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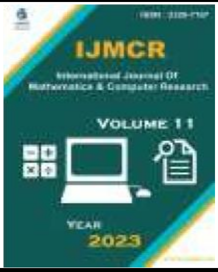

Subclass of Analytic Functions Associated with Linear Operator

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ABSTRACT
In this work, we introduce and study a new subclass of analytic functions defined by a linear operator and obtained coefficient estimates, growth and distortion theorems, radii of starlikeness, convexity and close-to-convexity are obtained. Furthermore, we obtained integral means inequalities for the class.

KEYWORDS : analytic, coefficient bounds, starlike, distortion.
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1. INTRODUCTION

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

which are analytic in the open unit disk

$$E = \{z \in \mathbb{C}: |z| < 1\}.$$

A function f in the class A is said to be in the class $ST(\alpha)$ of starlike functions of order α in E, if it satisfy the inequality

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (0 \le \alpha < 1), (z \in E) \qquad (1.2)$$

Note that $ST(0) = ST$ is the class of starlike functions.
Denote by T the subclass of A consisting of functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \ (a_n \ge 0). \qquad (1.3)$$

This subclass was introduced and extensively studied by Silverman [4].

Recently, Atshan and Buti [1] introduced a Rafid operator of $f \in R$ for $0 \le \lambda < 1$ and $0 \le m < 1$. It is denoted by $G_\lambda^m f(z)$ and defined as follows:

$$G_\lambda^m f(z) = \frac{1}{(1-\lambda)^{m+1}\Gamma(m+1)}\int_0^\infty t^{\lambda-1} e^{-\left(\frac{t}{1-\lambda}\right)} f(zt)dt \qquad 1.4)$$

Thus, if $f \in A$ has the form (1.1), then it follows from (1.4) that

$$G_\lambda^m f(z) = z + \sum_{n=2}^{\infty} \phi_n(\lambda, m) a_n z^n \qquad (1.5)$$

Where $\phi_n(\lambda, m) = (1-\lambda)^{m-1}\frac{\Gamma(n+m)}{\Gamma(m+1)}$

In this paper, using the operator $G_\lambda^m f(z)$, we define the following new class motivated by Murugusunderamoorthy and Magesh [3].

Definition 1. The function $f(z)$ of the form (1.1) is in the class $S_\lambda^m(\mu, \gamma, \varsigma)$ if it satisfies the inequality

$$Re\left\{\frac{z(G_\lambda^m f(z))'}{(1-\mu)z + \mu G_\lambda^m f(z)} - \gamma\right\} > \varsigma\left|\frac{z(G_\lambda^m f(z))'}{(1-\mu)z + \mu G_\lambda^m f(z)} - 1\right|$$

for $0 \le \lambda \le 0$, $0 \le \gamma \le 1$ and $\varsigma \ge 0$.
Further we define $TS_\lambda^m(\mu, \gamma, \varsigma) = S_\lambda^m(\mu, \gamma, \varsigma) \cap T$.
The aim of this paper is to study the coefficient bounds , radii of close-to-convex and starlikeness
convex linear combinations for the class $TS_\lambda^m(\mu, \gamma, \varsigma)$.Furthermore, we obtained integral means inequalities for the functions in $TS_\lambda^m(\mu, \gamma, \varsigma)$.

Theorem 1: A function *f(z)* of the form (1.1) is in $S_\lambda^m(\mu, \gamma, \varsigma)$

$$\sum_{n=2}^{\infty}[n(1+\varsigma) - \mu(\gamma+\varsigma)]\phi_n(\lambda, m)|a_n| \le 1-\gamma \qquad (2.1)$$

where $0 \le \mu \le 1$, $0 \le \gamma \le 1$ $\varsigma \ge 0$ and $\phi_n(\lambda, m)$ is given by (1.5).

Proof: It suffices to show that

$$\varsigma\left|\frac{z(G_\lambda^m f(z))'}{(1-\mu)z + \mu G_\lambda^m f(z)} - 1\right| - Re\left\{\frac{z(G_\lambda^m f(z))'}{(1-\mu)z + \mu G_\lambda^m f(z)} - 1\right\} \le 1 - \gamma \text{ -}$$

We have

$$\varsigma\left|\frac{z(G_\lambda^m f(z))'}{(1-\mu)z+\mu G_\lambda^m f(z)}-1\right| - Re\left\{\frac{z(G_\lambda^m f(z))'}{(1-\mu)z+\mu G_\lambda^m f(z)}-1\right\}$$

$$\leq (1+\varsigma)\left|\frac{z(G_\lambda^m f(z))'}{(1-\mu)z+\mu G_\lambda^m f(z)}-1\right|$$

$$\leq (1+\varsigma)\frac{\sum_{n=2}^{\infty}(n-\mu)\phi_n(\lambda,m)|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}\mu\phi_n(\lambda,m)|a_n||z|^{n-1}}$$

$$\leq (1+\varsigma)\frac{\sum_{n=2}^{\infty}(n-\mu)\phi_n(\lambda,m)|a_n|}{1-\sum_{n=2}^{\infty}\mu\phi_n(\lambda,m)|a_n|}$$

The last expression is bounded above by $(1-\gamma)$ if

$$\sum_{n=2}^{\infty}[n(1+\varsigma)-\mu(\gamma+\varsigma)]\,\phi_n(\lambda,m)|a_n| \leq 1-\gamma$$

and the proof is complete.

Theorem 2: Let $0 \leq \mu \leq 1,\ 0 \leq \gamma \leq 1$ and $\varsigma \geq 0$ then a function f of the form (1.3) to be in the class $TS_\lambda^m(\mu,\gamma,\varsigma)$ if and only if

$$\sum_{n=2}^{\infty}[n(1+\varsigma)-\mu(\gamma+\varsigma)]\,\phi_n(\lambda,m) \leq 1-\gamma \qquad (2.2)$$

where $\phi_n(\lambda,m)$ are given by (1.5)

Proof: In view of Theorem 1, we need only to prove the necessity. If $f \in TS_\lambda^m(\mu,\gamma,\varsigma)$ and z is real then

$$Re\left\{\frac{1-\sum_{n=2}^{\infty}n\phi_n(\lambda,m)a_n z^{n-1}}{1-\sum_{n=2}^{\infty}\mu\phi_n(\lambda,m)a_n z^{n-1}}-\gamma\right\} > \varsigma\left|\frac{\sum_{n=2}^{\infty}(n-\mu)\phi_n(\lambda,m)a_n z^{n-1}}{1-\sum_{n=2}^{\infty}\mu\phi_n(\lambda,m)a_n z^{n-1}}\right|$$

Letting $z \to 1$ along the real axis, we obtain the desired inequality

$$\sum_{n=2}^{\infty}[n(1+\varsigma)-\mu(\gamma+\varsigma)]\,\phi_n(\lambda,m)|a_n| \leq 1-\gamma,$$

where $0 \leq \mu < 1,\ 0 \leq \gamma \leq 1\ \varsigma \geq 0$ and $\phi_n(\lambda,m)$ are given by (1.6).

Corollary 1. If $f(z) \in TS_\lambda^m(\mu,\gamma,\varsigma)$, then

$$|a_n| \leq \frac{1-\gamma}{[n(1+\varsigma)-\mu(\gamma+\varsigma)]\phi_n(\lambda,m)} \qquad (2.3)$$

where $0 \leq \mu < 1,\ 0 \leq \gamma \leq 1\ \varsigma \geq 0$ and $\phi_n(\lambda,m)$ are given by (1.5). Equality holds for the function

$$f(z) = z - \frac{1-\gamma}{[n(1+\varsigma)-\mu(\gamma+\varsigma)]\phi_n(\lambda,m)}z^n. \qquad (2.4)$$

Theorem 3. Let

$$f_1(z) = z \text{ and}$$

$$f_n(z) = z - \frac{1-\gamma}{[n(1+\varsigma)-\mu(\gamma+\varsigma)]\phi_n(\lambda,m)}z^n,\ n \geq 2. \qquad (2.5)$$

Then $f(z) \in TS_\lambda^m(\mu,\gamma,\varsigma)$, if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} w_n f_n(z), \quad w_n \geq 0, \sum_{n=1}^{\infty} w_n = 1. \qquad (2.6)$$

Proof. Suppose $f(z)$ can be written as in (2.6).Then

$$f(z) = z - \sum_{n=2}^{\infty} w_n \frac{1-\gamma}{[n(1+\varsigma)-\mu(\gamma+\varsigma)]\phi_n(\lambda,m)}z^n.$$

Now,

$$\sum_{n=2}^{\infty} w_n \frac{(1-\gamma)[n(1+\varsigma)-\mu(\gamma+1)]\phi_n(\lambda,m)}{(1-\gamma)[n(1+\varsigma)-\mu(\gamma+1)]\phi_n(\lambda,m)} = \sum_{n=2}^{\infty} w_n = 1-w_1 \leq 1.$$

Thus $f(z) \in TS_\lambda^m(\mu,\gamma,\varsigma)$. Conversely , let us have $f(z) \in TS_\lambda^m(\mu,\gamma,\varsigma)$.Then by using (2.3) , we get

$$w_n = \frac{[n(1+\varsigma)-\mu(\gamma+1)]\phi_n(\lambda,m)}{(1-\gamma)}a_n,\ n \geq 2$$

and $w_1 = 1-\sum_{n=2}^{\infty} w_n$. Then we have $f(z) = \sum_{n=1}^{\infty} w_n f_n(z)$ and hence this completes the proof of Theorem.

Theorem 4. The class $TS_\lambda^m(\mu,\gamma,\varsigma)$ is a convex set.

Proof. Let the function

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n,\ a_{n,j} \geq 0, \text{j}=1,2 \qquad (2.7)$$

be in the class $TS_\lambda^m(\mu,\gamma,\varsigma)$. It sufficient to show that the function $h(z)$ defined by

$$h(z) = \xi f_1(z) + (1-\xi)f_2(z),\ 0 \leq \xi < 1,$$

is in the class $TS_\lambda^m(\mu,\gamma,\varsigma)$. Since

$$h(z) = z - \sum_{n=2}^{\infty}[\xi a_{n,1} + (1-\xi)a_{n,2}]z^n,$$

An easy compution with the aid of of Theorem 2, gives

$$\sum_{n=2}^{\infty}[n(1+\varsigma)-\mu(\gamma+\varsigma)]\,\xi\phi_n(\lambda,m)a_{n,1} + \sum_{n=2}^{\infty}[n(1+\varsigma)-\mu(\gamma+\varsigma)]\,(1-\xi)\phi_n(\lambda,m)a_{n,2}$$

$$\leq \xi(1-\gamma) + (1-\xi)(1-\gamma)$$

$$\leq (1-\gamma),$$

which implies that $h \in TS_\lambda^m(\mu,\gamma,\varsigma)$.

Hence $TS_\lambda^m(\mu,\gamma,\varsigma)$ is convex.

Next we obtain the radii of close –to-convexity ,starlikeness and convexity for the class $TS_\lambda^m(\mu,\gamma,\varsigma)$.

Theorem 5. Let the function $f(z)$ defined by (1.3) belong to the class $TS_\lambda^m(\mu,\gamma,\varsigma)$.Then $f(z)$ is close-to-convex of order δ $(0 \leq \delta < 1)$ in the disc $|z| < r_1$, where

$$r_1 = \inf_{n\geq 2}\left[\frac{(1-\delta)\sum_{n=2}^{\infty}[n(1+\varsigma)-\mu(\gamma+\varsigma)]\phi_n(\lambda,m)}{n(1-\gamma)}\right]^{\frac{1}{n-1}}, n \geq 2. \quad (2.8)$$

The result is sharp, with the extremal function $f(z)$ is given by (2.5)

Proof. Given $f \in T$, and f is close-to-convex of order δ, we have

$$|f'(z)-1| < 1-\delta \quad (2.9)$$

For the left hand side of (2.9) we have

$$|f'(z)-1| \leq \sum_{n=2}^{\infty} na_n |z|^{n-1}$$

The last expression is less than $1-\delta$

$$\sum_{n=2}^{\infty}\frac{n}{1-\delta}a_n|z|^{n-1} \leq 1.$$

Using the fact, that $f(z) \in TS_\lambda^m(\mu,\gamma,\varsigma)$ if and only if

$$\sum_{n=2}^{\infty}\frac{[n(1+\varsigma)-\mu(\gamma+\varsigma)]\phi_n(\lambda,m)}{(1-\gamma)}a_n \leq 1,$$

We can (2.9) is true if

$$\frac{n}{1-\delta}|z|^{n-1} \leq \frac{[n(1+\varsigma)-\mu(\gamma+\varsigma)]\phi_n(\lambda,m)}{(1-\gamma)}$$

or, equivalently,

$$|z| \leq \left\{\frac{(1-\delta)[n(1+\varsigma)-\mu(\gamma+\varsigma)]\phi_n(\lambda,m)}{n(1-\gamma)}\right\}^{\frac{1}{n-1}}$$

which completes the proof.

Theorem 6. Let the function $f(z)$ defined by (1.3) belong to the class $TS_\lambda^m(\mu,\gamma,\varsigma)$.Then $f(z)$ is starlike of order of order δ $(0 \leq \delta < 1)$ in the disc $|z| < r_2$, where

$$r_2 = \inf_{n\geq 2}\left[\frac{(1-\delta)\sum_{n=2}^{\infty}[n(1+\varsigma)-\mu(\gamma+\varsigma)]\phi_n(\lambda,m)}{(n-\delta)(1-\gamma)}\right]^{\frac{1}{n-1}} \quad (2.10)$$

The result is sharp,with extremal function $f(z)$ is given by (2.5).

Proof. Given $f \in T$, and f is starlike of order δ, we have

$$\left|\frac{zf'(z)}{f(z)}-1\right| < 1-\delta$$

(2.11)

For the left hand side of (2.11) we have

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq \sum_{n=2}^{\infty}\frac{(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}a_n|z|^{n-1}}$$

The last expression is less than 1-δ if

$$\sum_{n=2}^{\infty}\frac{n-\delta}{1-\delta}a_n|z|^{n-1} < 1.$$

Using the fact that $f(z) \in TS_\lambda^m(\mu,\gamma,\varsigma)$ if and if

$$\sum_{n=2}^{\infty}\frac{[n(1+\varsigma)-\mu(\gamma+\varsigma)]\phi_n(\lambda,m)}{(1-\gamma)}a_n \leq 1,$$

We can say (2.11) is true if

$$\sum_{n=2}^{\infty}\frac{n-\delta}{1-\delta}|z|^{n-1} \leq \frac{[n(1+\varsigma)-\mu(\gamma+\varsigma)]\phi_n(\lambda,m)}{(1-\gamma)}$$

or equilently

$$|z|^{n-1} \leq \frac{(1-\delta)[n(1+\varsigma)-\mu(\gamma+\varsigma)]\phi_n(\lambda,m)}{(n-\delta)(1-\gamma)}$$

which yields the starlikeness of the family.

Integral Means Inequalities

In [6], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality conjuctured [5] and settled in [6], that

$$\int_0^{2\pi}\left|f(re^{i\varphi})\right|^\eta d\varphi \leq \int_0^{2\pi}\left|f_2(re^{i\varphi})^\eta\right| \; d\varphi,$$

for all $f \in T$, $\eta > 0$ and $0 < r < 1$. In [5] , he also proved his conjucture for the subclasses $T^*(\alpha)$ and $C(\alpha)$ of T.

Now, we prove Silverman 's conjecture for the class of functions $TS_\lambda^m(\mu,\gamma,\varsigma)$.

We need the concept of subordination between analytic functions and a subordination

theorem of Littlewood [2].

Two functions f and g, which are analytic in E, the function f is said to be

subordinate to g in E if there exists a function w analytic in E with

$w(0) = 0$, $|w(z)| < 1$, $(z \in E)$ Such that $f(z) = g(w(z))$, $(z \in E)$.

We denote this subordination by $f(z) \prec g(z)$. ($\prec$denotes subordination).

Lemma 1. If the functions $f$and $g$are analytic in E with $f(z) \prec g(z)$,then for $\eta > 0$and$z = re^{i\varphi}$ $0 < r < 1$,

$$\int_0^{2\pi}\left|g(re^{i\varphi})\right|^\eta d\varphi \leq \int_0^{2\pi}\left|f\;\;(re^{i\varphi})\right|^\eta d\varphi$$

Now, we discuss the integral means inequalities for functions f in $TS_\lambda^m(\mu,\gamma,\varsigma)$

$$\int_0^{2\pi}\left|g(re^{i\varphi})\right|^\eta d\varphi \leq \int_0^{2\pi}\left|f\;\;(re^{i\varphi})\right|^\eta d\varphi$$

Theorem 7. Let $f \in TS_\lambda^m(\mu,\gamma,\varsigma)$, $0 \leq \mu < 1$, $0 \leq \gamma \leq 1$, and $f_2(z)$ be defined by

$$f_2(z) = z - \frac{1-\gamma}{\varphi_2(\lambda,m,\mu,\varsigma,\gamma)}z^2$$

(2.12)

Proof. For $f(z) = z - \sum_{n=2}^{\infty}a_n z^n$, (2.12) is equivalent to

$$\int_0^{2\pi}\left|1-\sum_{n=2}^{\infty}a_n z^{n-1}\right|^{\eta} d\varphi \leq \int_0^{2\pi}\left|1-\frac{1-\gamma}{\varphi_2(\lambda,m,\mu,\varsigma,\gamma)}z\right|^{\eta} d\varphi$$

By Lemma 1, it is enough to prove that

$$1-\sum_{n=2}^{\infty}a_n z^{n-1} \prec 1-\frac{1-\gamma}{\varphi_2(\lambda,m,\mu,\varsigma,\gamma)}z$$

Assuming

$$1-\sum_{n=2}^{\infty}a_n z^{n-1} \prec 1-\frac{1-\gamma}{\varphi_2(\lambda,m,\mu,\varsigma,\gamma)}w(z)\,,$$

and using (2.2) we obtain

$$|w(z)| = \left|\sum_{n=2}^{\infty}\frac{\varphi_2(\lambda,m,\mu,\varsigma,\gamma)}{1-\gamma}a_n z^{n-1}\right| \leq |z|\sum_{n=2}^{\infty}\frac{\varphi_2(\lambda,m,\mu,\varsigma,\gamma)}{1-\gamma}a_n \leq |z|$$

where $\varphi_n(\lambda,m,\mu,\varsigma,\gamma) = [n(1+\varsigma)-\mu(\gamma+\varsigma)]\phi_n(\lambda,m)$

This completes the proof.

REFERENCES

1. W.G. Athsan and R. H. Buti, Fractional calclus of a class of univalent functions, Eur.J.Pure Appl.Math., 4(2), 162-173, 2011.
2. J.E. Littlewood, On inequalities in the theory of functions, Proc. London Math. Soc., 23(2), 481-519. 1925.
3. G. Murugusundarmoorthy and N. Magesh, Certain sub-classes of starlike functions of complex order involving generalized hypergeometric functions. Int. J. Math. Math. Sci., art ID 178605, 12, 2010.
4. H. Silverman, Univalent functions with negative coefficients, Proc. Amer. Math. Soc. 51, 109-116, 1975.
5. H. Silverman, A survey with open problems on univalent functions whose coefficient are negative, Rocky Mountain J. Math., 21(3), 1099-1125, 1991.
6. H.Silverman, Integral means for univalent functions with negative c oefficient, Houston J. Math., 23(1), 169-174, 1997.