International Journal of Mathematics and Computer Research

ISSN: 2320-7167

Volume 12 Issue 02 February 2024, Page no. – 4051-4054

Index Copernicus ICV: 57.55, Impact Factor: 8.316

DOI: 10.47191/ijmcr/v12i2.07

On The Existence and Uniqueness of Solutions to a Non-Homogeneous Second-Order Difference Equation of Accretive Type in 2-Banach Spaces

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ARTICLE INFO

Published Online:
23 February 2024
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ABSTRACT

In this study we present the existence and uniqueness of a non-homogeneous second order difference equation of accretive type in linear 2-Banach spaces. We use the 2-norm contraction mapping and its fixed point to establish our results.

KEYWORDS: Linear 2-Banach space, linear 2-normed space, accretive operator, fixed point.

1. INTRODUCTION

The concept of accretive operator dates back to the study by Browder in 1967, during an investigation into non-expansive and accretive operators in Banach spaces (Sari 2015) [1]. After that, many authors flooded the field with a lot of findings. For instance, Barbu (1976)[2], presented the basic meaning of accretive operators and their uses in handling problems involving nonlinear semi-group and differential equations in Banach spaces. Also, Barbu [19], provded the basic results of the theory of nonlinear operators of monotone type as well as their equivalent dynamics generated in Banach spaces. It can be recalled that the nonlinear maximal monotone operators coincide with the m-accretive operative operators.

Garcia-Falset and Reich (2006)[3], investigated the necessary and sufficient condition for an accretive operator and its asymptotic behavior. The properties of accretive operators were studied by Sari (2015)[1] Jung (2016)[5] presented some results of his findings on zeros of accretive mappings, to mention but a few.

Studies have also been carried out on accretive operators in association with the second-order difference equation of accretive type in Banach spaces. Among many other authors, Apreutesei(2003)[6] and Rouhani *et al.*(2019)[7], investigated the homogeneous part of the equation. Findings on the non-homogeneous aspect of the second-order difference equation of accretive type in Banach spaces have been done by Jamshinedhad and Saeidi(2018)[10], Rouhani *et al.* (2019)[7], as well as Apreutesei and Apreutesei(2012), Rouhani and Khatibzadeh(2011)[8]; and host of others.

In making more explorations in the field of accretive operators, some authors have carried out their studies in 2-dimensional normed spaces known as the 2-normed spaces (2-Banach spaces)[15]. The concept of 2-Banach spaces was introduced by S. Gahler in 1963[12]. In this regard Kir [11] studied accretive operators arising from 2-Banach spaces, and their links with the classes of non-expansive mappings. Harikrishnan and Ravindran [12], investigated some properties of resolvents of accretive operators in linear 2-normed spaces and gave attention to contractive mappings and the unique fixed points of contraction mappings in linear 2-normed spaces.

Our focus is on the non-homogeneous second order difference equation of accretive type adopted from Rouhani *et al.* (2019)[7]. This is an equation of the form:

$$\begin{cases} u_{m+1} - (1+\theta_m)u_m + \theta_m u_{m-1} \in c_m A u_m + f_m, & 1 \le m \in \mathbb{N} \\ u_0 = v, \quad \sup\{\| u_m \| : m \ge 0\} < \infty \end{cases} \quad \mathbf{(1.1)}$$

Where A is a nonlinear m-operative operator in real 2-Banach spaces $(B, \|.,.\|)$, $c_m > 0$, $\theta_m > 0$. We explore the existence and uniqueness of (1.1) in 2-Banach spaces using 2-Banach fixed point principle, otherwise known as 2-norm contraction mapping, to obtain a fixed point of the mapping.

2. PRELIMINARIES

In this section, we look at the basic terms which are relevant to our findings on the existence and uniqueness of the non-homogeneous second-order difference of accretive type (1.1).

Definition (2.1): Linear 2-normed space [14]

By linear 2-normed space, we refer to a pair $(B, \|.,.\|)$ such that B is at least a two-dimensional real linear space and a function $\|.,.\|\ B \times B \longrightarrow \mathbb{R}$ having the following properties:

(i) $\| u, v \| = 0$ if and only if u and v are linearly dependent;

(ii) $\| u, v \| = \| v, u \|$ for all $u, v \in B$;

(iii) $\| u, v + y \| \leq \| u, v \| + \| u, y \|$ for $u, v, y \in B$(triangle inequality);

(iv) $\| \alpha u, v \| = |\alpha| \| u, v \|, \alpha \in \mathbb{R}$ and $u, v \in B$.

Definition(2.2) Convergent sequence in 2-normed space [13]

A sequence $\{u_m\}$ in a 2-normed space $(B, \|.,.\|)$ is said to be a convergence sequence if there is a $u \in B$ such that $\lim_{m\to\infty} \| u_m - u, y \| = 0, \textit{for all } y \in B$.

Definition (2.3) Cauchy sequence in a 2-normed space [13]

A sequence $\{u_m\}$ in a 2-normed space $(B, \|.,.\|)$ is said to be if

$$\lim_{m,n\longrightarrow\infty} \| u_{m,} - u_n , y \| = 0, \text{ for all } y \in B.$$

If every Cauchy sequence in B converges to some $u \in B$, then B is said to be compete with respect to the 2-norm. Any complete 2-normed space is called a 2-Banach space.

Definition (2.4) Accretive operator[11]

Let $(B, \|.,.\|)$ be a 2-normed space. If A is a non-linear operator mapping a subset of B. An operator $A: D(A) \subset B \rightarrow B)$ is said to be accretive if for every $y \in D(A)$ is $\| u - v, y \| \leq \| (u - v) + \alpha(Au - Av), y \| \textit{ for all } u, v \in A(D)$ and $\alpha > 0$.

An operator $A: D(A) \subset B \rightarrow B$ is said to be m-accretive if $R(I + \alpha A) = B$, for all $\alpha > 0$.

Definition (2.5)Uniformly convex 2-normed space $(B, \|.,.\|)$ [16]

A 2-normed space $(B, \|.,.\|)$ if for every $\varepsilon \epsilon (0,2)$ and $y \neq 0 \in B$, there exists $\alpha < 0$ such that $\| u - v, y \leq \frac{1}{2}(1-\alpha)$.

Remark (2.6)

For a linear 2-normed space $(B, \|.,.\|)$ we keep in mind the following: (1) The space B is strictly convex, (2) For every non-zero element u, the associated normed space is strictly convex in the usual sense, (3) The mapping $(A: B \longrightarrow B)$ is non-expansive and the set of fixed points of B is a convex set; and (4)The dual of B, that is, B^* is also convex[17, 18].

In this study, we refer to linear 2-Banach spaces as 2-normed spaces and vice-versa as done by some authors one of which is [15].

3. RESULTS

3.1 Contraction mapping in $(B, \|.,.\|)$ [9]

Let $(B, \|.,.\|)$ be a linear 2-normed space, then the mapping $T: B \longrightarrow B$ is said to be a contraction mapping if there exists some $k \in (0,1)$ such that $\| Tu - Tv, y \| \leq |k| \| u - v, y \|$, for all $u, v, y \in B$.

Theorem (3.1)

Let $(B, \|.,.\|)$ be a uniformly convex linear 2-normed space, and let $A : B \rightarrow B$ be accretive such that $A^{-1}(0) = \emptyset$. Then (1.1) has a unique solution.

Proof: We first show that (1.1) has a contraction mapping T. Let the mapping $T: B \rightarrow B$, where B is a 2-Banach space. We define (1.1) by

$$T(u_m) = u_{m+1} - (1 + \theta_m)u_m + \theta_m u_{m-1} - c_m Au_m - f_m \quad \mathbf{(3.1)}$$

We know from [7] that the sequences θ_m, c_m and f_m are bounded. We need to show that for any $u, v \in B$, there exists a constant k with $0 \leq k < 1$ such that

$$\| Tu - Tv, y \| \leq k \| u - v, y \| \text{ for all } u, v, y \in B \quad \mathbf{(3.2)}$$

From algebraic processes and properties of 2-norm, we show that the operator T is a contraction mapping with

$$k = \max\{|1|, |\theta|, |c|\} \quad \mathbf{(3.3)}$$

Considering the difference between $T(u_m)$ and $T(v_m)$, from (3.1) we have

$$T(u_m) - T(v_m) = (u_{m+1} - v_{m+1}) - (1+\theta_m)(u_m - v_n) + \theta_m(u_{m-1} - v_{m-1}) + c_m A(u_m - v_m) + (f_m - f_m) \quad \mathbf{(3.4)}$$

Simplifying (3.4), we have

$$T(u_m) - T(v_m) = (u_{m+1} - v_{m+1}) - (1+\theta_m)(u_m - v_n) + \theta_m(u_{m-1} - v_{m-1}) + c_m A(u_m - v_m) \quad \mathbf{(3.5)}$$

Taking the norm of both sides of (3.5) and applying 2-norm principle, we have $\| T(u_m) - T(v_m), y \| = \| u_{m+1} - v_{m+1}, y \| + \| (1+\theta_m)(u_m - v_n), y \| + \| \theta_m(u_{m-1} - v_{m-1}), y \| + \| c_m A(u_m - v_m), y \|, for\ all\ u, v, y \in B$ $\quad \mathbf{(3.6)}$

Thus, let $u_m = \{u_{m,} u_{m-1} \ldots\}$, $v_m = \{v_m, v_{m-1} \ldots\}$ for all $u, v \in B$. Applying the triangular inequality on (3.6), we have

$$\| T(u) - T(v), y \| \leq \| u_m - v_m, y \| + |1+\theta_m| \, \| u_m - v_m, y \| + |\theta_m| \, \| u_m - v_m, y \| + |c_m A| \, \| u_m - v_m, y \|, for\ all\ u, v, y \in B \quad \mathbf{(3.7)}$$

Applying the sub-additivity of (3.7) we have $\| Tu - Tv, y \| \leq |1+\theta| \, \| u - v,\ y \| + |\theta| \, \| u - v, y \| + |c| \, \| u - v, y \|$, for all $u, v, y \in B$ $\quad \mathbf{(3.8)}$

Choosing k such that $0 \leq k < 1$ to ensure our contradiction property, we consider the following conditions

$$\begin{cases} |\theta| + |c| + k \leq |1| \\ k < |1| + |\theta| + |c| \\ k = \max\{|1|, |\theta|, |c|\} \end{cases} \quad \mathbf{(3.9)}$$

From (3.8) and (3.9) we conclude that $\| Tu - Tv, y \| \leq k \| u - v, y \|$ for all $u, v, y \in B$ which shows that T is a contraction mapping. Assuming that $T = A(A$ being the accretive operator in (1.1), then me have $\| Au - Av, y \| \leq k \| u - v, y \|$ for all $u, v, y \in B$ $\quad \mathbf{(3.10)}$

Definition (3.2): Sequentially continuous 2-normed space[12]

Let $(B, \|., .\|)$ be a linear 2-normed space and $A: B \longrightarrow B$ a contraction mapping. Then A is said to be sequentially continuous if there exists a constant $k \in (0,1)$ such that for all $u, v, y \in B$, $\| Au_m - Au, y \| \leq k \| u_m - u, y \| \longrightarrow 0$ as $m \longrightarrow \infty \Longrightarrow Au_m - Au$ $\quad \mathbf{(3.11)}$

Then , is sequentially continuous.

Lemma (3.2)

Let $(B, \|., .\|)$ be a linear uniformly and strictly convex 2-Banach space. Let the mapping $A: B \longrightarrow B$ be a contraction in (1.1), then (1.1) has a unique solution in B.

Proof: We recall that A is a contraction in (1.1). Hence, there exists some constant $k \in (0,1)$ such that for all $u, v, y \in B$, $k \in \mathbb{R}$, $\| Au - Av, y \| \leq k \| u - v, y \|$, we have

$$\| A^3 u - A^3 v, y \| = \| A^2(Au) - A^2(Av), y \| \leq k \| Au - Av, y \| \leq k^3 \| u - v, y \| \text{ for all } y \in B.$$

In the same vein, $\| A^m u - A^m v, y \| \leq k^m \| u - v, y \|$ for all $y \in B$. Letting $u_0 \in B$, we construct a sequence on u_0. Let $u_1 = Au_0, u_2 = Au_1, u_3 = Au_2, u_4 = Au_3, \ldots, u_m = Au_{m-1}$, then $u_1 = Au_0, u_2 = A^2 u_0, u_3 = A^3 u_0, \ldots, u_m = A^m u_0$.

We first prove that $\{u_m\}$ is Cauchy in B. Let $g, m > 0$ with $g > m$. Taking $g = m + w$, then for any $y \in B$, $\| u_m - u_g, y \| = \| u_m - u_{m+w}, y \|$

$$\begin{aligned}
&= \| (u_m - u_{m+1}) + (u_{m+1} - u_{m+2}) + u_{m+2} - \ldots + (u_{m+w-1} - u_{m+w}), y \| \\
&\leq \| u_m - u_{m+1}, y \| + \| u_{m+1} - u_{m+2}, y \| + \cdots + \| u_{m+w-1} - u_{m+w}, y \| \\
&\leq \| A^m u_0 - A^m u_1, y \| + \| A^{m+1} u_0 - A^{m+1} u_1, y \| + \cdots + \| A^{m+w-1} u_0 - A^{m+w-1} u_1, y \| \\
&\leq k^m \| u_0 - u_1, y \| + k^{m+1} \| u_0 - u_1, y \| + \cdots + k^{m+w-1} \| u_0 - u_1, y \| \\
&\leq k^m \| u_0 - u_1, y \| (1 + k + k^2 + k^3 + \cdots) \\
&= \left(\frac{k^m}{1-k}\right) \| u_0 - u_1, y \|
\end{aligned}$$

That is, $\| u_m - u_{w,} y \| \leq \left(\frac{k^m}{1-k}\right) \| u_0 - u_1, y \|$ for all $y \in B$.

We know that every linear 2-Banach space is bounded. Hence, B is bounded . Thus, there exists an $M > 0$ such that $\| u_0 - u_1, y \| \leq M$ for all $y \in B$. It follows that $\| u_m - u_g,\ y \| \leq \left(\frac{k^m M}{1-k}\right)$ for all $y \in B$. This implies that $\| u_m - u_g, y \| \longrightarrow 0$ as $m \longrightarrow \infty$. That is to say, since $k \in (0,1)$ implies that $\{u_m\}$ is Cauchy in B, $\{u_m\} \longrightarrow u \in B$.

3.1.1 The fixed point of the accretive operator A

We have to show that A has a fixed point $u \in B$. From Definition 3.3, we recall that A is sequentially continuous, meaning that $Au = lim Au_m = lim Au_{m+1} = u$ as $m \longrightarrow \infty$. Thus, A has a fixed point $u \in B$

3.1.2 Uniqueness of the fixed point

To prove the uniqueness, we assume that for all $u, v, y \in B$ and $k \in (1,0)$, $\| Au - Av, y \| \leq k \| u - v, y \| \Longrightarrow k \leq 1$, which is a contradicts $k \in (0,1)$. It follows that for every $y \in B$, $\| Au - Av, y \| = 0$ which implies that $\| Au - Av, y \| = 0$ which implies that $Au = Av$, meaning that $u = v$. Hence, the fixed point of A is unique, meaning tha the solution exists and is unique.

CONCLUSION

Solution to the non-homogeneous second order difference equation of accretive type (1.1) exists in 2-Banach spaces, and the solution is unique.

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