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An Efficient Algorithm for the Inverse of P-Diagonal Toeplitz Matrices

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1 Abstract

In this recurring paper, we provide a compact and more general algorithm for obtaining the inverse of p-diagonal matrices. We implemented it on a more complex structure, a nona-diagonal matrix and tested it,to test its efficiency using the same method. Currently, this extension not only showcases the flexibility of our method,but also shows that we can further improve the computational performance when working with more general matrix structures.

Keywords : Diagonal, Inverse, Nona-diagonal.

2 Introduction

Diagonal matrices are more of a mathematical tool that they are used a lot in a lot of fields from numerical analysis to engineering, physics, and applied mathematics. They are well known for their structured format, with non-zero entries occurring only on the main diagonal and a specified number of adjacent diagonals; which allows for very efficient computation. Such matrices are especially common in the numerical solution of ordinary and partial differential equations (ODEs and PDEs) where they typically result from discretization methods like finite differences or finite elements. They also be used in interpolation problem when itcomes to dealing with a smooth approximation to our data, and when we need to deal with boundary value problems BVP $\rightarrow$ here this functions will be useful in order to get a differential equations solution with some boundaries on it.

In this paper, we investigate the case of p-diagonal matrices, a special case of diagonal matrices where the non-zero entries lie along the main diagonal as well as the p adjacent diagonals, for a small, constant parameter p, which leads to structures that are both simple yet highly computable. The main goal is to propose a general method for inverting p-diagonal matrices. The importance of the inverse of these matrices lies in the fact that solving linear systems — which are abundant in scientific computations and numerical simulations — could be performed efficiently.

We commence with providing some theoretical background into the characteristics of these so-called p-diagonal matrices, and their inverses. Then, we unfold a stage-by-stage algorithm with exploiting the particular structure of these matrices to speed up the calculation. It has low time- and space complexity and is also implemented as a step by step process which is why this algorithm is an excellent option for applications that operate on large datasets.

In order to test and proved the practical functionality of our algorithm, we apply it to a more complex class of matrices where non-zero elements are in 9 diagonals, known as non-diagonal matrices. This extension has two roles to play; one is to show that our approach can be applied to matrices of higher bandwidth, and the other is to understand the scalability of the algorithm to more complex matrix patterns. In this paper, we demonstrate that our method is robust and flexible to be applied intobroader problems, by importing our method into nona-diagonal matrices.

The results in this paper will help to both, develop the theory of p-diagonal and nona-diagonal matrices as well as provide tools to address the real problems. They are explored practically through numerical computations arising from differential equations, interpolation, boundary value problems,etc., which may benefit significantly from the proposed algorithm. This work is uniquely positioned to be a resource for both theoretical and applied researchers and practitioners working in computational mathematics, the promotion and demonstration of which is certainly needed for the broader community.

3 Inverse of p-diagonal matrix

Definition:
Consedering the p-diagonal matrix as bellow:

$$D = \begin{pmatrix} d & a_1 & a_2 & \cdots & a_p & 0 & \cdots & 0 \\ b_1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ b_2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & a_p \\ b_p & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & a_1 \\ 0 & \cdots & 0 & b_p & \cdots & b_2 & b_1 & d \end{pmatrix} \tag{1}$$

Where $D \in \mathcal{M}_{n\times n}(\mathbb{K})$.
Assuming that D is non-singular and :

$$D^{-1} = [C_1, C_2, \cdots, C_n]$$

Where $(C_i)_{1\leq i\leq n}$ are the columns inverse of D^{-1}. From the relation $DD^{-1} = I_n$ (where I_n denotes the identity matrix) we deduce the relations:

$$C_{n-p} = \frac{1}{a_p}(E_n - a_{p-1}C_{n-p+1} - a_{p-2}C_{n-p+2} - \cdots - dC_n)$$

For $n-p-1 \leq j \leq p-1$

$$C_{j-p} = \frac{1}{a_p}(E_j - a_{p-1}C_{j-p+1} - a_{p-2}C_{j-p+2} - \cdots - b_pC_{j+p1})$$

Consedering the p numbers of the sequences of numbers $(A_{i,j})_{1\leq i\leq p;1\leq j\leq n}$ defined as:
For $0 \leq k \leq p$

$$A_{p,p-k} = 1$$

For $1 \leq i \leq p$

$$dA_{i,0} + a_1A_{i,1} + \cdots + a_pA_{i,p} = 0$$

$$b_1A_{i,0} + dA_{i,1} + a_1A_{i,1} + \cdots + a_pA_{i,p+1} = 0$$

$$\vdots$$

$$b_pA_{i,0} + b_{p-1}A_{i,1} + \cdots + a_pA_{i,n-1} = 0$$

For $p \leq q \leq n-p$ and $p \leq j \leq 1$.

$$b_jA_{i,q-p} + b_{j-1}A_{i,q-p+1} + \cdots + a_jA_{i,q+p-1} = 0$$

And

$$b_pA_{i,n-2} + \cdots + a_{p-1}A_{i,n-1} + A_{i,n} = 0$$

$$\vdots$$

$$b_pA_{i,n-p-1} + \cdots + dA_{i,n-p+2} + A_{i,n-p+1} = 0$$

We define for such $1 \leq i \leq p$ and $0 \leq j \leq n+p-1$:

$$Q_{1,j} = \begin{pmatrix} A_{1,n-p+1} & \cdots & A_{1,n+p-2} & A_{1,j} \\ \vdots & \cdots & \vdots & \vdots \\ \vdots & \cdots & \vdots & \vdots \\ A_{p,n-p+1} & \cdots & A_{p,n+p-2} & A_{p,j} \end{pmatrix} \tag{2}$$

$$\vdots \tag{3}$$

$$Q_{p-1,j} = \begin{pmatrix} A_{1,n-p+1} & A_{1,j} & A_{1,n-p+3} & \cdots & A_{1,n+p-1} \\ \vdots & \cdots & \vdots & \vdots & \\ \vdots & \cdots & \vdots & \vdots & \\ \vdots & \cdots & \vdots & \vdots & \\ A_{p,n-p+1} & A_{p,j} & A_{p,n-p+3} & \cdots & A_{p,n+p-1} \end{pmatrix} \tag{4}$$

$$Q_{p,j} = \begin{pmatrix} A_{1,j} & A_{1,n-p+2} & \cdots & A_{1,n+p-1} \\ \vdots & \cdots & \vdots & \vdots \\ \vdots & \cdots & \vdots & \vdots \\ A_{p,j} & A_{p,n-p+2} & \cdots & A_{p,n+p-1} \end{pmatrix} \quad (5)$$

WE can write:

$$DQ_1 = -Q_{1,n+p-1}E_{n-p+1}$$

$$\vdots$$

$$DQ_p = -Q_{p,n}E_n$$

Theorem 1 *Assuming that $Q_{n+p-1} \neq 0$, then D is non-singular and:*

$$C_n = \frac{-1}{Q_{1,n+p-1}}[Q_{1,0}, Q_{1,1}, \cdots, Q_{1,n-1}]^t \quad (6)$$

$$\vdots$$

$$C_{n-p+1} = \frac{-1}{Q_{p,n-p+1}}[Q_{p,0}, Q_{p,1}, \cdots, Q_{p,n-1}]^t \quad (7)$$

4 Numerical experiments

In this section we applied the generalized algorithm in a Nona-diagonal Toeplitz matrix defined as:

$$D = \begin{pmatrix} d & a_1 & a_2 & a_3 & a_4 & 0 & \cdots & 0 \\ b_1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ b_2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ b_3 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & a_4 \\ b_4 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & a_3 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & a_2 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & a_1 \\ 0 & \cdots & 0 & b_4 & b_3 & b_2 & b_1 & d \end{pmatrix} \quad (8)$$

We assume that D is non-singular then:

$$D^{-1} = [C_1, \cdots, C_n]$$

Where $(C_i)_{1\leq i\leq n}$ are the columns of the inverse D^{-1} From the relation $DD^{-1} = I_n$ we get:

$$C_{n-4} = \frac{1}{a_4}(E_n - a_3C_{n-3} - a_2C_{n-2} - a_1C_{n-1} - dC_n)$$

$$C_{j-4} = \frac{1}{a_4}(E_j - a_3C_{j-3} - a_2C_{j-2} - a_1C_{j-1} - dC_j - b_1C_{j+1} - b_2C_{j+2} - b_3C_{j+3} - b_4C_{j+4}) \quad for\, n-5 \leq j \leq 3$$

Consider the sequence of numbers $(X_i)_{(0\leq i\leq n)}$, $(Y_i)_{(0\leq i\leq n)}$, $(V_i)_{(0\leq i\leq n)}$ and $(W_i)_{(0\leq i\leq n)}$ characterized by a term recurrence relation:

$$X_0 = 0$$
$$X_1 = 0$$
$$X_2 = 0$$
$$X_3 = 1$$
$$dX_0 + a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 = 0$$
$$b_1X_0 + dX_1 + a_1X_2 + a_2X_3 + a_3X_4 + a_4X_5 = 0$$
$$b_2X_0 + b_1X_1 + dX_2 + a_1X_3 + a_2X_4 + a_3X_5 + a_4X_6 = 0$$
$$b_3X_0 + b_2X_1 + b_1X_2 + dX_3 + a_1X_4 + a_2X_5 + a_3X_6 + a_4X_7 = 0$$
$$b_4X_0 + b_3X_1 + b_2X_2 + b_1X_3 + dX_4 + a_1X_5 + a_2X_6 + a_3X_7 + a_4X_8 = 0$$
$$b_4X_{j-4}+b_3X_{j-3}+b_2X_{j-2}+b_1X_{j-1}+dX_j+a_1X_{j+1}+a_2X_{j+2}+a_3X_{j+3}+a_4X_{j+4} = 0 \quad for\ 5 \leq j \leq n-5$$
$$b_4X_{n-9}b_3X_{n_8}+b_2X_{n-7}+b_1X_{n-6}+dX_{n-5}+a_1X_{n-4}+a_2X_{n-3}+a_3X_{n-2}+a_4X_{n-1}+X_n = 0$$
$$b_4X_{n-8}b_3X_{n_7}+b_2X_{n-6}+b_1X_{n-5}+dX_{n-4}+a_1X_{n-3}+a_2X_{n-2}+a_3X_{n-1}+a_4X_n+X_{n+1} = 0$$
$$b_4X_{n-7}b_3X_{n_6}+b_2X_{n-5}+b_1X_{n-4}+dX_{n-3}+a_1X_{n-2}+a_2X_{n-1}+a_3X_n+a_4X_{n+1}+X_{n+2} = 0$$
$$b_4X_{n-6}b_3X_{n_5}+b_2X_{n-4}+b_1X_{n-3}+dX_{n-2}+a_1X_{n-1}+a_2X_n+a_3X_{n+1}+a_4X_{n+2}+X_{n+3} = 0$$

And:

$$Y_0 = 0$$
$$Y_1 = 0$$
$$Y_2 = 1$$
$$Y_3 = 0$$
$$dY_0 + a_1Y_1 + a_2Y_2 + a_3Y_3 + a_4Y_4 = 0$$
$$b_1Y_0 + dY_1 + a_1Y_2 + a_2Y_3 + a_3Y_4 + a_4Y_5 = 0$$
$$b_2Y_0 + b_1Y_1 + dY_2 + a_1Y_3 + a_2Y_4 + a_3Y_5 + a_4Y_6 = 0$$
$$b_3Y_0 + b_2Y_1 + b_1Y_2 + dY_3 + a_1Y_4 + a_2Y_5 + a_3Y_6 + a_4Y_7 = 0$$
$$b_4Y_0 + b_3Y_1 + b_2Y_2 + b_1Y_3 + dY_4 + a_1Y_5 + a_2Y_6 + a_3Y_7 + a_4Y_8 = 0$$
$$b_4Y_{j-4}+b_3Y_{j-3}+b_2Y_{j-2}+b_1Y_{j-1}+dY_j+a_1Y_{j+1}+a_2Y_{j+2}+a_3Y_{j+3}+a_4Y_{j+4} = 0 \quad for\ 5 \leq j \leq n-5$$
$$b_4Y_{n-9}+b_3Y_{n_8}+b_2Y_{n-7}+b_1Y_{n-6}+dY_{n-5}+a_1Y_{n-4}+a_2Y_{n-3}+a_3Y_{n-2}+a_4Y_{n-1}+Y_n = 0$$

$b_4Y_{n-8}+b_3Y_{n_7}+b_2Y_{n-6}+b_1Y_{n-5}+dY_{n-4}+a_1Y_{n-3}+a_2Y_{n-2}+a_3Y_{n-1}+a_4Y_n+Y_{n+1}=0$

$b_4Y_{n-7}+b_3Y_{n_6}+b_2Y_{n-5}+b_1Y_{n-4}+dY_{n-3}+a_1Y_{n-2}+a_2Y_{n-1}+a_3Y_n+a_4Y_{n+1}+Y_{n+2}=0$

$b_4Y_{n-6}+b_3Y_{n_5}+b_2Y_{n-4}+b_1Y_{n-3}+dY_{n-2}+a_1Y_{n-1}+a_2Y_n+a_3Y_{n+1}+a_4Y_{n+2}+Y_{n+3}=0$

Also we have:

$$V_0=0$$

$$V_1=1$$

$$V_2=0$$

$$V_3=0$$

$$dV_0+a_1V_1+a_2V_2+a_3V_3+a_4V_4=0$$

$$b_1V_0+dV_1+a_1V_2+a_2V_3+a_3V_4+a_4V_5=0$$

$$b_2V_0+b_1V_1+dV_2+a_1V_3+a_2V_4+a_3V_5+a_4V_6=0$$

$$b_3V_0+b_2V_1+b_1V_2+dV_3+a_1V_4+a_2V_5+a_3V_6+a_4V_7=0$$

$$b_4V_0+b_3V_1+b_2V_2+b_1V_3+dV_4+a_1V_5+a_2V_6+a_3V_7+a_4V_8=0$$

$b_4V_{j-4}+b_3V_{j-3}+b_2V_{j-2}+b_1V_{j-1}+dV_j+a_1V_{j+1}+a_2V_{j+2}+a_3V_{j+3}+a_4V_{j+4}=0 \quad for\ 5\leq j\leq n-5$

$b_4V_{n-9}+b_3V_{n_8}+b_2V_{n-7}+b_1V_{n-6}+dV_{n-5}+a_1V_{n-4}+a_2V_{n-3}+a_3V_{n-2}+a_4V_{n-1}+V_n=0$

$b_4V_{n-8}+b_3V_{n_7}+b_2V_{n-6}+b_1V_{n-5}+dV_{n-4}+a_1V_{n-3}+a_2V_{n-2}+a_3V_{n-1}+a_4V_n+V_{n+1}=0$

$b_4V_{n-7}+b_3V_{n_6}+b_2V_{n-5}+b_1V_{n-4}+dV_{n-3}+a_1V_{n-2}+a_2V_{n-1}+a_3V_n+a_4V_{n+1}+V_{n+2}=0$

$b_4V_{n-6}+b_3V_{n_5}+b_2V_{n-4}+b_1V_{n-3}+dV_{n-2}+a_1V_{n-1}+a_2V_n+a_3V_{n+1}+a_4V_{n+2}+V_{n+3}=0$

Finnaly:

$$W_0=1$$

$$W_1=0$$

$$W_2=0$$

$$W_3=0$$

$$dW_0+a_1W_1+a_2W_2+a_3W_3+a_4W_4=0$$

$$b_1W_0+dW_1+a_1W_2+a_2W_3+a_3W_4+a_4W_5=0$$

$$b_2W_0+b_1W_1+dW_2+a_1W_3+a_2W_4+a_3W_5+a_4W_6=0$$

$$b_3W_0+b_2W_1+b_1W_2+dW_3+a_1W_4+a_2W_5+a_3W_6+a_4W_7=0$$

$$b_4W_0+b_3W_1+b_2W_2+b_1W_3+dW_4+a_1W_5+a_2W_6+a_3W_7+a_4W_8=0$$

$b_4W_{j-4}+b_3W_{j-3}+b_2W_{j-2}+b_1W_{j-1}+dW_j+a_1W_{j+1}+a_2W_{j+2}+a_3W_{j+3}+a_4W_{j+4}=0 \quad for\ 5\leq j\leq n-5$

$$b_4W_{n-9}+b_3W_{n_8}+b_2W_{n-7}+b_1W_{n-6}+dW_{n-5}+a_1W_{n-4}+a_2W_{n-3}+a_3W_{n-2}+a_4W_{n-1}+W_n=0$$

$$b_4W_{n-8}+b_3W_{n_7}+b_2W_{n-6}+b_1W_{n-5}+dW_{n-4}+a_1W_{n-3}+a_2W_{n-2}+a_3W_{n-1}+a_4W_n+W_{n+1}=0$$

$$b_4W_{n-7}+b_3W_{n_6}+b_2W_{n-5}+b_1W_{n-4}+dW_{n-3}+a_1W_{n-2}+a_2W_{n-1}+a_3W_n+a_4W_{n+1}+W_{n+2}=0$$

$$b_4W_{n-6}+b_3W_{n_5}+b_2W_{n-4}+b_1W_{n-3}+dW_{n-2}+a_1W_{n-1}+a_2W_n+a_3W_{n+1}+a_4W_{n+2}+W_{n+3}=0$$

Considering for such $0 \leq i \leq n+3$

$$P_i = det\begin{pmatrix} X_n & X_{n+1} & X_{n+2} & X_i \\ Y_n & Y_{n+1} & Y_{n+2} & Y_i \\ V_n & V_{n+1} & V_{n+2} & V_i \\ W_n & W_{n+1} & W_{n+2} & W_i \end{pmatrix}$$

$$L_i = det\begin{pmatrix} X_n & X_{n+1} & X_i & X_{n+3} \\ Y_n & Y_{n+1} & Y_i & Y_{n+3} \\ V_n & V_{n+1} & V_i & V_{n+3} \\ W_n & W_{n+1} & W_i & W_{n+3} \end{pmatrix}$$

$$M_i = det\begin{pmatrix} X_n & X_i & X_{n+2} & X_{n+3} \\ Y_n & Y_i & Y_{n+2} & Y_{n+3} \\ V_n & V_i & V_{n+2} & V_{n+3} \\ W_n & W_i & W_{n+2} & W_{n+3} \end{pmatrix}$$

$$N_i = det\begin{pmatrix} X_i & X_{n+1} & X_{n+2} & X_{n+3} \\ Y_i & Y_{n+1} & Y_{n+2} & Y_{n+3} \\ V_i & V_{n+1} & V_{n+2} & V_{n+3} \\ W_i & W_{n+1} & W_{n+2} & W_{n+3} \end{pmatrix}$$

Theorem 2 *We supposed that $P_{n+3} \neq 0$, then D is non-singular and:*

$$C_n = \frac{-1}{P_{n+3}}[P_0, \cdots, P_{n-1}]$$

$$C_{n-1} = \frac{-1}{L_{n+2}}[L_0, \cdots, L_{n-1}]$$

$$C_{n-2} = \frac{-1}{M_{n+1}}[M_0, \cdots, M_{n-1}]$$

$$C_{n-3} = \frac{-1}{N_n}[N_0, \cdots, N_{n-1}]$$

5 Exemple

The table compares 'Toeplitz-Hessenberg' andour algorithm (implemented in MATLAB R2024b) execution time. Execution time (in seconds) for the two considered proposed algorithms evaluated in MATLAB R2024b.

Table 1: The running time

Size of the matrix (n)	Algorithm	LU method
100	0.036954	0.918161
200	0.061992	2.547606
300	0.090051	6.816085
500	0.149696	24.165349
1000	0.314484	149.750575

6 Conclusion

To sum up, this paper presents a method to compute p-diagonal matrix inverse with a low computational overhead andfurther generalizes the result for the application of nona-diagonal matrices. Thanks to thestructured nature of these matrices, it is a significant tool for solving problems in numerical analysis, differential equations and more by improving exponentially the efficiency of the algorithm. We note that customized algorithms for structured matrices are important, and thus, this work could serve asa springboard to extend even more matrix classes and leading to future work in high-performance numerical algorithms.

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