International Journal of Mathematics and Computer Research

ISSN: 2320-7167

Volume 12 Issue 03 March 2024, Page no. – 4097-4102

Index Copernicus ICV: 57.55, Impact Factor: 8.316

DOI: 10.47191/ijmcr/v12i3.03

Strong and Δ-Convergence Results for Generalized Non-Expansive Type Map pings through JF-Iteration Process in Hyperbolic Spaces.

A. S. Saluja[1], Aarti Patel[2]

[1] Department of mathematics, Institute for Excellence in Higher Education, IEHE, Bhopal (M.P.), 462016, India

[2] Department of mathematics, Babulal Gour Govt. P. G. College BHEL, Bhopal (M.P.), 462022, India

ARTICLE INFO	ABSTRACT
Published Online: 20 March 2024 **Corresponding Author:** A. S. Saluja	In this paper, we prove some strong and Δ-convergence results for generalized non-expansive mappings through JF-iterative process in hyperbolic spaces.

KEYWORDS: Generalized non-expansive mappings, Fixed Point, JF-iterative scheme, hyperbolic spaces.

1. INTRODUCTION

The concept of generalized non-expansive mappings was introduced by Hardy and Rogers [16]. Further, generalized non-expansive mapping was introduced by Suzuki's or called condition (c)[29]. It can be defined in many settings of metric spaces. Let G: Y→Y be a self-map on a nonempty subset Y of a Banach space X. It is known that if G has a fixed point then G is quasi non-expansive mapping. The class of generalized non-expansive mappings is larger than the class of non-expansive mapping and smaller than the class of quasi non-expansive mappings was defined by Fukhar-ud-din and Saleh [8]. The class of mappings satisfying Suzuki's condition (c) is larger than the class of non-expansive mappings and smaller than the class quasi non-expansive mappings was defined by Suzuki's [29]. The existence and convergence theorems for non-expansive, generalized non-expensive and suzuki's condition (c) have been studied by several authors, e.g. see Bogin [4], Wong [34], Goebel et al. [10],

Gursoy et al. [14], Gursoy et al. [15], Thakur et al. [30], Dhomphonhsa et al. [7], Ali et al. [2], Uddin and Imdad (a)[31], Uddin and Imdad (b)[32], Uddin and Imdad [33].

Let G: Y →Y be a self-map on a nonempty subset Y of a Banach space X and $\{r_n\}$and $\{s_n\}$ real sequences in (0,1) for all $n \geq 0$. Non-expensive mappings of approximate fixed point for an iteration scheme introduced by Mann [23] which is generated by an arbitrary point $p_1 \epsilon Y$

$$p_{n+1} = (1 - r_n)p_n + r_nGq_n, \qquad n \in Y$$

(1.1)

Where $\{r_n\}$real sequence in (0,1). It is known as Mann iterative scheme which fails to converge to a fixed point of pseudo contractive mappings. Pseudo contractive mappings of approximate fixed point two steps iteration scheme was introduced by Ishikhawa [17] which is generated by arbitrary point $p_1 \epsilon Y$

$$\begin{cases} p_{n+1} = (1 - r_n)p_n + r_nGq_n \\ q_n = (1 - s_n)p_n + s_nGp_n, \end{cases} \qquad n \in Y$$

(1.2)

Where $\{r_n\}$ and $\{s_n\}$real sequence in (0,1). In the past few decades, large number of iterative schemes were introduced and studied by several authors i e. Noor [25], S.Agrawal[1] Picard -S Gursoy and karakaya[12], Gursoy[13] and Thakur et. al.[30] respectively, which are generated by an arbitrary $p_1 \epsilon Y$

$$\begin{cases} p_{n+1} = (1 - r_n)p_n + r_nGq_n \\ q_n = (1 - s_n)p_n + s_nGw_n \\ w_n = (1 - t_n)p_n + t_nGp_n, \end{cases} \qquad n \in Y$$

(1.3)

$$\begin{cases} p_{n+1} = (1 - r_n)Gp_n + r_nGq_n \\ q_n = (1 - s_n)p_n + s_nGp_n, \end{cases} \qquad n \in Y$$

(1.4)

$$\begin{cases} p_{n+1} = Gq_n \\ q_n = (1 - r_n)Gp_n + r_nGw_n \\ w_n = (1 - s_n)p_n + s_nGp_n, \end{cases} \qquad n \in Y$$

(1.5)

$$\begin{cases} p_{n+1} = Gq_n \\ q_n = G((1 - r_n)Gp_n + r_nw_n) \\ w_n = (1 - s_n)p_n + s_nGp_n, \end{cases} \qquad n \in Y$$

(1.6) Where $\{r_n\}$,$\{s_n\}$and $\{t_n\}$real sequence in (0,1). Recently

In 2020 a new iteration process called JF-iteration scheme was introduced by Ali [3] which defined as follow

$$\begin{cases} p_{n+1} = G((1-r_n)q_n + r_n G q_n) \\ q_n = G w_n \\ w_n = G((1-s_n)p_n + s_n G p_n), \quad n \in Y \end{cases}$$

(1.7)

They obtained some basic properties for Generalized non expensive mappings due to Hardy and Rogers [16]. Also, they proved some convergence results using JF-iteration scheme for Generalized non expensive mappings in uniformly convex Banach Space. Lim [22] introduced the concept of Δ-convergence. Motivated by above, we use JF-iteration process for proving some Δ-convergence and strong convergence theorems for mapping in hyperbolic spaces.

2. PRELIMINARIES

In this study, we discuss on the setting of hyperbolic spaces which was introduced by kohlenbach [19], containing normed linear spaces and convex subsets and Hadamrd manifolds [27], CAT(0) spaces in the sense of Gromov[11] and Hilbert ball equipped with the hyperbolic metric [27]. In this context we need some definitions, lemmas and prepositions which will be used in the sequel,

Definition[19] A hyperbolic space is a triple (X, d, W) where (X, d) is a metric space and $W: X^2 \times [0,1] \to X$ such that (W1) $d(w, W(u, v, \omega)) \leq (1-\omega)d(w, u) + \omega d(w, v)$

(W2) $d(W(u, v, \omega), d(u, v, \sigma)) = |\omega - \sigma| d(u, v)$,

(W3) $W(u, v, \omega) = W(v, u, (1-\omega))$,

(W4) $d(W(u, z, \omega), W(v, w, \omega) \leq (1-\omega)d(u, v) + \omega d(z, w)$

For all $u, v, w, z \in X$ and $\omega, \sigma \in [01]$

Definition [20] A hyperbolic space (X, d, W) is called uniformly convex, if for all $u, v, z \in X$, $r > 0$ and $\varepsilon \in (0\ 2]$ there exists $\delta \in (0\ 1]$, such that $d(v, u) \leq r$, $d(z, u) \leq r$ and $d(v, z) \leq \varepsilon r$. Then,

$$d(W(v, z, ½), u) \leq (1 - \delta)r. \quad (2.1)$$

Definition [20] A mapping $\mu: (0\ \infty) \times (0\ 2] \to (0\ 1)$ which provides $\delta = \mu(r, \varepsilon)$ for a given $r > 0$ and $\varepsilon \in (0\ 2]$ is well known as a modulus of uniform convexity of X. We call μ as a monotone if it decreases with r (for a fixed ε), i.e., for any given $\epsilon > 0$ and for any $r_2 > r_1 > 0$, we have $\mu(r_2, \varepsilon) \leq (r_1, \varepsilon)$

Definition [20] A nonempty subset Y of a hyperbolic space is said to be convex if $W(u, v, \omega) \in Y$ for any $u, v \in Y$ and $\omega \in [0\ 1]$. If $u, v \in X$ and $\omega \in [0, 1]$, then we use the notion $(1-\omega)u(+)\omega v$ for $W(u, v, \omega)$. In [20], it is remarked that any normed space $(X, \|.\|)$ is a hyperbolic space, with $(1-\omega)u(+)\omega v = (1-\omega)u + \omega v$. Hence, the class of uniformly convex hyperbolic spaces is a natural generalization of uniformly convex Banach spaces.

Firstly, the JF-iteration process is expressed in the Hyperbolic space as follow:

$$\begin{cases} p_n \in Y \\ p_{n+1} = W(Gq_n, q_n, r_n) \\ q_n = W(Gw_n, p_n, s_n) \\ w_n = W(p_n, Gp_n, t_n), \quad n \in Y \end{cases}$$

(2.2)

for all $n \geq 0$, $\{r_n\}$, $\{s_n\}$ & $\{t_n\}$ are real sequence in [0,1].

Let Y be a nonempty subset of metric space X. If $G(p) = p$, then p is said to be a fixed point of a mapping G. The set of all fixed points of G is denoted by F(G); $F(G) = \{x \in Y: Gx = x\}$.

Definition [24] A mapping $G: Y \to Y$ is said to be

i Non-expansive if $d(Gu, Gv) \leq d(u, v)$ for all $u, v \in Y$;

ii Quasi non-expansive if $F(G) \neq \varphi$ and $d(Gu, Gp) \leq d(u, p)$; for all $u \in Y$ and $p \in F(G)$.

iii [16] Generalized non-expansive if for all $u, v \in Y$

$$d(Gv, Gu) \leq a_1 d(v,u) + a_2 d(Gu, u) + a_3 d(Gv, v) + a_4 d(Gv, u) + a_5 d(Gu,v) \quad (2.3)$$

Where $a_1, \ldots, a_5$ are non-negative real numbers with $a_1 + a_2 + a_3 + a_4 + a_5 \leq 1$

C.F Fuster and Galves [9] defined the condition is equivalent to the following condition

$$d(Gv, Gu) \leq a\, d(v, u) + b\,(d(Gu, u) + d(Gv, v)) + c\,(d(Gv, u) + d(Gu, v)) \quad (2.4)$$

For all $u, v \in Y$, where a, b, c are non-negative constants with $a + 2b + 2c \leq 1$ and $a = a_1$, $b = a_2 + a_3/2$, $c = a_4 + a_5/2$

iv [29] Suzuki's or called condition (c), which is defined as follows if

$$½\, d(Gu, u) \leq d(v, u) \text{ implies } d(Gv, Gu) \leq d(v, u); \quad \forall u, v \in Y. \quad (2.5)$$

Lemma 2.1 [3] Let $G: Y \to Y$ be a generalized non-expansive mapping satisfying (2.4), where Y is a nonempty subset of hyperbolic space X. Then

$$d(Gv, u) \leq d(v, u) + \frac{1+b+c}{1-b-c} d(Gu, u); \text{ holds for all } u, v \in Y. \quad (2.6)$$

by[7],We require the following definition of convergence in hyperbolic space which called Δ- convergence. The principle results are obtained by it.

Let Y be nonempty, closed and convex subset of a Hyperbolic space X, $\{p_n\}$ a bounded sequence in X and $u \in Y$, we define a function $r(., \{p_n\}): X \to [0,\infty]$ by

$r(u, \{p_n\}) = \limsup_{n\to\infty} d(u, p_n)$

An asymptotic radius of $\{p_n\}$ relative to Y is defined by

$r(Y, \{p_n\}) = \inf\{r\{u,\{p_n\}) : u \in Y\}$.

An asymptotic centre of $\{p_n\}$ relative to Y is defined by

$AC(Y, \{p_n\}) = \{u \in Y : r(u,\{p_n\}) = r(Y, \{p_n\})\}$.

The sequence $\{p_n\}$ in X is said to Δ-convergence to $u \in Y$ if u is unique asymptotic centre of $\{w_n\}$ for every subsequence $\{w_n\}$ of $\{p_n\}$. In this case, we write $\Delta\text{-}\limsup_{n\to\infty} p_n = p$ and call p the Δ-lim of $\{p_n\}$.

Lemma 2.2 [21] Let X be a complete uniformly convex Hyperbolic space with a monotone modulus of uniform convexity μ.Then every bounded sequence $\{p_n\}$ in X has a unique asymptotic centre with respect to any nonempty closed convex subset Y of X.

Lemma 2.3 [18] Let X be a complete uniformly convex Hyperbolic space with a monotone modulus of uniform convexity μ. Let $u \in Y$ and $\{\alpha_n\}$ be a sequence in [a, b] for some a, b $\in$ (0, 1). If $\{p_n\}$ and $\{q_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(p_n, p) \leq \vartheta$, $\limsup_{n\to\infty} d(q_n, p) \leq \vartheta$ and $\lim_{n\to\infty} d(W(p_n, q_n, \alpha_n), p) = \vartheta$ for some $\vartheta \geq 0$. Then, $\lim_{n\to\infty} d(p_n, q_n) = 0$.

3. MAIN RESULTS

First, we obtain the following useful lemmas which help us to prove main results

Lemma 1 A Let G: Y → Y be a generalized non-expansive mapping satisfying (2.4), where Y is a nonempty closed & convex subset of a uniformly convex hyperbolic space X. Let $\{p_n\}$ be a sequence generated by (2.2); Then $\lim_{n\to\square} d(p, p_n)$ exists for all $p \in F(G)$.

Proof:- let $p \in F(G)$ & $p_n \in Y$; since G is generalized non-expansive mapping, we can easily obtain that

$d(Gp, Gp_n) = d(p, p_n) \leq d(p, p_n)$; for all $p_n \in Y$ & $p \in F(G)$

Thus using (2.2), we obtain that

$$\begin{aligned} d(w_n,p) &= d(W(p_n, Gp_n, t_n), p) \\ &\leq (1-t_n)d(p_n, p) + t_n\, d(Gp_n, p) \\ &= (1-t_n)d(p_n, p) + t_n\, d(Gp_n, G p) \\ &\leq (1-t_n)d(p_n, p) + t_n\, d(p_n, p) \end{aligned}$$

$$d(w_n, p) \leq d(p_n, p) \quad (3.1)$$

using (2.2) & (3.1)

$$\begin{aligned} d(q_n,p) &= d(W(Gw_n, w_n, s_n), p) \\ &\leq (1-s_n)d(Gw_n, p) + s_n\, d(w_n, p) \\ &= (1-s_n)d(Gw_n, Gp) + s_n\, d(w_n,p) \\ &\leq (1-s_n)d(w_n, p) + s_n\, d(w_n, p) \end{aligned}$$

$d(w_n, p) \leq d(q_n, p)$

$$d(w_n, p) \leq d(p_n, p) \quad (3.2)$$

using (2.2) & (3.2)

$$\begin{aligned} d(p_{n+1},p) &= d(W(Gq_n, q_n, r_n), p) \\ &\leq (1-r_n)d(Gq_n, p) + r_n\, d(q_n, p) \\ &= (1-r_n)d(Gq_n, Gp) + r_n\, d(q_n, p) \\ &\leq (1-r_n)d(q_n, p) + r_n\, d(p_n, p) \end{aligned}$$

$d(p_{n+1}, p) \leq d(q_n, p)$

$$d(p_{n+1}, p) \leq d(q_n, p) \quad (3.3)$$

Thus the sequence $\{d(p_n, p)\}$is bounded below & decreasing . Hence $\lim_{n\to\square} d(P_n, p)$ exists for all $p \in F(G)$.

Lemma 2 A Let G: Y → Y be a generalized non-expansive mapping satisfying (2.4), where Y is a nonempty closed & convex subset of a uniformly convex hyperbolic space X. Let $\{p_n\}$ be a sequence generated by (2.2). Then $F(G) \neq \varphi$, if and only if $\{p_n\}$ is bounded & $\lim_{n\to\square} d(Gp_n, P_n) = 0$.

Proof:- Assume that $F(G) \neq \varphi$, & $p \in F(G)$, by lemma 1$\{p_n\}$is bounded.

Next we will indicate that $\lim_{n\to\infty} d(Gp_n, p_n) = 0$

Since G is generalized non-expansive mapping, we have

$$d(p, Gp_n) = d(Gp, Gp_n) \leq d(p,p_n) \quad (3.4)$$

from lemma1 we achieve $\lim_{n\to\infty} d(p_n, p)$ exists for all $p \in F(G)$

Assume that $\lim_{n\to\infty} d(p_n, p) = \alpha$, $\alpha > 0$. then

$$\begin{aligned} d(w_n,p) &= d(W(p_n, Gp_n, t_n), p) \\ &\leq (1-t_n)d(p_n, p) + t_n\, d(Gp_n, p) \\ &= (1-t_n)d(p_n, p) + t_n\, d(Gp_n,G p) \\ &\leq (1-t_n)d(p_n, p) + t_n\, d(p_n, p) \end{aligned}$$

$d(w_n, p) \leq d(p_n, p)$

Taking limsup as $n\to\infty$

$$\limsup_{n\to\infty} d(w_n, p) \leq \limsup_{n\to\infty} d(p_n, p) = \alpha \quad (3.5)$$

From (3.1) & (3.3)

$d(p_{n+1}, p) \leq d(q_n, p) \leq d(w_n, p)$

$d(p_{n+1}, p) \leq d(w_n, p)$

Taking liminf as $n\to\infty$

$$\alpha \leq \liminf_{n\to\infty} d(p_{n+1}, p) \leq \liminf_{n\to\infty} d(w_n, p) \quad (3.6)$$

From (3.5) & (3.6)

$\liminf_{n\to\infty} d(w_n, p) = \alpha$, we get that

$$\limsup_{n\to\infty} d(w_n, p) \leq \limsup_{n\to\infty} d(p_n, p) = \alpha \quad (3.7)$$

It follows from lemma 2.3, (3.6) & (3.7)

$\lim_{n\to\infty} d(Gp_n, p_n) = 0$

Conversely, assume that $\{p_n\}$ is bounded and $\lim_{n\to\infty} d(Gp_n, p_n) = 0$. Let $p \in AC(Y, \{p_n\})$;

Using lemma 2.1, we have

$$\begin{aligned} r(Gp, \{p_n\}) &= \limsup_{n\to\infty} d(Gp, p_n) \\ &\leq \limsup_{n\to\infty} d(p, p_n) + \frac{1+b+c}{1-b-c}\, d(Gp, p_n);\ \text{holds for all } u, v \in Y. \\ &= \limsup_{n\to\infty} d(p, p_n) \\ &= r(p,\{p_n\}) = r(Y, \{p_n\}). \end{aligned}$$

That is $Gp \in AC(Y,\{p_n\})$. Since X is uniformly convex, $AC(Y,\{p_n\})$ is singleton, implying that Gp = p.

Now we prove Δ-convergence theorem for generalized non-expansive mappings in Hyperbolic space.

Theorem 3.1 Let Y be a nonempty closed, convex subset of x and G: Y→Y be a generalized non-expansive mapping which satisfying condition (2.4) with $F(G) \neq \varphi$,let{pn} Δ-converges to a fixed points of G.

Proof:- It follows from lemma 2 that $\{p_n\}$ is a bonded sequence. Thus,$\{p_n\}$has a Δ-convergent subsequence. Now, we are going to show that every Δ-convergent subsequence of$\{p_n\}$ has a unique Δ-limit in F(G).

Let u and v be Δ-limits of the sequences $\{p_{nj}\}$and $\{p_{nk}\}$ of $\{p_n\}$respectively. From lemma 2.2, we have

$AC(Y,\{p_{nj}\}) = \{u\}$ & $AC(Y,\{p_{nk}\}) = \{v\}$

By lemma 2, we obtain that $\lim_{n\to\infty} d(p_{nj}, Gp_n) = 0$ & $\lim_{n\to\infty} d(p_{nk}, Gp_n) = 0$.

Next we prove that u & v are fixed points of G & u, v should be are unique, since G satisfies the condition (2.6)

$d(Gu,\{p_{nj}\}) \leq d(p_{nj}, u) + \frac{1+b+c}{1-b-c} d(Gu, u)$ (3.8)

Letting limsup $n \to \infty$ on both side of the above inequality, we get

$r(Gu,\{p_{nj}\}) = \limsup_{n \to \infty} d(p_{nj}, Gu)$

$\leq \limsup_{n \to \infty} d(p_{nj}, u) + \frac{1+b+c}{1-b-c} d(Gu, p_{nj})$

` $\leq \limsup_{n \to \infty} d(p_{nj}, u) = r(u,\{p_{nj}\})$

The uniqueness of the asymptotic centre implies Gu = u. Thus, u is a fixed point of G.

Similarly, we also have v as a fixed point of G

Finaly, we show that u =v. Suppose u v, and so by the uniqueness of an asymptotic centre, we have

$\limsup_{n \to \infty} d(p_n, u) = \limsup_{n \to \infty} d(p_{nj}, u)$

$< \limsup_{n \to \infty} d(p_{nj}, v)$

$= \limsup_{n \to \infty} d(p_n, u)$

$= \limsup_{n \to \infty} d(p_{nk}, v)$

$< \limsup_{n \to \infty} d(p_{nk}, u)$

$= \limsup_{n \to \infty} d(p_n, u)$

This is a contradiction. Thus u = v. Then $\{p_n\}$ Δ-converges ta a fixed point of G.

Next, we prove some strong convergence theorems-

Theorem 3.2 Let Y is a nonempty closed & convex subset of a uniformly convex hyperbolic space X & G : Y → Y be a self-mapping satisfying (2.4) with $F(G) \neq \varphi$. Then the sequence $\{p_n\}$ generated by iterative scheme (2.2) converge to the a point of F(G) if and only if $\liminf_{n\to\infty} d(p_n,F(G))=0$ where $d(p_n, F(G)) = \inf\{d(p_n, p); p \in F(G)\}$.

Proof:- Assume that $\{p_n\}$ converges to $p \in F(G)$ so, $\lim_{n \to \infty} d(p_n, p) = 0$, becaues

$0 \leq d(p_n, F(G) \leq d(p_n, p)$ for all $p \in F(G)$

Therefore $\liminf_{n \to \infty} d(p_n, F(G)) = 0$

Conversely, assume that $\liminf_{n \to \infty} d(p_n, F(G)) = 0$ & $p \in F(G)$, from lemma1 $\lim_{n \to \infty} d(p_n, p)$ exists for all $p \in F(G)$, therefore $\lim_{n \to \infty} d(p_n, F(G)) = 0$ by the assumption.

Now it is enough to show that $\{p_n\}$ is Cauchy sequence in Y

Therefore $\lim_{n \to \infty} d(p_n, F(G)) = 0$, for a given $\varepsilon > 0$ there exists $m_0 \in N$ such that for all $n \geq m_0$

$d(p_n, F(G)) < \varepsilon/2$

$\inf \{d(p_n, p; p \in F(G)\} < \varepsilon/2$

In particular, $\inf \{d(p_{m0}, p; p \in F(G)\} < \varepsilon/2$, therefore there exists $p \in F(G)$ such that

$d(p_{m0}, p) < \varepsilon/2$

Now for $m, n \geq m_0$

$d(p_{m+n}, p) \leq d(p_{m+n}, p) + d(p_n, p)$

$\leq d(p_{m0}, p) + d(p_{m0}, p)$

$= 2\, d(p_{m0}, p)$

$d(p_{m+n}, p) < \varepsilon$

Thus $\{p_n\}$is a Cauchy sequence in Y, since Y is closed there is a point $q \in Y$ such that $\lim_{n \to \infty} p_n = q$. Now $\lim_{n \to \infty} d(p_n, F(G)) = 0$. gives that $d(q, F(G))$, that is $q \in F(G)$.

Theorem 3.3 Let Y is a nonempty closed & convex subset of a uniformly convex hyperbolic space X & G:Y →Y be a self-mapping satisfying (2.4) with $F(G) \neq \varphi$ .Then the sequence$\{p_n\}$ generated by iterative scheme (2.2) converges strongly to a fixed point of G.

Proof:- From the lemma 2.3, G has a fixed point. Now from lemma 2 we have

$\liminf_{n \to \infty} d(p_n, Gp_n) = 0$, since Y is compact there is a sub sequence $\{p_{nj}\}$of $\{p_n\}$ such that $p_{nj} \to p_n$ strongly for some $p \in Y$. by lemma 2.1, we have

$d(Gp, p_{nj}) \leq d(p, p_{nj}) + \frac{1+b+c}{1-b-c} d(p_{nj}, Gp_{nj})$; $\forall j \geq 1$

letting $j \to \infty$, we get $p_{nj} \to Gp$. Thus Gp = p, i.e. $p \in F(G)$. Also $\lim_{n \to \infty} d(p, p_n)$ exists by lemma 1. Hence p is the strong limit of $\{p_n\}$. Condition(I) was introduced by Senter & Dotson [29] as a requirement for mapping which is defined as follow

A mapping G: Y→Y is said to satisfy condition (I). If there exists a non-decreasing function $g: R_+ \to R_+$ with $g(0) = 0$ & $g(t) > 0$, for all $t > 0$ such that $d(u, Gu) \geq g(d(u, F(G))$, for all $u \in Y$. Here R_+ denotes the set of all non-negative real numbers.

Now we prove a strong convergence result using condition(I)

Theorem 3.4 Let Y be a nonempty closed, convex subset of X and G: Y→Y be a generalized non-expansive mapping which satisfying condition (2.4) & condition (I). Then the sequence$\{p_n\}$ generated by (2.2) converges strongly to a fixed points of G

Proof:- we proved the following in lemma 2

$\lim_{n \to \infty} d(Gp_n, p_n) = 0$ (3.9)

Using condition (I) & (3.9), we get

$0 \leq \lim_{n \to \infty} g(d(p_n, F(G)) \leq \lim_{n \to \infty} d(Gp_n, p_n) = 0$ implies $\lim_{n \to \infty} g(d(p_n, F(G)) = 0$.From $g: R_+ \to R_+$ with $g(0) = 0$ & $g(t) > 0$, for all $t > 0$ we have

$\lim_{n \to \infty} d(p_n, F(G) = 0$

By applying Theorem 3.2, we obtain the desired result; therefore, the sequence $\{p_n\}$ converges strongly to a fixed point of G.

4. NUMERICAL EXAMPLE

Example 4.1 Let X = R with metric d(u, v) = |u-v| and Y=[0,1] be a non-empty compact convex subset of X. Define uniformly hyperbolic space with monotone modulus of uniform convexity. Let a mapping G: [0,1] → [0,1] defined by $G(u) = \frac{u+7}{8}$ for all $u \in [0,1]$. Need to establish that G generalized non- expansive mapping due to hardy and rogers.

Verification: if $u = \frac{7}{23}$,$v = \frac{1}{8}$ and $a= \frac{1}{2}$, $b = \frac{2}{5}$ and $c = 0$, we see that

||Gu – Gv|| ≤ a ||u-v|| + b(||u-Gu|| + ||v-Gv||) +c(||u-Gv|| + ||v-Gu||)

$\| \frac{178}{200} - \frac{57}{64} \| \leq \frac{1}{2} \| \frac{7}{23} - \frac{1}{8} \| + \frac{2}{5} [\| \frac{7}{23} - \frac{178}{200} \| + \| \frac{1}{8} - \frac{57}{64} \|]$

$0.022418 \leq 0.4769014$

Hence, for a = $\frac{1}{2}$, b = $\frac{2}{5}$ and c = 0 (a +2b+ 2c = $\frac{9}{10}$ < 1) G is a generalized non-expansive mapping. With the help of manual computation, we compute that the sequence $\{p_n\}$ generated by JF iteration scheme converges to a fixed point 0.99999 of G, where an initial point $p_0 = u_0 = 0.9$ and for all n ≥ 0, we choose real sequence in [0,1] as $t_n = \frac{1}{10n+2}$,r_n =1 and s_n = 1 which is shown by the Table 1 and G has a unique fixed point 0.999999.Which is shown by the Table 1.

Table 1: Sequence generated by generalized JF- iteration scheme

Iterate	Generalized JF- iteration scheme
p_0	0.9
p_1	0.999121
p_2	0.999997
p_3	0.999999
p_4	0.999999
p_5	0.999999
p_6	0.999999

5. CONCLUSION

Our results extend the corresponding results of Ali [3] & P. Chuadchawna[5] in two ways; first, from M-iterative process to JF-iterative process, Second, from Banach spaces to the general setting of hyperbolic spaces.

REFERENCES

1. Agrawal R P, O'Regan D, Sahu D R(2007) Iterative construction of fixed points of nearly asymptotically non-expansive mappings. J Nonlinear Convex Anal 8(1):61–79.
2. Ali J, Ali F, Kumar P(2019) Approximation of fixed points for Suzuki's generalized non-expansive mappings. Mathematics 7(6): 522.
3. Ali F, Ali J, Nieto J(2020) Some observation on generalized non-expansive mappings with an application. Comput Appl Math39:74.
4. Bogin J(1976) A generalization of a fixed points theorem of Goebel,Kirk and Shimi.Can Math Bull 19:7-12.
5. Chuadchawna P, Ali F, Kaewcharoen A(2020) fixed point approximate of generalized non-expansive mapping via generalized-iterative process in hyperbolic spaces. I J Math Sci 6435043: p6s.
6. Dhomphongsa S, Panyanak B(2008) Δ-Convergence theorem in CAT(0) spaces. Comput Math Appl 56(10) :2572–2579.
7. Dhomphongsa S, Inthakon W, Kaewkhao A(2009) Edelstein's method and fixed point theorems for some generalized non-expansive mappings. J Math Anal Appl 350(1):12–17.
8. Fukhar-ud-din H, Saleh K(2018) One-step iterations for a finite family of generalized non-expansive mappings in CAT(0) spaces. Bull Malays Math Sci Soc 41(2):597–608.
9. Fuster E L, Gàlvez E M(2011) The fixed point theory for some generalized non-expansive mappings, Abst Appl Anal 2011:p15s.
10. Goebel K, Kirk W A, Shimi T N(1973) A fixed point theorem in uniformly convex spaces. Boll Un Mat Ital.7:67–75.
11. Gromov M(2001) Mesoscopic curvature and Hyperbolicity. Cont math 288:58-69
12. Gursoy F, Karakaya V(2014) A Picard-S hybrid type iteration method for solving a differential equation with retarded argument. arXiv:1403.2546v2.
13. Gürsoy F(2016) A Picard-S iterative method for approximating fixed point of weak-contraction mappings. Filomat 30(10):2829–2845.
14. Gürsoy F, Khan A R, Ertürk M, Karakaya V(2018) Convergence and data dependency of normal-S iterative method for discontinuous operators on Banach space. Numer Funct Anal Optim 39(3):322–345.
15. Gürsoy F, Eksteen J A, Khan A R, Karakaya V(2019) An iterative method and its application to stable inversion. Soft Comput 23(16):7393–7406.
16. Hardy G F,Rogers T D(1973) A generalization of a fixed-point theorem of Reich.Can Math Bull16:201-206.
17. Ishikawa S(1974) Fixed points by a new iteration method. Proc Am Math Soc 44:147–150.
18. Khan R A, Fukhar-ud-din H(2012) An implicit algorithm for two finite families of non-expansive maps in hyperbolic spaces. Fixed point theory A54.
19. Kohlenbach U(2005), Some logical meta theorems with applications in functional analysis. Trans Ameri Math Soci 357(1):89–129.
20. Leustean L(2007) A quadratic rate of asymptotic regularity for CAT(0) spaces. J Math Anal Appl 235:386–399.
21. Leustean L(2010) Non-expansive iteration in uniformly convex -hyperbolic spaces, in Nonlinear Anal Opt I. Nonlinear Anal. Conte Math, Leizarowitz A, Mordukhovich B S, Shafrir I, Zaslavski A, Eds., Ramat Gan Am Math Soci.
22. Lim T C(1976) Remark on some fixed point theorems. Proc Amer Math Soc 60:179–182.
23. Mann W R(1953) Mean value methods in iteration. Proc Amer Math Soc 4:506–510.

24. Markin J(1973), Continuous depence of fixed point sets. Proc Am Math Soci 38:545–547.
25. Noor M A(2000) New approximation schemes for general variational inequalities.J Math Appl 251(1):217-229.
26. Picard E(1890) Memoire sur la theorie des equations aux derivees partielles et la methode desapproximations successives. J Math Pures Appl 6:145–21.
27. Reich S, Shafrir I(1990) Non-expansive iterations in hyperbolic spaces.Nonlinear Anal Theo,Meth Appl 15(6):537–558.
28. Senter H F, Dotson W G(1974) Approximating fixed points of non-expansive mappings. Proc Am Math Soc 44(2):375–380.
29. Suzuki T(2008) Fixed point theorems and convergence theorems for some generalized non-expansive mappings. J Math Anal Appl 340(2):1088–1095.
30. Thakur B S, Thakur D, Postolache M(2016) A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized non-expansive mappings. Appl Math Comput 275:147–155.
31. Uddin I, Imdad M(2015) Some convergence theorems for a hybrid pair of generalized non-expansive mappings in CAT(0) spaces. J Nonlinear Convex Anal 16(3):447–457.
32. Uddin I, Imdad M(2015) On certain convergence of S-iteration scheme in CAT(0) spaces. Kuwait J Sci 42(2):93–106.
33. Uddin I, Imdad M(2018) Convergence of SP-iteration for generalized non-expansive mapping in Hadamard spaces. Hacet J Math Stat 47(6):1595–1604.
34. Wong C S(1974) Generalized contractions and fixed point theorems. Proc Am Math Soc 42:409–41.