u_{2n}, Iu_{2n+1}), d(H^2u_{2n}, EHu_{2n}), d(Iu_{2n+1}, Fu_{2n+1}), d(EHu_{2n}, Iu_{2n+1}), d(H^2u_{2n}, Fu_{2n+1})\}]^{\eta}$$

Letting $n \to \infty$ and using (3.4), (3.5), and (3.6) we obtain,

$$d(Es, s) \leq [\max\{d(Es, s), d(Es, Es), d(s, s), d(Es, s), d(Es, s)\}]^{\eta}$$
$$= [\max\{d(Es, s), 1, 1, d(Es, s), d(Es, s)\}]^{\eta}$$
$$= [d(Es, s)]^{\eta}$$
$$d(Es, s) \leq [d(Es, s)]^{\eta}.$$

This implies that, $d(Es, s) = 1$, i.e., $Es = s$

Now, $s = Es \in E(M) \subset I(M)$, so there exist $\alpha \in M$ s.t.,

$$s = Es = I\alpha \quad (3.7)$$

On putting $u = Hu_{2n}$ and $v = \alpha$ in inequality (2) we get,

$$d(EHu_{2n}, F\alpha) \leq [\max\{d(H^2u_{2n}, I\alpha), d(H^2u_{2n}, EHu_{2n}), d(I\alpha, F\alpha), d(EHu_{2n}, I\alpha), d(H^2u_{2n}, F\alpha)\}]^{\eta}$$

Letting $n \to \infty$ and using (3.5), (3.6), and (3.7) we obtain,

$$d(Es, F\alpha) \leq [\max\{d(Es, Es), d(Es, Es), d(Es, F\alpha), d(Es, Es), d(Es, F\alpha)\}]^{\eta}$$
$$= [d(Es, F\alpha)]^{\eta}$$
$$d(s, F\alpha) \leq [d(s, F\alpha)]^{\eta}.$$

This implies that, $s = F\alpha$ (3.8)

Since (I, F) is a pair of compatible of type (P) mappings and $I\alpha = s = F\alpha$. Then by preposition 2.10, we have $IF\alpha = FI\alpha$, and hence,

$$Is = IF\alpha = FI\alpha = Fs \quad (3.9)$$

On putting $u = u_{2n}$ and $v = s$ in inequality (2) we get,

$$d(Eu_{2n}, Fs) \leq [\max\{d(Hu_{2n}, Is), d(Hu_{2n}, Eu_{2n}), d(Is, Fs), d(Eu_{2n}, Is), d(Hu_{2n}, Fs)\}]^{\eta}$$

Letting $n \to \infty$ and using (3.9) we obtain,

$$d(s, Fs) \leq [\max\{d(s, Fs), d(s, s), d(Fs, Fs), d(s, Fs), d(s, Fs)\}]^{\eta}$$
$$= [\max\{d(s, Fs), 1, 1, d(s, Fs), d(s, Fs)\}]^{\eta}$$
$$= [d(Es, s)]^{\eta}$$
$$d(s, Fs) \leq [d(s, Fs)]^{\eta}.$$

This implies that, $d(s, Fs) = 1$, i.e., $Fs = s$

Now, $s = Fs \in F(M) \subset H(M)$, so there exist a point $\alpha_2 \in M$ such that,

$$s = Fs = H\alpha_2 \quad (3.10)$$

On putting $u = \alpha_2$ and $v = s$ in inequality (2) and using (3.10) we get,

$$d(E\alpha_2, s) = d(E\alpha_2, Fs)$$
$$\leq [\max\{d(H\alpha_2, Is), d(H\alpha_2, E\alpha_2), d(Is, Fs), d(E\alpha_2, Is), d(H\alpha_2, Fs)\}]^{\eta}$$
$$= [d(E\alpha_2, s)]^{\eta}$$
$$d(E\alpha_2, s) \leq [d(E\alpha_2, s)]^{\eta}$$

This implies that, $d(E\alpha_2, s) = 1$ i.e., $E\alpha_2 = s$.

Since, (H, E) is a pair of compatible of type (P) mappings and $E\alpha_2 = s = H\alpha_2$, then it follows from preposition 2.10 that $Hs = HE\alpha_2 = EH\alpha_2 = Es$. Hence,

$$Hs = Es = Is = Fs = s$$

i.e., s is a *CFP* of mappings H, I, E, and F.

Case (ii). Suppose that F is continuous, then this can be proven similar to case (i).

Case (iii). Suppose that H is continuous then,

$$\lim_{n\to\infty} HEu_{2n} = \lim_{n\to\infty} H^2u_{2n} = Hs \tag{3.11}$$

Since, (H, E) is a pair of Compatible of type (P) mappings, so it follows from the preposition 2.11, we have,

$$\lim_{n\to\infty} E^2u_{2n} = Hs \tag{3.12}$$

On putting $u = Eu_{2n}$ and $v = u_{2n+1}$ in inequality (2) we get,

$$d(E^2u_{2n}, Fu_{2n+1}) \le [\max\{d(HEu_{2n}, Iu_{2n+1}), d(HEu_{2n}, E^2u_{2n}), d(Iu_{2n+1}, Fu_{2n+1}), d(E^2u_{2n}, Iu_{2n+1}), d(HEu_{2n}, Fu_{2n+1})\}]^{\eta}$$

Letting n → ∞ and using (3.4), (3.11), and (3.12) we obtain,

$$d(Hs, s) \le [\max\{d(Hs, s), d(Hs, Hs), d(s, s), d(Hs, s), d(Hs, s)\}]^{\eta}$$
$$= [\max\{d(Hs, s), 1, 1, d(Hs, s), d(Hs, s)\}]^{\eta}$$
$$= [d(Hs, s)]^{\eta}$$
$$d(Hs, s) \le [d(Hs, s)]^{\eta}.$$

This implies that, $d(Hs, s) = 1$, i.e., $Hs = s$ (3.13)

On putting $u = s$ and $v = u_{2n+1}$ in inequality (2) we get,

$$d(Es, Fu_{2n+1}) \le [\max\{d(Hs, Iu_{2n+1}), d(Hs, Es), d(Iu_{2n+1}, Fu_{2n+1}), d(Es, Iu_{2n+1}), d(Hs, Fu_{2n+1})\}]^{\eta}$$

Letting n → ∞ and using (3.4) and (3.13) we obtain,

$$d(Es, s) \le [\max\{d(s, s), d(s, Es), d(s, s), d(Es, s), d(s, s)\}]^{\eta}$$
$$= [\max\{1, d(s, Es), 1, d(Es, s), 1\}]^{\eta}$$
$$= [d(Es, s)]^{\eta}$$
$$d(Es, s) \le [d(Es, s)]^{\eta}.$$

This implies that,

$$d(Es, s) = 1, \text{ i.e., } Es = s \tag{3.14}$$

Now, $s = Es \in E(M) \subset I(M)$, so there exist $\alpha_3 \in M$ such that,

$$s = Es = I\alpha_3 \tag{3.15}$$

On putting $u = s$ and $v = \alpha_3$ in (2) and using (3.14) and (3.15) we have,

$$d(s, F\alpha_3) = d(Es, F\alpha_3)$$
$$\le [\max\{d(Hs, I\alpha_3), d(Hs, Es), d(I\alpha_3, F\alpha_3), d(Es, I\alpha_3), d(Hs, F\alpha_3)\}]^{\eta}$$
$$= [\max\{d(s, s), d(s, s), d(s, F\alpha_3), d(s, I\alpha_3), d(s, F\alpha_3)\}]^{\eta}$$
$$= [d(s, F\alpha_3)]^{\eta}$$
$$d(s, F\alpha_3) \le [d(s, F\alpha_3)]^{\eta}.$$

This implies that,

$$d(s, F\alpha_3) = 1, \text{ i.e., } F\alpha_3 = s \tag{3.16}$$

Since (I, F) is a pair of compatible of type (P) mappings and $F\alpha_3 = s = I\alpha_3$, then from preposition 2.10, we get $IF\alpha_3 = FI\alpha_3$ and so, we have

$$Is = IF\alpha_3 = FI\alpha_3 = Fs \tag{3.17}$$

On putting $u = s$ and $v = s$ in inequality (2) and using (3.15) and (3.17) we get,

$$d(Es, Fs) \le [\max\{d(s, Fs), d(s, s), d(Fs, Fs), d(s, Fs), d(s, Fs)\}]^{\eta}$$
$$= [d(s, Fs)]^{\eta}$$
$$d(s, Fs) \le [d(s, Fs)]^{\eta}.$$

This implies that, $d(s, Fs) = 1$, i.e., $Fs = s$. Therefore,

$$Hs = Es = Is = Fs = s \tag{3.18}$$

Hence, s is a *CFP* of mappings H, I, E, and F.

Case (iv). Suppose that I is continuous, then this can be proven similar to case (iii).

Uniqueness: let s_1 is another *CFP* of mappings H, I, E, and F, then on putting $u = s_1$ and $v = s$ in inequality (2) and using (3.18), we get,

$$d(s_1, s) = d(Es_1, Fs)$$
$$\le [\max\{d(Hs_1, Is), d(Hs_1, Es_1), d(Is, Fs), d(Es_1, Is), d(Hs_1, Fs)\}]^{\eta}$$
$$= [\max\{d(s_1, s), d(s_1, s_1), d(s, s), d(s_1, s), d(s_1, s)\}]^{\eta}$$
$$= [\max\{d(s_1, s), 1, 1, d(s_1, s), d(s_1, s)\}]^{\eta}$$
$$= [d(s_1, s)]^{\eta}$$
$$d(s_1, s) \le [d(s_1, s)]^{\eta}.$$

This implies that, $d(s_1, s) = 1$ i.e., $s_1 = s$.
Hence, mappings $H, I, E,$ and F have a unique *CFP*.

Example 3.2. Let $K = R$ and $L = \{u \in K: u \geq 1\}$ be a multiplicative cone in K. Let $d: M \times M \to K$, where $M = [1, \infty)$ is a multiplicative metric defined as:

$$d(u, v) = \left|\frac{u}{v}\right|, \quad \forall\, u, v \in M.$$

Then (M, d) is clearly a complete multiplicative cone metric space. Also, let the following four self-mappings $E, F, H, I: M \to M$ of multiplicative cone metric space (M, d) such that,

$$Eu = u, \qquad Fu = u^2, \qquad Hu = 2u^2 - 1, \quad Iu = 2u^4 - 1, \qquad \forall\, u \in M.$$

Then, we can easily see that,

(1) Since $E(M) = F(M) = I(M) = H(M) = M$, so $E(M) \subset I(M)$, $F(M) \subset H(M)$.

(2) Let $\eta = \frac{1}{3} \in (0, \frac{1}{2})$, then from the inequality (2) of Theorem 3.1,

$$d(Eu, Fv) \leq [\max\{d(Hu, Iv), d(Hu, Eu), d(Iv, Fv), d(Eu, Iv), d(Hu, Fv)\}]^{\eta}$$

is satisfied for all $u, v \in M$.

(3) $H, I, E,$ and F all are continuous mappings.

(4) Consider a sequence $u_n = 1 + \frac{1}{n}$, for $n \geq 1$. Then we have,

$$\lim_{n\to\infty} u_n = 1 \quad \text{and} \quad \lim_{n\to\infty} Hu_n = \lim_{n\to\infty} Eu_n = \lim_{n\to\infty} Iu_n = \lim_{n\to\infty} Fu_n = 1 = \lambda \in M.$$

Also, $\lim_{n\to\infty} d(HHu_n, EEu_n) = 1$, $\quad \lim_{n\to\infty} d(IIu_n, FFu_n) = 1$,

Therefore, we can see that, (H, E) and (I, F) are pairs of mappings of type (P) compatible.
Hence, all requirements of Theorem 3.1, are fulfilled, and $H1 = I1 = E1 = F1 = 1$, i.e., 1 is the unique *CFP* of mappings $H, I, E,$ and F.

4. CONCLUSION

This paper aims to introduce compatible mappings of type (P) to multiplicative cone metric space and by using the properties, develop and generalize the results of common fixed points to multiplicative cone metric space. Our presented result generalizes numerous prevailing fixed point results in the literature and also extends the scope of study of common fixed point theorems in multiplicative cone metric space.

REFERENCES

1. C. B. Ampadu, A coupled version of the higher-order Banach contracton principle in multiplicative cone metric space, JP Journal of Applied Mathematics, 15 (2017), 63-88.
2. C. B. Ampadu, A Fixed Point Theorem of the Hardy and Rogers Kind Endowed with Multiplicative Cone-C Class Functions, Earthline Journal of Mathematical Sciences, 2 (2019), 169-179.
3. D. Ilic and V. Rakocevic, Common fixed points for maps on cone metric space, Journal of Mathematical Analysis and Applications, 341 (2008), 876-882.
4. G. Jungck, Commuting mappings and fixed points, The American Mathematical Monthly, 83 (1976), 261-263.
5. G. Jungck, Compatible mappings and common fixed points, International journal of mathematics and mathematical sciences, 9 (1986), 771-779.
6. G. Song, X. Sun, Y. Zhao and G. Wang, New common fixed point theorems for maps on cone metric spaces, Applied Mathematics Letters, 23 (2010), 1033-1037.
7. H.K. Pathak, Y.J. Cho, S.M. Kang, and B.S. Lee, Fixed point theorems for compatible mappings of type (P) and applications to dynamic programming, Le Matematiche, 50 (1995), 15-33.
8. K. Prudhvi, Common fixed point theorem in Cone metric spaces, Ammerican Journal of Applied Mathematics and Statistics, 3 (2015), 68-70.
9. K. Fallahi, A. Petrusel and G. S. Rad, Fixed point results for pointwise Chatterjea type mappings with respect to a c-distance in cone metric spaces endowed with a graph, UPB Sci. Bull.(Series A), 80 (2018), 47-54.
10. L. G. Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractive

mappings, Journal of mathematical Analysis and Applications, 332 (2007), 1468-1476.
11. M. Abbas and G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, Journal of Mathematical Analysis and Applications, 341 (2008), 416-420.
12. M. Arshad, A. Azam and P. Vetro, Some common fixed point results in cone uniform spaces, Fixed Point Theory Appl, 11(2009).
13. S. Janković, Z. Kadelburg and S. Radenović, On cone metric spaces: a survey, Nonlinear Analysis: Theory, Methods & Applications, 74 (2011), 2591-2601.
14. S. Radenović and F. Vetro, Some remarks on Perov type mappings in cone metric spaces, Mediterranean Journal of Mathematics, 14 (2017), 1-15.
15. S. Sessa, On a weak commutativity condition of mappings in fixed point considerations, Publ. Inst. Math, 32 (1982), 149-153.
16. S. U. Rehman, Y. Li, S. Jabeen and T. Mahmood, Common fixed point theorems for a pair of self-mappings in fuzzy cone metric spaces, In Abstract and Applied analysis, Vol. 36 (2019) 55-67.
17. S. Wang and B. GU, Distance in cone metric spaces and common fixed point theorems, Applied Mathematics Letters, 24 (2011), 1735-1739.
18. X. T. Shi and F. Gu, Common fixed point theorem for two pairs of converging commuting mappings in a cone metric space, J. Hangzhou Norm. Univ., Nat. Sci. Ed, 11 (2012), 433-435.
19. Z. Kadelburg, S. Radenović and B. Rosić, Strict contractive conditions and common fixed point theorems in cone metric spaces, Fixed Point Theory and Applications, 23 (2009), 1-14.