



Extended Modified Sub-Equation Method for Solving Nonlinear Travelling Wave Equation

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ARTICLE INFO	ABSTRACT
<p>Published Online: 20 January 2025</p> <p>Corresponding Author: Prof. Dr. Abdel Radi Abdel Rahman Abdel Gadir Abdel Rahman</p>	<p>We considered the third order nonlinear Schrodinger equation (TONSE) that models the wave pulse transmission in a time period less than one-trillionth of a second. We extended modified sub-equation method to obtain numerous exact travelling wave solutions containing sets of generalized hyperbolic. Trigonometric and rational solutions that are more general than classical ones. We followed analytical mathematical method by constructed the transformation groups which help us using vector fields and Lie symmetry groups. We discussed the dynamic behavior and structure of the exact solutions for distinct solution of arbitrary constants. We obtained the symmetry reductions and exact of the equation. In addition to group-invariant solutions which are Jacobi elliptic function and exponential type.</p>
<p>KEYWORDS: Exact Travelling Wave Equation, Schrodinger Equation, Extended Modified Method, Lie Symmetry Groups, Invariant Solutions.</p>	

1. INTRODUCTION

It is observed that most of the physical phenomena occurring in nature are mathematically modeled by the evolution equations. However, we know from the empirical result that many important physical processes are the type of nonlinear evolution equations

$$F(x, t, u_t, u_x, u_{xx}, u_{xxx}, \dots) = 0 \tag{1}$$

[1]. Well-known Korteweg- de Vries equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{2}$$

Represents shallow water waves solution which is special solitary travelling wave-unchanged wave velocity and shape after interaction is a generalized wave packet. For these reasons, a travelling solution is used not only in water wave theory, but also in optical communication. Solutions are derived from the sensitive between nonlinear and dispersive terms. It is desirable that travelling wave solution transmission in communication systems should be high speed [7,8]. For example, the (1+1)-dimensional nonlinear Schrodinger equation

$$iq_t + 2|q|^2q + q_{xx} = 0 \tag{3}$$

We considered the third order equations

$$iq_x + \alpha_2(q_{tt} + 2q|q|^2) - i\alpha_3(q_{ttt} + 6q_t|q|^2) = 0 \tag{4}$$

From the hierarchy of the higher order given in [7]. Our main goal is to obtain exact analytical solutions of this equation. There are many methods in the literature to obtain the solutions of nonlinear Schrodinger equation. Recently, the sub-equation extended method are introduced in [8]. What makes this method interesting is that, unlike other methods that its solutions include the generalized type of hyperbolic and trigonometric functions. This method has a finite series expansion form balancing principle. In order to overcome this deficiency, this method which is very effective and practical for solving nonlinear differential equations in mathematical physics was used to obtain the solution under consideration. Another approach discussed in the study is Lie technique. In this algorithmic method based on the finding of transformation groups that leave the equation invariant, reduced equations and group invariant solutions can be obtained. The wave and group invariant solutions of equation (4) will be investigated with the help of these two methods. The reduction of equation (4) is given in (4), Lie groups method is employed to study equation (4).

2. MATHEMATICAL MODEL

In spite of the fact that equation (3) is successful in the describing a great number of nonlinear effects, it may be necessary to modify the experimental conditions. Therefore higher-order effects should be considered for the transmission of pulses to sub-picoseconds and femtoseconds which has better performance on the transmitting information. The higher-order integrable hierarchy can be presented as

$$iq_x + \alpha_2(q_{tt} + 2q|q|^2) - i\alpha_3(q_{ttt} + 6q_t|q|^2) + \alpha_4(q_{tttt} + 6q_t^* + 4q|q_t|^2 + 8|q|^2q_{tt} + 6|q|^4q) - i\alpha_5(q_{ttttt} + 10|q|^2q_{ttt} + 30|q|^4q_t + 10qq_tq_{tt}^* + 10qq_{tt}^* + 10qq_t^*q_{tt} + 10q_t^2q_t^* + 20q^*q_tq_{tt}) + \dots = 0, \tag{5}$$

where $q(x, t)$ represents the normalized complex amplitude of the optical pulse envelope, asterisk represents the conjugation, $\alpha_I (I = 2,3,4, \dots)$ are real constant parameters, x denotes the propagation variable and t denotes the transverse variable (time in a moving frame) [11]. We investigated the equation (4) which we have obtained by taking $\alpha_m = 0, m = 4,5, \dots$. we aimed to simplify the equation (4). Thus we are seeking solutions of (4) with the following structure

$$q(x, t) = p(\zeta)e^{i\varphi(x,t)}, \varphi(x, t) = -kx + \omega t + \theta, \tag{6}$$

where $\zeta = x - vt$ is the wave variable and $p(\zeta)$ is an amplitude component of the soliton solution. Here v and k are the velocity and frequency of the soliton, respectively. ω is the soliton wave number and θ is phase constant. If we use the transformation in the equation (4) and separate the real and imaginary parts, a pair of relations emerges. The real part equation gives

$$(2\alpha_2 + 6\alpha_3\omega)p^3 + (k - \alpha_3\omega^3 - \alpha_2\omega^2)p + (\alpha_2 + 3\alpha_3\omega)v^2p'' = 0, \tag{7}$$

And imaginary part equation reads

$$\alpha_3p'''v^3 + p' + (6\alpha_3p^2p' - 2\alpha_2\omega p' - 3\alpha_3\omega^2p')v = 0. \tag{8}$$

Integrating equation (8) once and setting the integration constant to zero we obtain

$$2v\alpha_3p^3 + (1 + v(-2\alpha_2\omega - 3\alpha_3\omega^2))p + \alpha_3v^3p'' = 0. \tag{9}$$

Equations (7) and (9) will be equivalent provided that

$$\frac{2v\alpha_3}{(2\alpha_2 + 6\alpha_3\omega)} = \frac{1 + v(-2\alpha_2\omega - 3\alpha_3\omega^2)}{k - \alpha_3\omega^3 - \alpha_2\omega^2} = \frac{\alpha_3v^3}{(\alpha_2 + 3\alpha_3\omega)v^2}.$$

Hence, one can find the following parametric constraints,

$$\alpha_3 = \frac{\alpha_2}{-3\omega + v'}k = -\frac{-4v\alpha_2\omega^2 + 2v^2\alpha_2\omega^3 + 3\omega - v}{-3\omega + v}. \tag{10}$$

Eventually, equations (7) and (9) can be rearranged to be in the form

$$p'' + \frac{2}{v^2}p^3 + \frac{(1 + v(-2\alpha_2\omega - 3\alpha_3\omega^2))}{\alpha_3v^3}p = 0. \tag{11}$$

The solutions of the equation (11) will be examined using the extended modified sub-equation method.

3. BASIC IDEAS OF THE EXTENDED MODIFIED SUB-EQUATION METHOD

We presented briefly the main steps of the extended modified sub-equation method for finding travelling wave equations [7]. Firstly, we considered the general of the type

$$P(u, u_t, u_x, u_{tt}, u_{xx}, \dots) = 0. \tag{12}$$

Using the wave transformation

$$u(x, t) = U(\zeta), \quad \zeta = x - tv,$$

we can rewrite equation (12) as the following nonlinear ordinary differential equation

$$Q(U, U', U'', U''', \dots) = 0. \tag{13}$$

Let us assume that the solution of ordinary differential equation (13) can be written as a polynomial of $R(\zeta)$ as follows:

$$U(\zeta) = \sum_{j=-n}^n b_j R^j(\zeta), \quad b_j \neq 0, \tag{14}$$

where $b_j (-n \leq j \leq n)$ are constants which we will be determined later. $R(\zeta)$ in (14) satisfies the nonlinear ordinary differential equation in the form

$$R'(\zeta) = \ln(A)(S_0 + S_1R(\zeta) + S_2R^2(\zeta)), \quad A \neq 0,1. \tag{15}$$

The coefficient classifications and corresponding solution forms of (15) are as follows:

Case1: If $\Delta = S_1^2 - 4S_0S_2 < 0, S_2 \neq 0$ then $R_1(\zeta) = \frac{S_1}{2S_2} + \frac{\sqrt{-\Delta} \tan_A\left(\frac{\sqrt{-\Delta}\zeta}{2}\right)}{2S_2}$

$$R_2(\zeta) = -\frac{S_1}{2S_2} - \frac{\sqrt{-\Delta} \cot_A\left(\frac{\sqrt{-\Delta}\zeta}{2}\right)}{2S_2}, \quad R_3(\zeta) = \frac{S_1}{2S_2} + \frac{\sqrt{-\Delta} \left(\tan_A(\sqrt{-\Delta}\zeta) \pm \sqrt{r} \operatorname{psec}_A(\sqrt{-\Delta}\zeta) \right)}{2S_2},$$

$$R_4(\zeta) = -\frac{S_1}{2S_2} - \frac{\sqrt{-\Delta} \left(\cot_A(\sqrt{-\Delta}\zeta) \pm \sqrt{r} \operatorname{P}_A(\sqrt{-\Delta}\zeta) \right)}{2S_2},$$

$$R_5(\zeta) = -\frac{S_1}{2S_2} + \frac{-\sqrt{-\Delta} \left(\tan_A \left(\frac{\sqrt{-\Delta}}{4} \zeta \right) - \cot_A \left(\frac{\sqrt{-\Delta}}{4} \zeta \right) \right)}{4S_2}.$$

Case2: If $\Delta = S_1^2 - 4S_0S_2 > 0, S_2 \neq 0$, then

$$R_6(\zeta) = -\frac{S_1}{2S_2} - \frac{\sqrt{\Delta} \tanh_A \left(\frac{\sqrt{\Delta}}{2} \zeta \right)}{2S_2}, R_7(\zeta) = -\frac{S_1}{2S_2} - \frac{\sqrt{\Delta} \coth_A \left(\frac{\sqrt{\Delta}}{2} \zeta \right)}{2S_2},$$

$$R_8(\zeta) = -\frac{S_1}{2S_2} - \frac{\sqrt{\Delta} \left(\coth_A(\sqrt{\Delta}\zeta) \pm \sqrt{r} \operatorname{pcsch}_A(\sqrt{\Delta}\zeta) \right)}{2S_2},$$

$$R_9(\zeta) = -\frac{S_1}{2S_2} - \frac{\sqrt{\Delta} \left(\tanh_A(\sqrt{\Delta}\zeta) \pm \sqrt{r} \operatorname{pcsch}_A(\sqrt{\Delta}\zeta) \right)}{2S_2},$$

$$R_{10}(\zeta) = -\frac{S_1}{2S_2} - \frac{\sqrt{\Delta} \left(\tanh_A \left(\frac{\sqrt{\Delta}}{4} \zeta \right) + \coth_A \left(\frac{\sqrt{\Delta}}{4} \zeta \right) \right)}{4S_2}.$$

Case 3: If $S_0 = S_2, S_1 = 0$. Then

$$R_{11}(\zeta) = \tan_A(S_0\zeta), R_{12}(\zeta) = -\cot_A(S_0\zeta), R_{13}(\zeta) = \tan_A(2S_0\zeta) \pm \sqrt{r} \operatorname{psec}_A(2S_0\zeta)$$

$$R_{14}(\zeta) = -\cot_A(2S_0\zeta) \pm \sqrt{r} \operatorname{p}_A(2S_0\zeta),$$

$$R_{15}(\zeta) = \frac{1}{2} \tan_A \left(\frac{S_0}{2} \zeta \right) - \frac{1}{2} \coth_A \left(\frac{S_0}{2} \zeta \right).$$

Case 4: If $S_0 = -S_2, S_1 = 0$, then

$$R_{16}(\zeta) = -\tanh_4(S_0\zeta), R_{17}(\zeta) = -\coth_4(S_0\zeta), R_{18}(\zeta) = \tanh_4(2S_0\zeta) \pm \sqrt{r} \operatorname{psech}_4(2S_0\zeta),$$

$$R_{19}(\zeta) = -\coth_4(2S_0\zeta) \pm \sqrt{r} \operatorname{pcsch}_4(2S_0\zeta),$$

$$R_{20}(\zeta) = -\frac{1}{2} \tanh_4 \left(\frac{S_0}{2} \zeta \right) - \frac{1}{2} \coth_4 \left(\frac{S_0}{2} \zeta \right).$$

Case 5: If $S_1^2 - 4S_0S_2 = 0$ then

$$R_{21}(\zeta) = -2 \frac{S_0(S_1\zeta \ln(A) + 2)}{S_1^2\zeta \ln(A)}.$$

Case 6: If $S_1 = \lambda, S_0 = m\lambda, m \neq 0$ and $S_2 = 0$ then

$$R_{22}(\zeta) = A^{\lambda\zeta} - m.$$

Case 7: If $S_1 = 0, S_2 = 0$ then

$$R_{23}(\zeta) = S_0\zeta \ln(A).$$

Case 8: If $S_0 = 0, S_1 = 0$ then

$$R_{24}(\zeta) = -\frac{1}{S_2 \ln(A)\zeta}.$$

Case 9: If $S_0 = 0, S_1 \neq 0$ then

$$S_{25}(\zeta) = -\frac{rS_1}{S_2(\cosh_A(S_1\zeta) - \sinh_A(S_1\zeta) + r)},$$

$$R_{26}(\zeta) = -\frac{(\cosh_A(S_1\zeta) + \sinh_A(S_1\zeta))S_1}{S_2(\cosh_A(S_1\zeta) + \sinh_A(S_1\zeta) + p)}.$$

Case 10: If $S_1 = \lambda, S_2 = m\lambda, m \neq 0$ and $S_0 = 0$ then

$$R_{27}(\zeta) = \frac{rA^{\lambda\zeta}}{p - mrA^{\lambda\zeta}}.$$

The generalized trigonometric and hyperbolic functions used in the families given above are defined as follows:

$$\tan_A(\xi) = \frac{-i(rA^{i\xi} - pA^{-i\xi})}{rA^{i\xi} + pA^{-i\xi}}, \tanh_A(\xi) = \frac{i(rA^\xi - pA^{-\xi})}{rA^\xi + pA^{-\xi}},$$

$$\cot_A(\xi) = \frac{i(rA^{i\xi} + pA^{-i\xi})}{rA^{i\xi} - pA^{-i\xi}}, \coth_A(\xi) = \frac{rA^\xi + pA^{-\xi}}{rA^\xi - pA^{-\xi}},$$

$$\cos_A(\xi) = \frac{rA^{i\xi} + pA^{-i\xi}}{2}, \cosh_A(\xi) = \frac{rA^\xi + pA^{-\xi}}{2},$$

$$\sin_A(\xi) = \frac{-i(rA^{i\xi} - pA^{-i\xi})}{2}, \sinh_A(\xi) = \frac{rA^\xi - pA^{-\xi}}{2},$$

$$\begin{aligned} \operatorname{csc}_A(\xi) &= \frac{2i}{rA^{i\xi} - pA^{-\xi}}, & \operatorname{csch}_A(\xi) &= \frac{2}{rA^\xi - pA^{-\xi}}, \\ \operatorname{sec}_A(\xi) &= \frac{2}{rA^{i\xi} + pA^{-i\xi}}, & \operatorname{sech}_A(\xi) &= \frac{2}{rA^\xi + pA^{-\xi}}. \end{aligned} \quad (16)$$

In equation (16), ξ is an independent variable $r, p > 0$ constants are deformation parameters n in (14) is a positive integer that can be determined by the balancing procedure constructed taking into account the highest order nonlinear terms and the highest order linear terms in the resulting equation. By using equation (14) and equation (15) into equation (13), an equation consisting of the power of $R(\xi)$ has to be equal to zero. Hence we obtained an algebraic system of equations in terms of $b_{-n}, \dots, b_{-1}, b_0, b_1, \dots, b_n$. By determining these parameters and rewriting the equation (14) using determined parameters an analytic solution $u(x, t)$ is obtained in closed form.

4. EXACT MODIFIED SOLUTIONS OF TRAVELLING WAVE EQUATION

We obtained the analytical solutions for the amplitude of the travelling wave equations by using the extended modified sub-equation method. Substituting $p(\zeta) = \sum_{i=-n}^n b_j R^i(\zeta)$ into equation (11) and balancing p'' with p^3 yields $n = 1$. Therefore equation (11) admits the use of

$$p(\zeta) = b_{-1}R(\zeta)^{-1} + b_0 + b_1R(\zeta). \quad (17)$$

Substituting equation (17) into equation (11) through equation (15) and collecting the coefficients of different powers of $R(\zeta)$ setting each coefficient to zero we get the system of algebraic equations. By solving the resulting system with the help of maple the following results are achieved.

Set 1

After the hung calculations we deduce the following relations between parameters appearing algebraic

$$\begin{aligned} \omega &= \frac{2v^2\alpha_2 + 3 + \sqrt{4v^4\alpha_2^2 + 9 - 24v^4\alpha_2^2(\ln(A))^2}}{6v\alpha_2}, \\ b_{-1} &= iS_0 \ln(A)v, b_0 = \frac{iv \ln(A)S_i}{2}, b_1 = 0, \end{aligned}$$

Where α_2, α_3, v, k are arbitrary constants we now can construct the exact solutions of equation (4) easily for these parameters set through the classification cases which is given in (3).

Case1: If $\Delta = S_1^2 - 4S_0S_2 < 0, S_2 \neq 0$, then we have

$$\begin{aligned} q_1(x, t) &= \left(\frac{iS_1 \ln(A)v}{2} + iv \ln(A)S_0 \times \left(\frac{-S_1}{2S_2} - \frac{\sqrt{-\Delta} \cot_A \left(\frac{\sqrt{-\Delta}(x-vt)}{2} \right)}{2S_2} \right)^{-1} \right) \times e^{i(-kx+\omega t+\theta)}, \\ q_2(x, t) &= \left(\frac{iS_1 \ln(A)v}{2} + iv \ln(A)S_0 \times \left(\frac{-S_1}{2S_2} - \frac{\sqrt{-\Delta} \cot_A(x-vt)}{2S_2} \right)^{-1} \right) \times e^{i(-kx+\omega t+\theta)} \\ q_3(x, t) &= \left(\frac{iS_1 \ln(A)v}{2} + iv \ln(A)S_0 \times \left(-\frac{S_1}{2S_2} + \frac{\sqrt{-\Delta}(\tan_A(\sqrt{-\Delta}(x-vt))) \pm \sqrt{r} p \operatorname{sec}_A(\sqrt{-\Delta}(x-vt))}{2S_2} \right)^{-1} \right) \times e^{i(-kx+\omega t+\theta)} \\ q_4(x, t) &= \left(\frac{iS_1 \ln(A)v}{2} + iv \ln(A)S_0 \times \left(\frac{-S_1}{2S_2} - \frac{\sqrt{-\Delta}(\cot_A(\sqrt{-\Delta}(x-vt))) \pm \sqrt{r} P_A(\sqrt{-\Delta}(x-vt))}{2S_2} \right)^{-1} \right) \times e^{i(-kx+\omega t+\theta)} \\ q_5(x, t) &= \left(\frac{iS_1 \ln(A)v}{2} + iv \ln(A)S_0 \times \left(-\frac{S_1}{2S_2} + \frac{\sqrt{-\Delta}(\tan_A(\frac{\sqrt{-\Delta}(x-vt)}{4}))}{4S_2} - \frac{\cot_A(\sqrt{-\Delta}(x-vt))}{4S_2} \right)^{-1} \right) \times e^{i(-kx+\omega t+\theta)}. \end{aligned}$$

Case 2: If $\Delta = S_1^2 - 4S_0S_2 > 0, S_2 \neq 0$, then we obtain

$$q_6(x, t) = \left(\frac{iS_1 \ln(A)v}{2} + iv \ln(A)S_0 \times \left(-\frac{S_1}{2S_2} - \frac{\sqrt{\Delta} \tanh_A \left(\frac{\sqrt{\Delta}(x-vt)}{2} \right)}{2S_2} \right)^{-1} \right) \times e^{i(-kx+\omega t+\theta)},$$

$$q_7(x, t) = \left(\frac{iS_1 \ln(A)v}{2} + iv \ln(A)S_0 \times \left(-\frac{S_1}{2S_2} - \frac{\sqrt{\Delta} \coth_A \left(\frac{\sqrt{\Delta}(x-vt)}{2} \right)}{2S_2} \right)^{-1} \right) \times e^{i(-kx+\omega t+\theta)},$$

$$q_8(x, t) = \left(\frac{iS_1 \ln(A)v}{2} + iv \ln(A)S_0 \times \left(-\frac{S_1}{2S_2} - \frac{\sqrt{\Delta} \left(\coth_A(\sqrt{\Delta}(x-vt)) \right)_{\pm}}{\sqrt{r} \operatorname{pcsch}_A(\sqrt{\Delta}(x-vt))} \right)^{-1} \right) \times e^{i(-kx+\omega t+\theta)},$$

$$q_9(x, t) = \left(\frac{iS_1 \ln(A)v}{2} + iv \ln(A)S_0 \times \left(-\frac{S_1}{2S_2} - \frac{\sqrt{\Delta} \left(\tanh_A \left(\frac{\sqrt{\Delta}(x-vt)}{4} \right) \right) + \coth_A \left(\frac{\sqrt{\Delta}(x-tv)}{4} \right)}{4S_2} \right)^{-1} \right) \times e^{i(-kx+\omega t+\theta)}.$$

Case 3: If $S_0 = S_2, S_1 = 0$, then we yield

$$q_{10}(x, t) = \frac{iv \ln S_0 e^{i(-kx+\omega t+\theta)}}{\tan_A(S_0(x-vt))},$$

$$q_{11}(x, t) = -\frac{iv \ln(A)S_0 e^{i(-kx+\omega t+\theta)}}{\cot_A(S_0(x-vt))},$$

$$q_{12}(x, t) = \frac{iv \ln(A)S_0 e^{i(-kx+\omega t+\theta)}}{\tan_A(2S_0(x-vt)) \pm \sqrt{r} \operatorname{tsec}_A(2S_0(x-vt))},$$

$$q_{13}(x, t) = \frac{iv \ln(A)S_0 e^{i(-kx+\omega t+\theta)}}{-\cot_A(2S_0(x-vt)) \pm \sqrt{r} \operatorname{pcsch}_A(2S_0(x-vt))},$$

$$q_{14}(x, t) = \frac{2iv \ln(A)S_0 e^{i(-kx+\omega t+\theta)}}{\tan_A \left(\frac{S_0(x-vt)}{2} \right) - \cot_A \left(\frac{S_0(x-vt)}{2} \right)}.$$

Case 4:P If $S_0 = -S_2, S_1 = 0$, then one obtains

$$q_{15}(x, t) = \frac{-iv \ln(A)S_0 e^{i(-kx+\omega t+\theta)}}{\tan_A(S_0(x-vt))},$$

$$q_{16}(x, t) = \frac{-iv \ln(A)S_0 e^{i(-kx+\omega t+\theta)}}{\coth_A(S_0(x-vt))},$$

$$q_{17}(x, t) = \frac{iv \ln(A)S_0 e^{i(-kx+\omega t+\theta)}}{-\coth_A(2S_0(x-vt)) \pm \sqrt{r} \operatorname{pcsch}_A(2S_0(x-vt))}.$$

Case 5: If $S_1^2 - 4S_0S_2 = 0$ then we attain

$$q_{18}(x, t) = \left(\frac{iS_1 \ln(A)v}{2} - \frac{iv(\ln(A))^2 S_1^2(x-vt)}{2(S_1(x-vt)\ln(A) + 2)} \right) \times e^{i(-kx+\omega t+\theta)}. \tag{19}$$

Case 6: If $S_1 = \lambda, S_0 = m\lambda, m \neq 0$ and $S_2 = 0$ then we derive

$$q_{19}(x, t) = \left(\frac{iS_1 \ln(A)v}{2} + \frac{iv \ln(A)S_0}{A^{\lambda(x-vt)} - m} \right) e^{i(-kx+\omega t+\theta)}.$$

Case 9: If $S_0 = 0, S_1 \neq 0$ then we construct

$$q_{20}(x, t) = (iS_1 \ln(A)v) 2^{-1} e^{i(-kx+\omega t+\theta)}.$$

Case 10: If $S_1 = \lambda, S_2 = m\lambda, m \neq 0$ and $S_0 = 0$ then we get

$$q_{21}(x, t) = (i\lambda \ln(A)v) 2^{-1} e^{i(-kx+\omega t+\theta)}.$$

Set 2

After some calculations, the following relations are obtained between the parameters in the system of algebraic equations:

$$\omega = \frac{2v^2 + 3 + \sqrt{4v^4\alpha_2^2 + 9 - 24v^4\alpha_2^2(\ln(A))^2 S_2 S_0 + 6v^4\alpha_2^2(\ln(A))^2 S_1^2}}{6v\alpha_2},$$

$$b_{-1} = 0, b_0 = \frac{iv \ln(A) S_1}{2}, b_1 = iS_2 \ln(A)v,$$

where α_2, α_3, v, k are arbitrary constants. According to classification cases for these parameters in (3) we can construct the exact solutions of equation(4) as follows:

case 1: If $\Delta = S_1^2 - 4S_0S_2 < 0, S_2 \neq 0$, then

$$q_{22}(x, t) = \left(\frac{1}{2} i \ln(A) v \sqrt{-\Delta} \tan_A \left(\frac{\sqrt{-\Delta}(x - vt)}{2} \right) \right) \times e^{i(-kx + \omega t + \theta)},$$

$$q_{23}(x, t) = \left(\frac{1}{2} i \ln(A) v \sqrt{-\Delta} \cot_A \left(\frac{\sqrt{-\Delta}(x - vt)}{2} \right) \right) \times e^{i(-kx + \omega t + \theta)},$$

$$q_{24}(x, t) = \left(\frac{i \ln(A) v \sqrt{-\Delta} \left(\tan_A \left(\frac{\sqrt{-\Delta}(x - vt)}{2} \right) \right)}{\pm \sqrt{-\Delta} \sec_A \left(\frac{\sqrt{-\Delta}(x - vt)}{2} \right)} \right) \times e^{i(-kx + \omega t + \theta)},$$

$$q_{25}(x, t) = \left(\frac{i \ln(A) v \sqrt{-\Delta} \left(\cot_A \left(\frac{\sqrt{-\Delta}(x - vt)}{2} \right) \right)}{\pm \sqrt{-\Delta} \csc_A \left(\frac{\sqrt{-\Delta}(x - vt)}{2} \right)} \right) \times e^{i(-kx + \omega t + \theta)},$$

$$q_{26}(x, t) = \left(\frac{i \ln(A) v \sqrt{-\Delta} \left(\tan_A \left(\frac{\sqrt{-\Delta}(x - vt)}{2} \right) \right)}{-\cot_A \left(\frac{\sqrt{-\Delta}(x - vt)}{2} \right)} \right) \times e^{i(-kx + \omega t + \theta)}.$$

Case 2: If $\Delta = S_1^2 - 4S_0S_2 > 0, S_2 \neq 0$ then

$$q_{27}(x, t) = \left(-i \ln(A) v \sqrt{\Delta} \tanh_A \left(\frac{\sqrt{\Delta}(x - vt)}{2} \right) \right) 2^{-1} \times e^{i(-kx + \omega t + \theta)},$$

$$q_{28}(x, t) = \left(-i \ln(A) v \sqrt{\Delta} \coth_A \left(\frac{\sqrt{\Delta}(x - vt)}{2} \right) \right) 2^{-1} \times e^{i(-kx + \omega t + \theta)},$$

$$q_{29}(x, t) = \left(\frac{i \ln(A) v \sqrt{\Delta} \left(\coth_A \left(\frac{\sqrt{\Delta}(x - vt)}{2} \right) \right)}{\pm \sqrt{\Delta} \operatorname{csch}_A \left(\frac{\sqrt{\Delta}(x - vt)}{2} \right)} \right) \times e^{i(-kx + \omega t + \theta)},$$

$$q_{30}(x, t) = \left(\frac{-i \ln(A) v \sqrt{\Delta} \left(\tanh_A \left(\frac{\sqrt{\Delta}(x - vt)}{4} \right) \right)}{+\coth_A \left(\frac{\sqrt{\Delta}(x - vt)}{4} \right)} \right) \times e^{i(-kx + \omega t + \theta)}. \quad (20)$$

Case 3: If $S_0 = S_2, S_1 = 0$, then

$$q_{31}(x, t) = iS_0 \ln(A) v \tan_A(S_0(x - vt)) \times e^{i(-kx + \omega t + \theta)},$$

$$q_{32}(x, t) = -iS_0 \ln(A) v \cot_A(S_0(x - vt)) \times e^{i(-kx + \omega t + \theta)},$$

$$q_{33}(x, t) = iS_0 \ln(A) v \left(\tan_A(2S_0(x - vt)) \right) \pm \sqrt{r p} \sec_A(2S_0(x - vt)) e^{i(-kx + \omega t + \theta)},$$

$$q_{34}(x, t) = iS_0 \ln(A) v \left(-\cot_A(2S_0(x - vt)) \right) \pm \sqrt{r p} \csc_A(2S_0(x - vt)) e^{i(-kx + \omega t + \theta)},$$

$$q_{35}(x, t) = \frac{iS_0 \ln(A)v}{2} \left(\tan_A \left(\frac{S_0(x - vt)}{2} \right) \right) - \cot_A \left(\frac{S_0(x - vt)}{2} \right) e^{i(-kx + \omega t + \theta)}.$$

Case 4: If $S_0 = -S_2, S_1 = 0$ then

$$q_{36}(x, t) = \left(-iS_2 \ln(A)v \tanh_A(S_0(x - vt)) \right) \times e^{i(-kx + \omega t + \theta)},$$

$$q_{37}(x, t) = \left(iS_2 \ln(A)v \left(-\coth_A(2S_0(x - vt)) \right) \right) \pm \sqrt{r} \operatorname{pcsch}_A(2S_0(x - vt)) e^{i(-kx + \omega t + \theta)}.$$

Case 5: If $S_1^2 - 4S_0S_2 = 0$ then

$$q_{38}(x, t) = \left(\frac{-2iS_2vS_0(S_1(x - vt)\ln(A) + 2)}{S_1^2(x - vt)} + \frac{iv \ln(A)S_1}{2} \right) e^{i(-kx + \omega t + \theta)}.$$

Case 6: If $S_1 = \lambda, S_0 = m\lambda, m \neq 0$ and $S_2 = 0$ then

$$q_{39}(x, t) = \frac{iv \ln(A)\lambda}{2} e^{i(-kx + \omega t + \theta)}.$$

Case 7: If $S_0 = 0, S_1 = 0$ then

$$q_{40}(x, t) = \frac{-iv}{(x - vt)} e^{i(-kx + \omega t + \theta)}.$$

Case 8: If $S_0 = 0, S_1 \neq 0$ then

$$q_{41}(x, t) = \left(\frac{-i \ln(A)vrS_1}{\cosh_A(S_1(x - vt)) - \sinh_A(S_1(x - vt) + r)} + \frac{iv \ln(A)S_1}{2} \right) e^{i(-kx + \omega t + \theta)} \quad (21)$$

5. DISCUSSION AND RESULTS :

We applied Lie symmetry analysis for equation (1)

$$iq_1 + \alpha_2(q_\alpha + 2q|q|^2) - i\alpha_3(q_{ttt} + 6q_t + |q|^2) = 0 \quad (1)$$

Firstly, we assumed that : $q(x, t) = u(x, t)e^{iv(x, t)}$,

(2) where u and v are real valued functions and we

substituted equation (2) into and split up real imaginary parts so we obtained

$$-v_x u + \alpha_2 u_{tt} - \alpha_2 u v_t^2 + 2\alpha_2 u^3 + 3\alpha_3 u_{tt} v_t + 3\alpha_3 u_t v_{tt} + \alpha_3 u v_{ttt} - \alpha_3 u v_t^3 + 6\alpha_3 u^3 v_t = 0,$$

$$2\alpha_2 u_t v_t + \alpha_2 u v_{tt} + 3\alpha_3 u_t v_t^2 - \alpha_3 u_{ttt} - 6\alpha_3 u^2 u_t + u_x + 3\alpha_3 u v_{tt} v_t = 0 \quad (3)$$

With a small parameter (3), the corresponding vector field for these transformations is

$$X = \zeta(x, t, u, v) \frac{\partial}{\partial x} + t(x, t, u, v) \frac{\partial}{\partial t} + \eta(x, t, u, v) \frac{\partial}{\partial u} + \phi(x, t, u, v) \frac{\partial}{\partial v} \quad (4)$$

When (4) is a vector field (or generator) the transformation group of the equation or system considered is

$$\frac{d\hat{x}}{d\epsilon} = \zeta(\hat{x}, \hat{t}, \hat{u}, \hat{v}), \quad \hat{x}|_\epsilon = 0 = x,$$

$$\frac{d\hat{t}}{d\epsilon} = \tau(\hat{x}, \hat{t}, \hat{u}, \hat{v}), \quad \hat{t}|_\epsilon = 0 = t,$$

$$\frac{d\hat{u}}{d\epsilon} = \eta(\hat{x}, \hat{t}, \hat{u}, \hat{v}), \quad \hat{u}|_\epsilon = 0 = u.$$

$$\frac{d\hat{v}}{d\epsilon} = \phi(\hat{x}, \hat{t}, \hat{u}, \hat{v}), \quad \hat{v}|_\epsilon = 0 = v,$$

The third prolongation formula $Pr^{(3)}X$ is

$$Pr^{(3)}X = x + \phi^x \frac{\partial}{\partial v_x} + \phi^t \frac{\partial}{\partial v_t} + \eta^t \frac{\partial}{\partial u_t} + \eta^{tt} \frac{\partial}{\partial u_{tt}} + \phi^{tt} \frac{\partial}{\partial v_{tt}} + \phi^{ttt} \frac{\partial}{\partial v_{ttt}} + \eta^x \frac{\partial}{\partial u_x} + \phi^t \frac{\partial}{\partial v_t} + \eta^t \frac{\partial}{\partial u_t} + \phi^{tt} \frac{\partial}{\partial v_{tt}} + \eta^{ttt} \frac{\partial}{\partial u_{ttt}} \quad (5)$$

Where $\eta^x, \phi^x, \phi^t, \eta^t, \phi^{tt}, \phi^{ttt}, \eta^{tt}, \phi^{ttt}$ are extended infinitesimal. Hence the system of (3) has the following invariant conditions

$$\eta(-v_x - \alpha_2 v_t^2 + 6\alpha_2 u^2 + \alpha_3 v_{ttt} - \alpha_3 v_t^3 + 18\alpha_3 v_t u^2) - \phi^x u + \eta^{tt}(\alpha_3 + 3\alpha_3 v_2) + \phi^t(-2\alpha_2 u v_t - 3\alpha_3 u_{tt} - 3\alpha_3 u v_t^2 + 6\alpha_3 u^3) + \alpha_3(3\eta^t v_{tt} + 3\phi^{tt} u_t + \phi^{ttt} u) = 0,$$

$$\eta(\alpha_2 v_{tt} - 12\alpha_3 u_t u + 3\alpha_3 v_t v_{tt}) + \eta^t(2\alpha_2 v_2 + 3\alpha_3 v_t^2 - 6\alpha_3 u^2) + \phi^t(2\alpha_2 v_t + 6\alpha_3 u_t v_t + 3\alpha_3 v_{tt}) + \phi^{tt}(\alpha_2 u + 3\alpha_3 u v_t) - \eta^{ttt} \alpha_3 + \eta^x = 0.$$

With the help of the obtained equation pair and the values of extended infinitesimals we got an over determined system of partial differential equation. Solving overdetermined system of PDEs we can be obtained the following :

$$\zeta = -3c_1 x + c_3,$$

$$\tau = -c_1 t + \frac{2\alpha_2^2 c_{11}}{3\alpha_3} x + c_4,$$

$$\eta = c_1 u,$$

$$\phi = \frac{\alpha_2 c_1}{3\alpha_3} t + c_2, \quad (6)$$

From what we have discussed we can arrive at the following facts : There are many methods that we can use to solve Travelling wave equation but we preferred to use the exact modified sub-equation method because it's a method that works to solve mathematical problems in a physical manner and describes the follow of water in different geometric shapes. It also works to move from one solution to another solution in a stable , smoot , logical and symmetrical way for nonlinear partial differential equations which based on the vector field method and its characterized by accuracy

6. CONCLUSION

We considered the third ordinary nonlinear Schrodinger equation which enables studies and advances in the speed of information transmission that plays a major role in fields such as ultra short pulses, optical fiber applied physics, communication system, etc,... To contribute to the studies of the higher order Schrodinger equation and the special cases of this equation in the literature .We considered equation (4) which was considered $\alpha_2 = \frac{1}{2}$ and we studied the nonautonomous characteristics of the W-Shaped solutions and have modified the darboux transformation method to the first and second orders respectively. As far we knew the exact invariant solutions of this equation which include generalized hyperbolic and trigonometric functions which we investigated for the first time. We believe that the solutions we have obtained are new. One of the advantages of the applied methods in that it contains more general solutions than most of the methods in the literature. The results obtained by the application of these methods have shown that this method is effective, strong and applicable to other problems in mathematical physics.

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