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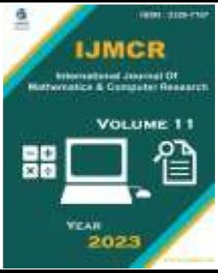

Degree Sum Distance Spectra and Energy of Graphs

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ABSTRACT

In this paper motivated by the Schultz index for a connected graph, the degree sum distance matrix and degree sum distance energy are defined. We also obtain some bounds for the degree sum distance energy and deduce the degree sum distance energy of certain graphs.

KEYWORDS : Eigenvalue, Spectra, Degree sum distance matrix, Degree sum distance energy.

I. INTRODUCTION

We will focus on simple connected graphs, which are graphs without loops and multiple edges. Let G be a connected graph of order n with vertex set V. We can denote d_i as the degree of a vertex v_i, which is the number of edges incident on it. $d(v_i, v_j)$ or d_{ij} represents the distance between two vertices v_i and v_j, which is defined as the length of the shortest path joining them. The Schultz Index introduced by Schultz [14] is defined as,

$$S(G) = \sum_{i,\,j=1}^{n} [d_i + d_j] d_{ij}.$$

For detailed work, see [15-18].

Motivated by previous research related to degree and distance in a graph such as distance energy [1, 2], degree sum energy [3, 4], degree square sum polynomial [8], complementary distance energy [5], degree exponent energy [6, 7], degree exponent sum energy [9], in order to upgrade, we now introduce the concept of degree sum distance energy of a connected graph. The purpose of this paper is to compute the characteristic polynomial, eigenvalues and energy of degree sum distance matrix of a graph. Also, we compute bounds for degree sum distance energy.

The degree sum distance matrix of a connected graph G is defined as, $DSD(G) = [dsd_{ij}]$, where

$$dsd_{ij} = \begin{cases} (d(v_i) + d(v_j) d_{ij}) & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases} \qquad (1.1)$$

We note that,

(1) Sum of the all elements in $DSD(G)=2S(G)$.

(2) $DSD(G)$ is real symmetric, so that the eigenvalues of $DSD(G)$ are real.

(3) If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the eigenvalues of $DSD(G)$ then, they can be arranged in a non-increasing order as $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$.

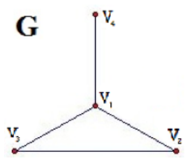
(4) $\sum_{i=1}^{n} \alpha_i = 0$ since trace $[DSD(G)]=0$.

(5) If we replace d_{ij} =1 for all $i \neq j$, we get the degree sum matrix.

Analogous to the energy of a graph defined by I.Gutman with respect to adjacency matrix, we define the degree sum distance energy of a graph as,

$$E_{DSD}(G) = \sum_{i=1}^{n} |\alpha_i|.$$

Example:



$$DSD(G) = \begin{pmatrix} 0 & 4 & 6 & 6 \\ 4 & 0 & 5 & 5 \\ 6 & 5 & 0 & 4 \\ 6 & 5 & 4 & 0 \end{pmatrix}.$$

Degree sum distance eigenvalues of graph G are α_1=15.0263, α_2= -3.8669, α_3= -4 and α_4= -7.1594. $E_{DSD}(G) = 30.0526$

II. PRELIMINARIES

We state some useful Lemmas for the derivations.

Lemma 2.1. Let G be a graph of order n, and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the eigenvalues of DSD(G) Then $\sum_{i=1}^{n}\alpha_i = 0$ and

$\sum_{i=1}^{n}\alpha_i^2 = 2M$,

Where

$$M = \sum_{\substack{i,j=1 \\ i<j}}^{n}([d_i + d_j]d_{ij})^2.$$

Lemma 2.2. The Cauchy-Schwartz inequality states that, if $(a_1, a_2, ..., a_p)$ and $(b_1, b_2, ..., b_p)$ are real p-vectors then,

$$\left(\sum_{i=1}^{p} a_i \cdot b_i\right)^2 \le \left(\sum_{i=1}^{p} a_i^2\right)\cdot\left(\sum_{i=1}^{p} b_i^2\right).$$

Lemma 2.3 [10]**.** Suppose that a_i and b_i , $1\le i\le n$, are positive real numbers. Then,

$$\left(\sum_{i=1}^{n} a_i^2\right)\cdot\left(\sum_{i=1}^{n} b_i^2\right) \le \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2\left(\sum_{i=1}^{n} a_i \cdot b_i\right)^2,$$

where $M_1 = \max_{1\le i\le n}(a_i)$, $M_2 = \max_{1\le i\le n}(b_i)$,
$m_1 = \min_{1\le i\le n}(a_i), m_2 = \min_{1\le i\le n}(b_i)$

Lemma 2.4 [11]**.** *Let* a_i and b_i , $1\le i\le n$, be non-negative real numbers. Then,

$$\left(\sum_{i=1}^{n} a_i^2\right)\cdot\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i \cdot b_i\right)^2 \le \frac{n^2}{4}(M_1M_2 - m_1m_2)^2,$$

where, $M_1 = \max_{1\le i\le n}(a_i)$, $M_2 = \max_{1\le i\le n}(b_i)$,
$m_1 = \min_{1\le i\le n}(a_i), m_2 = \min_{1\le i\le n}(b_i)$

Lemma 2.5 [12]**.** *Let* a_i and b_i , $1\le i\le n$, be non-negative real number. Then,

$$\left(\sum_{i=1}^{n} b_i^2\right) + rR\left(\sum_{i=1}^{n} a_i^2\right) \le (r + R)\sum_{i=1}^{n} a_i \cdot b_i,$$

where r and R are real constants, such that for each $i, 1\le i\le n$, $ra_i \le b_i \le Ra_i$, hold.

Lemma 2.6 [1]. If G is a r-regular graph of diameter two, then $D(G) = 2J_n - 2I_n - A(G)$ and thus the D- eigenvalues (eigenvalues of distance matrix) of G are $2n-r-2$, $-\alpha_n, -2, ..., -2$, arranged in a non-increasing order.

Lemma 2.7 [2]**.** Let G be a r-regular graph of diameter 2, and let its spectrum (ordinary) be

$spec(G) = (r, \alpha_2, ..., \alpha_n)$.

Then the D-spectrum of G is,

$spec_D(G) = ((2n - r - 2), -(\alpha_2 + 2), ..., -(\alpha_n + 2)).$

Lemma 2.8 [2]**.** Let G be a r-regular graph of diameter 1 or 2 with an adjacency matrix A and $spec(G) = (\alpha_1, \alpha_2, ..., \alpha_n)$. Then $H = G \times K_2$ *is* $(r+1)$-regular and of diameter 2 or 3 with,

$$spec_D(H) = \begin{pmatrix} 5n - 2(r+2) & -2(\alpha_i + 2) & -n & 0 \\ 1 & 1 & 1 & n-1 \end{pmatrix},$$

where $i = 2, ..., n.$

Lemma 2.9 [13]**.** Suppose that $[0, b_2, b_3, ..., b_n]$ is the first row of the adjacency matrix of a circulant graph G. Then the eigenvalues of G are,

$$\lambda^p = \sum_{j=2}^{n} b_j\omega^{(j-1)p},$$

where $p = 0, 1, ..., (n-1)$ *and* ω is n^{th} root of unity.

Lemma 2.10 [7]. If a, b, c and d are real numbers, then the determinant of the form,

$$\begin{vmatrix} (\alpha + a)I_{n_1} - aJ_{n_1} & -cJ_{n_1\times n_2} \\ -dJ_{n_2\times n_1} & (\alpha + b)I_{n_2} - bJ_{n_2} \end{vmatrix}$$

of order $n_1 + n_2$ can be expressed in the simplified form as,

$(\alpha + a)^{n_1-1}(\alpha + b)^{n_2-1}([\alpha - (n_1 - 1)a][\alpha - (n_2 - 1)b] - n_1n_2cd).$

Lemma 2.11. *If* $A = (a-b)I + bJ$, then the characteristic polynomial of A is, $|\lambda I - A| = [\lambda - a + b]^{n-1}[\lambda - a - (n-1)b]$.

where a and b are arbitrary constants, I is the identity matrix of order n *and* J *is* $n\times n$ *matrix with all entries* 1*'s*

III. BOUNDS ON DEGREE SUM DISTANCE ENERGY

In this section, we obtain some bounds on degree sum distance energy of any graph.

Proposition 3.1. Let G be a graph of order n and size m. Then,

$$E_{DSD}(G) \ge \sqrt{2Mn - \frac{n^2}{4}(\alpha_1 - \alpha_n)^2},$$

where α_1, α_n are maximum and minimum of α_i *'s.*

Proof. Suppose $\alpha_1, \alpha_2, ..., \alpha_n$ are the eigenvalues of *DSD(G).* We assume that a_i=1 and $b_i = \alpha_i$ then, by Lemma 2.4, we have,

$$\sum_{i=1}^{n} 1 \sum_{i=1}^{n}|\alpha_i|^2 - \left(\sum_{i=1}^{n}|\alpha_i|\right)^2 \le \frac{n^2}{4}(\alpha_1 - \alpha_n)^2$$

$$2Mn - (E_{DSD}(G))^2 \le \frac{n^2}{4}(\alpha_1 - \alpha_n)^2.$$

Hence,

$$E_{DSD}(G) \ge \sqrt{2Mn - \frac{n^2}{4}(\alpha_1 - \alpha_n)^2}.$$

Proposition 3.2. Let G be a graph of order n and suppose zero is not an eigenvalue of *DSD(G).*Then,

$$E_{DSD}(G) \geq \frac{2\sqrt{2Mn}\sqrt{\alpha_1\alpha_n}}{\alpha_1+\alpha_n},$$

where α_1,α_n are maximum and minimum of α_i 's.

Proof. Suppose $\alpha_1,\alpha_2,....,\alpha_n$ are the eigenvalues of $DSD(G)$. Let us assume that $a_i = |\alpha_i|$ and $b_i = 1$, by Lemma 2.3 we have,

$$\sum_{i=1}^{n}|\alpha_i|^2\sum_{i=1}^{n}1 \leq \frac{1}{4}\left(\sqrt{\frac{\alpha_n}{\alpha_1}}+\sqrt{\frac{\alpha_1}{\alpha_n}}\right)^2\left(\sum_{i=1}^{n}|\alpha_i|\right)^2$$

$$2Mn \leq \frac{1}{4}\left(\frac{(\alpha_1+\alpha_n)^2}{\alpha_1\alpha_n}\right)(E_{DSD}(G))^2.$$

Hence, $E_{DSD}(G) \geq \dfrac{2\sqrt{2Mn}\sqrt{\alpha_1\alpha_n}}{\alpha_1+\alpha_n}$

Proposition 3.3. Let G be a graph of order n and size m. Let $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ be a non-increasing sequence of eigenvalues of $DSD(G)$ .Then,

$$E_{DSD}(G) \geq \frac{\alpha_1 \cdot \alpha_n n + 2M}{\alpha_1+\alpha_n},$$

where α_1,α_n are maximum and minimum eigenvalues of $DSD(G)$.

Proof. Suppose $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the eigenvalues of $DSD(G)$. We assume that $b_i = |\alpha_i|, a_i = 1, r = \alpha_n = b$ and $R = \alpha_1$ then by Lemma 2.5, we have,

$$\sum_{i=1}^{n}|\alpha_i^2| + \alpha_1\alpha_n\sum_{i=1}^{n}1 \leq (\alpha_1+\alpha_n)\sum_{i=1}^{n}|\alpha_i|.$$

Since,

$$E_{DSD}(G) = \sum_{i=1}^{n}|\alpha_i|, \sum_{i=1}^{n}|\alpha_i^2| = 2M,$$

IV. DEGREE SUM DISTANCE ENERGY OF SOME GRAPHS

In this section, we obtain the degree sum distance energy of some graphs. Prior to it we discuss the connection between distance spectra and adjacency spectra, subsequently connecting degree sum distance spectra with adjacency spectra in case of regular graphs. The distance matrix $D(G)$ of a connected graph G is defined as,

$$d_{ij} = \begin{cases} 1, & \text{if } i \neq j \\ 0, & \text{if } i = j. \end{cases}$$

The collection of eigenvalues of a distance matrix of a graph G along with their multiplicity form the distance spectrum or D spectrum of G denoted by $spec_D(G)$. The following gives the relation between distance spectra and adjacency spectra for r regular graph.

Proposition 4.1. For any r-regular graph G, $DSD(G) = 2rD(G)$, where $D(G)$ is distance matrix of G.

Proposition 4.2. From Lemma 2.6, if G is r-regular graph G of diameter 2 with eigenvalues $(r,\alpha_2,....,\alpha_n)$ then degree sum distance spectra of G is, $spec_{DSD}(G) =$
$(2r(2n-r-2), -2r(\alpha_2+2), -2r(\alpha_3+2), ..., -2r(\alpha_n+2))$.

Proposition 4.3. *If* G be a r-regular graph of order n and diameter 1 *or* 2 then $H = G \times K_2$ *is* $(r+1)$-regular

$$spec_{DSD}(H) = \begin{pmatrix} 2(r+1)(5n-2r-4) & -4(r+1)(\alpha_i+2) & -2n(r+1) & 0 \\ 1 & 1 & 1 & n-1 \end{pmatrix},$$

where $i= 2, \ldots, n$.

From the above Proposition we have,

(1) $spec_{DSD}(K_n) = \begin{pmatrix} 2(n-1) & -2(n-1) \\ n-1 & n-1 \end{pmatrix}$.

Hence, $E_{DSD}(K_n) = 4(n-1)^2$.

(2) $spec_{DSD}(K_{n,n}) = \begin{pmatrix} 2n(3n-2) & 2n(n-2) & -4n \\ 1 & 1 & 2n-2 \end{pmatrix}$.

Hence, $E_{DSD}(K_{n,n}) = 16n(n-1)$.

(3) $spec_{DSD}(CP(n)) = \begin{pmatrix} 8n(n-1) & -8(n-1) & 0 \\ 1 & n & n-1 \end{pmatrix}$.

Hence, $E_{DSD}(CP(n)) = 16n(n-1)$.

Theorem 3.10. The degree sum distance energy of even and odd cycle are given by,
$E_{DSD}(C_{2n}) = 8n^2$ and $E_{DSD}(C_{2n+1}) = 8n(n+1)$ respectively.

Proof. (1) Consider an even cycle C_{2n}. Here the degree sum distance matrix is circulant with first row, $[0\ 4\ 8\ 12\ \ldots\ 4(n-1) 4n\ 4(n-1)\ \ldots\ 4]$.
By Lemma 2.9, extracting eigenvalues and adding their magnitudes, we get $E_{DSD}(C_{2n}) = 8n^2$.

(2) For an odd cycle C_{2n+1}, the degree sum distance matrix is circulant with first row, $[0\,4\,8\,12 \ldots.4n\,4n \ldots.4]$. By Lemma 2.9 extracting eigenvalues and adding their magnitudes, we get,
$E_{DSD}(C_{2n+1}) = 8n(n+1)$.

Theorem 3.11. *The degree sum distance energy of wheel graph* W_{n+1} *is,*
$E_{DSD}(W_{n+1}) = \sqrt{144(n-2)^2 + 4n(n+3)^2} + 12(n-2)$.

Proof. Let W_{n+1} be a wheel graph of order $(n+1)$.Starting with central vertex as first vertex (for 1 row/column), suitable labeling gives the degree sum distance matrix of W_{n+1} as,

$$DSD(W_{n+1}) = \begin{pmatrix} 0 & (n+3)J_{1\times n} \\ (n+3)J_{n\times 1} & MDSD(C_n) \end{pmatrix},$$

where J represents matrix of all *1's*, $MDSD(C_n)$ represents circulant matrix corresponding to C_n with first row

$[0\,6\,12\,12\ldots.12\,6]$. The degree sum distance polynomial of $DSD(W_{n+1})$ is then given by,

$$|\alpha I - DSD(W_{n+1})| = \begin{vmatrix} \alpha & -(n+3)J_{1\times n} \\ -(n+3)J_{n\times 1} & \alpha I_n - MDSD(C_n) \end{vmatrix}.$$

Adding to first column $\dfrac{n+3}{\alpha-12(n-2)}$. times addition of remaining columns gives

$$\left|\alpha I - DSD(W_{n+1})\right| = \begin{vmatrix} \alpha - \frac{n+3}{\alpha-12(n-2)} & -(n+3)J_{1\times n} \\ O_{n\times 1} & \alpha I_n - MDSD(C_n) \end{vmatrix}$$

which gives the characteristic polynomial as, $\left|\alpha I - DSD(W_{n+1})\right| = [\alpha^2 - 12(n-2)\alpha - n(n+3)^2] \times \left|\alpha I - MDSD(C_n)\right|$,

where, $\left|\alpha I - MDSD(C_n)\right|$ is without the factor $[\alpha - 12(n-2)]$

It can be shown that the eigenvalues of *MDSD(C$_n$)* are 12(*n*-2) and remaining all are negative so that the energy of *MDSD(C$_n$)* is 12(*n*-2).

By Lemma 2.9, we get,

$$E_{DSD}(W_{n+1}) = \sqrt{144(n-2)^2 + 4n(n+3)^2} + 12(n-2).$$

Hence the theorem.

Theorem 3.12. *The degree sum distance energy of the complete bipartite graph $K_{m,n}$(m ,n ≥2) is,*

$$E_{DSD}(K_{m,n}) = 8(2mn - m - n).$$

Proof. In $K_{m,n}$, m vertices have degree n and n vertices have degree m. The degree sum distance matrix is,

$$DSD(K_{m,n}) = \begin{pmatrix} 4nA(K_m) & (m+n)J_{m\times n} \\ (m+n)J_{n\times m} & 4mA(K_n) \end{pmatrix},$$

where J is matrix of all 1's and A the adjacency matrix. The degree sum distance polynomial is then given by,

$$|\ \alpha I - DSD(K_{m,n})\ | = \begin{vmatrix} \alpha I_m - 4nA(K_m) & -(m+n)J_{m\times n} \\ -(m+n)J_{n\times m} & \alpha I_n - 4mA(K_n) \end{vmatrix}.$$

Applying Lemma 2.10, the degree sum distance polynomial of $K_{m,n}$ is given by,

$$|\ \alpha I - DSD(K_{m,n})\ | = (\alpha+4n)^{m-1}(\alpha+4m)^{n-1}[\alpha^2 - 4(2mn-m-n)\alpha + 16(mn-m-n) - mn(m+n)^2].$$

We get, $spec_{DSD}(K_{m,n}) =$

$$\begin{pmatrix} -4n & -4m & 4mn-m-n \pm \sqrt{4(2mn-m-n)^2 - (16(mn-m-n) - mn(m+n)^2)} \\ m-1 & n-1 & 1 \end{pmatrix}$$

The theorem now follows by adding absolute eigenvalues.

On similar lines we state without proof, the following.

Theorem 3.13. The degree sum distance energy of the star graph $K_{1,n}$ is,

$$E_{DSD}(K_{1,n}) = 4(n-1) + 2\sqrt{4(n-1)^2 + n(n+1)^2}.$$

Theorem 3.14. The degree sum distance energy of the crown graph S_n^0 is,

$$E_{DSD}(S_n^0) = \begin{cases} 72 & if\ n = 3 \\ 16(n-1)^2 & if\ n \geq 4. \end{cases}$$

Proof. The crown graph S_n^0 is regular of degree n-1 So the degree sum distance matrix of S_n^0 for n ≥ 4 is,

$$DSD(S_n^0) = \begin{pmatrix} 4(n-1)A(K_n) & 6(n-1)I_n + 2(n-1)A(K_n) \\ 6(n-1)I_n + 2(n-1)A(K_n) & 4(n-1)A(K_n) \end{pmatrix},$$

where J is matrix of all 1's and A is the adjacency matrix. The degree sum distance polynomial is then given by,

$$|\alpha I - DSD(S_n^0)| = \begin{vmatrix} \alpha I_n - 4(n-1)A(K_n) & -6(n-1)I_n - 2(n-1)A(K_n) \\ -6(n-1)I_n - 2(n-1)A(K_n) & \alpha I_n - 4(n-1)A(K_n) \end{vmatrix}.$$

Applying Lemma 2.10, the degree sum distance polynomial is given by,

$$|\ \alpha I - DSD(S_n^0)\ |$$
$$= \alpha^{n-1}[\alpha + 8(n-1)]^{n-1}[\alpha - 6n(n-1)][\alpha - 2(n-1)(n-4)]$$

$$spec_{DSD}(S_n^0) = \begin{pmatrix} 0 & -8(n-1) & 6n(n-1) & 2(n-1)(n-4) \\ n-1 & n-1 & 1 & 1 \end{pmatrix}.$$

$for\ n = 3$, using Matlab. $spec_{DSD}(S_3^0) = \begin{pmatrix} 0 & 16 & 36 & -4 \\ 2 & 2 & 1 & 1 \end{pmatrix}$

Hence the theorem.

Theorem 3.15.

$$E_{DSD}(K_n + e) = (2n-2)(n-2) + |\alpha_1| + |\alpha_2| + |\alpha_3|,$$

where α_1, α_2 and α_3 are roots of the equation,

$$[\alpha^3 - 2(n-1)(n-2)\alpha^2 - ((n+1)^2 + (n-1)(8n^2 - 4n + 1))\alpha - 2(n^2-1)(3n^2 - n + 2)] = 0.$$

Proof. In K_n+e there is one vertex with degree n, one vertex with degree 1 and remaining n-1 vertices have degree n-1 so we get the degree sum distance matrix with suitable labeling as,

$$DSD(K_n + e) = \begin{pmatrix} 0 & n+1 & (2n-1)J_{1\times n-3} \\ n+1 & 0 & 2nJ_{1\times n-3} \\ (2n-1)J_{n-3\times 1} & 2nJ_{n-3\times 1} & 2(n-1)A(K_{n-3}) \end{pmatrix}.$$

The degree sum distance polynomial of K_n+e is then given by,

$$|\alpha I - DSD(K_n + e)| = \begin{vmatrix} \alpha & -(n+1) & -(2n-1)J_{1\times n-3} \\ -(n+1) & \alpha & -2nJ_{1\times n-3} \\ -(2n-1)J_{n-3\times 1} & -2nJ_{n-3\times 1} & \alpha I_{n-3} - 2(n-1)A(K_{n-3}) \end{vmatrix}.$$

Applying Lemma 2.10, the degree sum distance polynomial of K_n +e is given by,

$$|\alpha I - DSD(K_n + e)|$$
$$= [\alpha + (2n-2)]^{n-2} \times$$
$$[\alpha^3 - 2(n-1)(n-2)\alpha^2 - ((n+1)^2 + (n-1)(8n^2 - 4n + 1))\alpha - 2(n^2-1)(3n^2 - n + 2)].$$

so that,

$$spec_{DSD}(K_n + e) = \begin{pmatrix} -(2n-2) & \alpha_1 & \alpha_2 & \alpha_3 \\ n-2 & 1 & 1 & 1 \end{pmatrix},$$

where α_1, α_2, and α_3 are the roots of the equation,

$$[\alpha^3 - 2(n-1)(n-2)\alpha^2 - ((n+1)^2 + (n-1)(8n^2 - 4n + 1))\alpha - 2(n^2-1)(3n^2 - n + 2)] = 0.$$

Hence the theorem.

On similar lines, we obtain the degree sum distance spectra and energy of edge deleted complete graph $K_n - e$, as in the following theorem.

Theorem 3.16.

$$E_{DSD}(K_n - e) = 2n^2 - 4n - 2 + \sqrt{n^4 - 4n^3 + 10n^2 - 18n + 13}.$$

Definition 3.17. Vertex Coalescence: If G_1 and G_2 are any two graphs then the graph obtained by gluing G_1 and G_2 at a point is v called vertex coalescence denoted by $G_1 O_v G_2$.

Definition 3.18. Edge Coalescence: If G_1 and G_2 are any two graphs then the graph obtained by merging G_1 and G_2 on an edge e is called edge coalescence denoted by $G_1 O_e G_2$

Let K_n be a complete graph of order n then the vertex coalescence of K_n with K_n will be denoted by $K_n O_v K_n$ and the edge coalescence by $K_n O_e K_n$. $K_n O_v K_n$ has $2n$-1 vertices and $2 \times ({}^nC_2)$ edges whereas $K_n O_e K_n$ has $2n$-2 vertices and $2 \times ({}^nC_2 - 1)$ edges.

We now obtain DSD energy for . $K_n O_v K_n$ *and* $K_n O_e K_n$

Theorem 3.19.

$$E_{DSD}(K_n O_v K_n) = 4(n-1)(n-2) + 2n(n-1) + (n-1)\sqrt{(16n-8)^2 + 72(n-1)}.$$

Proof. The graph $K_n O_v K_n$ has one vertex of degree $2(n-1)$ and remaining $2(n-1)$ vertices of degree n-1.

With suitable labeling the degree sum distance matrix of $K_n O_v K_n$ takes the form,

$$DSD(K_n O_v K_n) = \begin{pmatrix} 0 & 3(n-1) & 3(n-1)J_{1\times n-2} \\ 3(n-1) & 0 & 2(n-1)J_{1\times n-2} \\ 3(n-1)J_{n-2\times 1} & 2(n-1)J_{n-2\times 1} & 2(n-1)A(K_{n-2}) \end{pmatrix}.$$

The degree sum distance polynomial of $K_n O_v K_n$ is,

$$|\alpha I - DSD(K_n O_v K_n)| = \begin{vmatrix} \alpha & -3(n-1) & -3(n-1)J_{1\times n-2} \\ -3(n-1) & \alpha & -2(n-1)J_{1\times n-2} \\ -3(n-1)J_{n-2\times 1} & -2(n-1)J_{n-2\times 1} & \alpha I_{n-2} - 2(n-1)A(K_{n-2}) \end{vmatrix}.$$

Using Lemma 2.10 we get the degree sum distance polynomial of $K_n O_v K_n$,

$$|\alpha I - DSD(K_n O_v K_n)| = [\alpha + 2n - 2]^{2n-4}[\alpha + 2n(n-1)][\alpha^2 - (6n-8)(n-1)\alpha - 18(n-1)^3].$$

We get,

$$spec_{DSD}(K_n O_v K_n) = \begin{pmatrix} -(2n-2) & -2n(n-1) & \frac{(6n-8)(n-1) \pm (n-1)\sqrt{(6n-8)^2 + 72(n-1)}}{2} \\ 2n-4 & 1 & 1 \end{pmatrix}.$$

Hence the theorem.

On similar lines we state without proof the following.

Theorem 3.20.

$$E_{DSD}(K_n O_e K_n) = (2n-2)(2n-6) + 2(n-1)^2 + (4n-6) + |\alpha_1| + |\alpha_2|,$$

where

$$\alpha_1 = \frac{2(n-1)(3n-7) + (4n-6) + \sqrt{[2(n-1)(3n-7) + (4n-6)]^2 - 4[2(n-1)(3n-7)(4n-6) - 4(3n-4)^2(n-2)]}}{2}$$

and

$$\alpha_2 = \frac{2(n-1)(3n-7) + (4n-6) - \sqrt{[2(n-1)(3n-7) + (4n-6)]^2 - 4[2(n-1)(3n-7)(4n-6) - 4(3n-4)^2(n-2)]}}{2}.$$

V. CONCLUSION

We discussed the degree sum distance energy of graphs. Also, we discussed bounds on degree sum distance energy. There is a scope to investigate degree sum distance energy of graphs with higher diameter, trees, unicyclic graphs, line graphs etc and also to construct degree sum distance equi-energetic graphs.

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