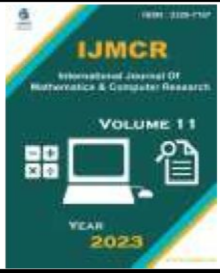
International Journal of Mathematics and Computer Research

ISSN: 2320-7167

Volume 11 Issue 11 November 2023, Page no. – 3882-3884

Index Copernicus ICV: 57.55, Impact Factor: 7.362

DOI: 10.47191/ijmcr/v11i11.08



On homogeneous Linear Recurrence Relations

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INFO ARTICLE

Published Online : 29 November 2023

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ABSTRACT

We use the Z-transform to motivate the Baldoni et al algorithm to solve homogeneous linear recurrence relations, with applications to Fibonacci numbers and Chebyshev polynomials.

KEYWORDS: Recurrence relations, Chebyshev polynomials, Fibonacci numbers, Baldoni et al method, Z-transform, Companion matrix, Characteristic equation, Lucas numbers

1. Introduction

Here we consider recurrence relations with the following structure:

$$f_{n+k} = a_{k-1} f_{n+k-1} + a_{k-2} f_{n+k-2} + \cdots + a_0 f_n , \qquad n, k \geq 0, \qquad (1)$$

and the initial values $f_0, f_1, \ldots, f_{k-1}$. In according with the Baldoni et al technique [1, 2], first we construct the companion matrix [3-5]:

$$A_{kxk} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & a_3 & \cdots & a_{k-1} \end{pmatrix}, \qquad (2)$$

with characteristic equation [4-10]:

$$\lambda^k - a_{k-1}\lambda^{k-1} - a_{k-2}\lambda^{k-2} - \cdots - a_1 \lambda - a_0 = 0, \qquad (3)$$

and we accept that it has distinct roots. Thus, the solution of (1) is given by:

$$f_n = b_1 \lambda_1^n + b_2 \lambda_2^n + \cdots + b_k \lambda_k^n, \qquad n = k, \ k+1, \ldots , \qquad (4)$$

where the b_j are determined by the initial values via the linear system:

$$b_1 \lambda_1^r + b_2 \lambda_2^r + \cdots + b_k \lambda_k^r = f_r , \qquad r = 0, 1, \ldots, k-1 . \qquad (5)$$

In Sec. 2 we employ the Z-transform [11, 12] to study (1) for the case $k = 2$and thus to motivate this Baldoni et al algorithm. The Sec. 3 has applications of this algorithm to Chebyshev polynomials [13-15] and Fibonacci numbers [16-19].

2 Solution of (1) via Z-transform

Here we obtain the solution of (1) for the case $k = 2$, that is:

$$f_{n+2} = a_1 f_{n+1} + a_0 f_n , \qquad n \geq 0, \qquad (6)$$

for given values of f_0 and f_1, whose Z-transform [11, 12] is immediate:

$$F(z) = \frac{z^2 f_0 + z f_1 - z a_1 f_0}{z^2 - a_1 z - a_0} = \frac{z (z f_0 + f_1 - a_1 f_0)}{(z - z_1)(z - z_2)} , \qquad a_0 = - z_1 z_2 , \quad a_1 = z_1 + z_2 . \qquad (7)$$

Thus, we observe the presence of the characteristic polynomial $z^2 - a_1 z - a_0$ in accordance with (3). The inverse Z-transform of (7) generates the sequence $\{f_j\}$:

$$f_n = \frac{1}{2\pi i} \oint F(z) z^{n-1} dz = \frac{1}{2\pi i} \oint \frac{z^n (z f_0 + f_1 - a_1 f_0)}{(z - z_1)(z - z_2)} dz , \qquad (8)$$

where the integration is realized for a contour that contains to z_1 and z_2, then the Cauchy's integral theorem [20] implies an expression with the structure (4):

$$f_n = b_1 z_1^n + b_2 z_2^n\,, \qquad b_1 = \frac{-z_1 f_0 - f_1 + a_1 f_0}{z_2 - z_1}\,, \qquad b_2 = \frac{z_2 f_0 + f_1 - a_1 f_0}{z_2 - z_1}\,, \qquad (9)$$

then are immediate the following relations verifying (5):

$$b_1 + b_2 = f_0\,, b_1 z_1 + b_2 z_2 = f_1\,. \qquad (10)$$

3. Applications of the Baldoni et al process

The Chebyshev-Lanczos polynomials $T_n(x)$ are defined by [14, 15, 21]:

$$T_{n+2} = 2x\,T_{n+1} - T_n\,, \qquad T_0 = 1, \qquad T_1 = x, \qquad -1 \le x \le 1, \qquad (11)$$

then (1) gives $k = 2,\ a_0 = -1,\ a_1 = 2x,$ hence from (2), (3) and (5):

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 2x \end{pmatrix}, \qquad \lambda^2 - 2x\,\lambda + 1 = 0, \qquad \lambda_1 = x + \sqrt{x^2 - 1}\,, \qquad \lambda_2 = x - \sqrt{x^2 - 1}\,, \qquad (12)$$

$$b_1 + b_2 = 1, \qquad b_1\lambda_1 + b_2\lambda_2 = x \qquad \therefore \qquad b_1 = b_2 = \frac{1}{2},$$

and from (4) we obtain the solution of (11) [$x = \cos\theta$]:

$$T_n = \frac{1}{2}(\lambda_1^n + \lambda_2^n) = \frac{1}{2}\left[\left(x + i\,\sqrt{1 - x^2}\right)^n + \left(x - i\,\sqrt{1 - x^2}\right)^n\right] = \frac{1}{2}\left(e^{in\theta} + e^{-in\theta}\right) = \cos(n\theta), \qquad (13)$$

which are important in problems of interpolation [22] and for the tau method of Lanczos-Ortiz [14, 22-24].

The Fibonacci numbers F_n [16] satisfy the recurrence relation [17-19]:

$$F_{n+2} = F_{n+1} + F_n\,, \qquad F_0 = 0, \qquad F_1 = 1\,, \qquad (14)$$

Then (1) and (3) give $k = 2,\ a_0 = a_1 = 1$ and $\lambda^2 - \lambda - 1 = 0,$ thus:

$$\lambda_1 = \frac{1}{2}\left(1 + \sqrt{5}\right), \quad \lambda_2 = \frac{1}{2}\left(1 - \sqrt{5}\right), \quad b_1 + b_2 = 0, b_1\,\lambda_1 + b_2\lambda_2 = 1, \qquad (15)$$

therefore $b_1 = -b_2 = \frac{1}{\sqrt{5}}$ and (4) implies the Binet formula:

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \quad n = 0, 1, 2, \ldots \qquad (16)$$

It is possible to write (16) in terms of the hypergeometric function [25]:

$$F_n = \frac{n}{2^{n-1}}\ {}_2F_1\left(\frac{1-n}{2}, \frac{2-n}{2}; \frac{3}{2}; 5\right). \qquad (17)$$

Finally, the Lucas numbers L_n [18, 26] verify the recurrence relation (14) with the initial values $L_0 = 2,$ $L_1 = 1,$ then this Baldoni et al algorithm leads to the formula:

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n. \qquad (18)$$

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