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Δ_{kl} − Statistical Convergence via Neutrosophic Normed Spaces for Double Sequences

B. G. Ahmadu[1], A. M. Brono[2]
[1,2]University of Maiduguri, Department of Mathematics, Borno State, Nigeria

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Corresponding Author:
B. G. Ahmadu

ABSTRACT
In this paper, we present the extension of Δ-statistically convergent and Δ-statistically Cauchy sequences via neutrosophic normed space (NNS) to double sequences. The study in analogy also define and introduce Δ_{kl} for which $st^{\mathfrak{R}}_{\Delta_{kl}} - \lim s_{kl} = s$ or $s_{kl} - s(S^{\mathfrak{R}}_{\Delta_{kl}})$, where $k \to \infty$. $S^{\mathfrak{R}}_{\Delta_{kl}}$ denote set of all Δ_{kl} − statistically convergent sequences. Furthermore, we present their feature utilizing double density and establish some inclusion relations between these concepts and prove some essentials analogous properties for double sequences.

KEYWORDS: Double sequences, Statistical sequence, Difference sequence, Neutrosophic normed space

1. INTRODUCTION

The first world publication related to the concept of neutrosophy of the usual notion of convergence. The idea of statistical convergence was given in the first edition (published in Warsaw in 1935) of the monograph of zygmund (1979), "who called it almost convergence". Formally the concept of statistical convergence was introduced by Fast (1951) and Steinhaus (1951) independently and later reintroduced by Schoenberg (1959). Although statistical convergence was introduced over nearly last sixty years, it has become an active area of research in recent years.

The concept of fuzzy set was originally introduced by Zadeh (1965). The fuzzy theory has become an area of active research for the last fifty years. It has a wide range of applications in the field of science and engineering, population dynamics (2000), chaos control (2004), computer programming (1980), nonlinear dynamical systems (2006), fuzzy physics (1992) and more. The intuitionistic fuzzy set on a universe X was first introduced by Atanssov in (1986) as a generalization of Fuzzy set, where besides the degree of membership of each element to a set, there was considered a degree non – membership. Taking into account the concept of fuzzy set and intuitionistic fuzzy set, Smarandache (2005) introduced the notion of Neutrosophic set (NS) which is a new version of the idea of the classical set. The first world publication related to the concept of neutrosophy was published in (1998) and included in the literature. On the other hand, Kaleva and Seikkala (1984) defined the fuzzy metric spaces (FMS) as a distance between two points to be a non-negative fuzzy number. After that, in (1994) some basic properties of FMS were studied and the Baire Category Consequently, FMS has used in the applied sciences such as fixed point theory, image and signal processing, medical imaging, decision making and more. After definition of the intuitionistic fuzzy set (IFS), it was used in all areas where FS theory was studied. Park (2004) introduced IF metric space (IFMS), that is a generalization of FMS. Then, Park used George and Veeramani's (1994) work for applying t-norm and t-conorm to FMS meanwhile defining IFMS and studying its basic properties. Moreover, Bera and Mahapatra introduced the neutrosophic soft linear spaces (NSLS) (2017). Later, neutrosophic soft normed linear spaces (NSNLS) was defined by Bera and Mahapatra (2018). Besides, In (2018), neutrosophic norm, Cauchy sequence in NSNLS, convexity of NSNLS, metric in NSNLS were defined and studied. Vakeel A. Khan (2021) used the notion of λ-statistical convergence in order to generalize these concepts. And also established some inclusion relations between them. They defined the statistical convergence and λ-statistical convergence in neutrosophic normed space. They gave the λ-statistically Cauchy sequence in neutrosophic normed space and presented the λ-statistically completeness in connection with a neutrosophic normed space. Some interesting examples are also displayed in support of the definitions and results. Kirisci Recently, Kirisci and Simsek (2020), introduced and studied the notion of statistical convergence in a neutrosophic normed spaces. Besides, they showed some interesting results. Recently, Carlos Granados and Alok Dhital (2021) studied and presented the idea of Statistical Convergence of Double Sequences in

Neutrosophic Normed Spaces. Quite recently, Nazmiye G. B. (2022) studied different types of convergence concepts and applied them to difference sequences. The concept of difference sequences was combined with structures that are advantageous to work like Lacunary sequences. In this work, we shall extend the notion of some concepts of statistical convergence via neutrosophic normed spaces by using double sequences. Moreover, we prove some of its properties and characterizations. Also this work aims to provide a solid foundation for studying statistical convergence properties of double sequences in a neutrosophic normed spaces setting.

2. PRELIMINARIES

Definition 2.1 (Fast and Steinhaus, 1951): A number sequence $x = (x_k)$ is said to be statistically convergent to the number L if for each $\epsilon > 0$, $\lim_n \frac{1}{n} |\{k \le n: |x_k - L| \ge \epsilon\}| = 0$

Definition 2.2 (Fast and Steinhaus, 1951): A sequences $x = (x_k)$ is said to be statistically Cauchy sequence if for every $\varepsilon > 0$, there exists a number $N = N(\varepsilon)$ such that

$$\lim_n \frac{1}{n} |\{k \le n: |x_k - x_N| \ge \varepsilon\}| = 0$$

Definition 2.3 (Fridy and Khan, 1998): A sequence(x_k) is said to be statistically bounded in X if there exist $x \in X$ and $M > 0$ such that $\Delta(\{k \in \mathbb{N}: \rho(x_k, x) \le M\}) = 1$.

Definition 2.4 (Fridy and Khan, 1998): A sequence (x_k) is a statistically Cauchy sequence in X if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $\Delta(A_{N(\varepsilon)}) = 1$, where

$$A_{N(\varepsilon)} = \{k \in \mathbb{N}: \rho(x_N, x_k) < \varepsilon\}.$$

2.2 Statistical Convergence of Double Sequences

The notion of statistical convergence of double sequences was defined by Mursaleen and Edely (2003).

Let $K \subset \mathbb{N} \times \mathbb{N}$ be two-dimensional set of positive integers and let be $K(n, m)$ the numbers of (j, k) in K such that $j \le n$ and $k \le m$. Then, the two-dimensional analogue of natural density can be defined as follows:

The lower asymptotic density of the set $K \subset \mathbb{N} \times \mathbb{N}$ is defined as:

$\underline{\delta_2}(K) = \lim_{n,m} inf \frac{K(n,m)}{nm}$. Similarly,

The upper asymptotic density of the set $K \subset \mathbb{N} \times \mathbb{N}$ is defined as:

$\overline{\delta_2}(K) = \lim_{n,m} sup \frac{K(n,m)}{nm}$.

In case that the sequence $\left(\frac{K(m,n)}{mn}\right)$ has a limit in Pringsheim's sense then we say that K has a double density and is defined as:

$$\lim_{n,m} \frac{K(n,m)}{nm} = \delta_2(K)$$

Statistical convergence for double sequence $x = (x_{kj})$ of real which was defined by Mursaleen and Edely (2003) as:

Definition 2.2.1 (Mursaleen and Edely, 2003): A real double sequence $x = (x_{kj})$ is said to be statistically convergent to the number L if for each $\epsilon > 0$, the set $\{(j, k), j \le m, k \le n: |x_{kj} - L| \ge \epsilon\}$, has a double natural density zero. In this case, we write $S_2 - \lim x_{kj} = L$.

Now, we write the essential definitions and concepts in the study

The concept of statistical convergence was defined by Fast and Steinhaus (1951) and also independently by Buck (1953) and Schornberg (1959) for real and complex sequences. Further this concept was studied by Salat (1980), Fridy (1985), Connor (1988) and later this notion was studied by various authors. Let K be a subset of N, then the asymptotic density of K, denoted by $d(K)$ is defined as follows:

$$d(k) = \lim_n \frac{1}{n} |\{k \le n: k \in K\}|,$$

where the vertical bars denote the cardinality of the enclosed set.

Definition 2.2.2 (Mursaleen and Edely, 2003): A real double sequence $x = (x_{jk})$ is said to be statistically Cauchy if for each $\varepsilon > 0$, there exist positive integers $m(\varepsilon)$ and $n(\varepsilon)$ such that for every $j, p \ge m$ and $k, q \ge n$, the set $\{(j, k), j \le m, k \le n: |x_{jk} - x_{pq}| \ge \varepsilon\}$ has double natural density zero.

Definition 2.2.3 (Mursaleen and Edely, 2003): A double sequence $x = (x_{jk})$ is said to be bounded if there exists a real number $M > 0$ such that $|x_{jk}| < M$ for each j and k, i.e, if $\|x\|_{(\infty,2)} = \sup_{jk} |x_{jk}| < \infty$. We shall denote the set of all bounded double sequence by l_2^∞.

Note that in contrast to the case of sequence a convergent double sequence need not be bounded.

Triangular norms (t-norms) (TN) were initiated by Menger K (1942). In the problem of computing the distance between two elements in space, Menger offered using probability distributions instead of using numbers for distance. TNs are used to generalize with the probability distribution of triangle inequality in metric space conditions. Triangular conorms (t-conorms) (TC) know as dual operations of TNs. TNs and TCs are very significant for fuzzy operations (intersections and unions).

Definition 2.2.4 (Kirisci and Simsek, 2020): Give an operation $\circ$: $[0,1] \times [0,1] \to [0,1]$. If the operation $\circ$ is satisfying the following conditions:

(1) $s \circ 1 = s$,

(2) If $s \le u$ and $t \le v$, then $s \circ t \le u \circ v$,

(3) $\circ$ is continuous,

(4) $\circ$ is continuous and associative.

Then, it is called that the operation $\circ$ is continuous TN, for $s, t, u, v \in [0,1]$.

Definition 2.2.5 (Kirisci and Simsek, 2020): Give an operation $\bullet$: $[0,1] \times [0,1] \to [0,1]$. If the operation $\bullet$ is satisfying the following conditions:

(1) $s \bullet 0 = s$,

(2) If $s \le u$ and $t \le v$, then $s \bullet t \le u \bullet v$,

(3) $\bullet$ is continuous,

(4) $\bullet$ is continuous and associative.

Then, it is called that the operation • is continuous TC, for $s,t,u,v \in [0,1]$.

Remark: From the above definitions, we can see that if we take $0 < \epsilon_1, \epsilon_2 < 1$ for $\epsilon_1 < \epsilon_2$, then there exist $0 < \epsilon_3, \epsilon_4 <$ 0,1 such that $\epsilon_1 \circ \epsilon_3 \geq \epsilon_2, \epsilon_1 \geq \epsilon_4 \bullet \epsilon_2$. Moreover, if we take $\epsilon_5 \in (0,1)$, then there exist $\epsilon_6, \epsilon_7 \in (0,1)$ such that $\epsilon_6 \circ \epsilon_6 \geq \epsilon_5$ *and* $\epsilon_7 \bullet \epsilon_7 \leq \epsilon_2$.

The notion of neutrosophic normed space (NNS) was defined by **(Kirisci and Simsek, 2020),** as well as the definition of statistical convergence with respect to NNS was given.

Definition 2.2.6 (Kirisci and Simsek, 2020): Take F as a vector space $N = \{< u, G(u), B(u), Y(u) >: u \in F\}$ be a normed space (NS) such that $N: F \times F^+ \to [0,1]$. Let ∘ and • show the continuous TN and continuous TC, respectively. If the following conditions are satisfied, then the four-tuple $V = (F, N, \circ, \bullet)$ is called neutrosophic normed space (NNS), for all $u, v, \in F, \lambda, \mu > 0$ *and for each* $\sigma \neq 0$:

(1) $0 \leq G(u,\lambda) \leq 1, 0 \leq B(u,\lambda) \leq 1, 0 \leq Y(u,\lambda) \leq 1$, for all $\lambda \in R^+$,

(2) $G(u,\lambda) + B(u,\lambda) + Y(u,\lambda) \leq 3$, for all $\lambda \in R^+$,

(3) $G(u,\lambda) = 1$, for $\lambda > 0$ if and only if $u = 0$,

(4) $G(\sigma u,\lambda) = G\left(u, \frac{\lambda}{|\sigma|}\right)$,

(5) $G(u,\lambda) \circ G(v,\mu) \leq G(u+v, \lambda+\mu)$,

(6) $Q(u,.)$ is continuous non decreasing function,

(7) $\lim_{\lambda\to\infty} G(u,\lambda) = 1$,

(8) $B(u,\lambda) = 0$, for $\lambda > 0$ if and only if $u = 0$,

(9) $B(\sigma u,\lambda) = B(u, \frac{\lambda}{|\sigma|})$,

(10) $B(u,\mu) \bullet B(v,\lambda) \geq B(u+v, \lambda+\mu)$,

(11) $B(u,.)$ is continuous non decreasing function,

(12) $\lim_{\lambda\to\infty} B(u,\lambda) = 0$,

(13) $Y(u,\lambda) = 0$, for $\lambda > 0$ if and only if $u = 0$,

(14) $Y(\sigma u,\lambda) = Y(u, \frac{\lambda}{|\sigma|})$,

(15) $Y(u,\lambda) \bullet Y(v,\lambda) \geq Y(u+v, \lambda+\mu)$,

(16) $Y(u,.)$ is continuous non decreasing function,

(17) $\lim_{\lambda\to\infty} Y(u,\lambda) = 0$,

(18) if $\lambda \leq 0$, then $G(u,\lambda) = 0$, $B(u,\lambda) = 1$ and $Y(u,\lambda) = 1$.

Then $N = (G, B, Y)$ is called neutrosophic norm (NN).

Example 2.2.1 (Kirisci and Simsek, 2020): Let $(F, \|.\|)$ be a NS. Give the operations ∘ and • as TN $u \circ v = uv$; TC $u \bullet v = u + v - uv$. For $\lambda > \|u\|$,

$$G(u,\lambda) = \frac{\lambda}{\lambda+\|u\|}, B(u,\lambda) = \frac{\|u\|}{\lambda+\|u\|}, Y(u,\lambda) = \frac{\|u\|}{\lambda},$$

for all $u, v, \in F$, *and* $, \lambda > 0$. If we take $\lambda \leq \|u\|$, then $G(u,\lambda) = 0$, $B(u,\lambda) = 1$ and $Y(u,\lambda) = 1$. Then, $(F, N, \circ, \bullet)$ is NNS such that $N: F \times F^+ \to [0,1]$.

Definition 2.2.7 (Kirisci and Simsek, 2020): Let V be a NNS and (a_n) be a sequence in V such that $0 < \epsilon < 1$ and $\lambda > 0$ Then, (a_n) converges to a if and only if there exists $n_0 \in \mathbb{N}$ such that $\mathcal{G}(a_n - a, \lambda) > 1 - \epsilon, \mathcal{B}(a_n - a, \lambda) < \epsilon$ and $\mathcal{Y}(a_n - a, \lambda) < \epsilon$. That is $\lim_{n\to\infty} \mathcal{G}(a_n - a, \lambda) = 1, \lim_{n\to\infty} \mathcal{B}(a_n - a, \lambda) = 0$ and $\lim_{n\to\infty} \mathcal{Y}(a_n - a, \lambda) = 0$ as $\lambda > 0$. In this case, the sequence (a_n) is said to be a convergent sequence in V. The convergent in NNS is denoted by $N - \lim a_n = L$.

2.3 $\Delta-$ Statistical Convergence and $\Delta-$ Lacunary Statistical Convergence

Definition 2.3.1 (Basarir, 1995): (s_k) is called to be $\Delta-$ statistically convergent to s, where $\delta(\{k \in \mathbb{N}: |\Delta s_k - s| \geq \varepsilon\}) = 0$, for all $\varepsilon > 0$ and $\Delta s_k = s_k - s_{k+1}$, 1.e., $\lim_{p\to\infty} \frac{1}{p} |\{k \leq p: |\Delta s_k - s| \geq \varepsilon\}| = 0$. Then, it is demonstrated $st - lim \Delta s_k = s$. S_Δ is denoted, set of all $\Delta-$ statistical convergence sequences.

Definition 2.3.2 (Fridy and Orhan, 1993): Let $\theta = \{k_r\}$ be a sequence of increasing integers, $k_0 = 0$ and also $\lim_{r\to\infty} k_r - k_{r-1} = \infty$. Then, θ is called to be Lacunary sequences. Let $A \subset \mathbb{N}, I_r = \left({}^{k_r}/_{k_{r-1}}\right)$ and $I_r = (k_{r-1}, k_r)$. $\delta^\theta(A) = \lim_{r\to\infty} \frac{1}{I_r} |k \in I_r: k \in A|$, is said to be the $\theta-$ density of A if limit is exhibited.**Definition 1.6.5.3 (Fridy and Orhan, 1993):** Let $A_\varepsilon = \{k \in I_r: |s_k - s| \geq \varepsilon\}$; for all $\varepsilon > 0$, if $\delta^\theta(A_\varepsilon) = \lim_{r\to\infty} \frac{1}{I_r} |\{k \in I_r: |s_k - s| \geq \varepsilon\}| = 0$, in this case, (s_k) is called to be $\Delta-$ Lacunary statistical convergent to s. Then, it is represented as $st^\theta - lim s_k = s$. S^θ is a denoted set of every Lacunary statistical convergence sequences.

Definition 2.3.3 (Nazmiye, 2022): Let $(\mathcal{U}, \mu_{(\Re,v)}, \varrho_{(\Re,\mathcal{U})}, \zeta_{(\Re,\mathcal{F})}, \otimes, \boxdot)$ be neutrosophic normed spaces and $\Delta_{s_k} = s_k - s_{k+1}$. (s_k) is called to be $\Delta-$ convergence to s according to neutrosophic normed if, for all $\varepsilon \in (0,1)$ and $r > 0$, there exists a $\tilde{k} \in \mathbb{N}$ such that, for every $k \geq \tilde{k}$,

$$\mu_{(\Re,v)}\left(\Delta_{s_k} - s, r\right) \leq 1 - \varepsilon, \varrho_{(\Re,\mathcal{U}),}\left(\left(\Delta_{s_k} - s, r\right) \geq \varepsilon, \zeta_{(\Re,\mathcal{F})}\left(\left(\Delta_{s_k} - s, r\right) \geq \varepsilon.\right.\right.$$

This sequences is shown with ${}^{\Re}_{\Delta}-\lim s_k = s$.

Definition 2.3.4 (Nazmiye, 2022): Let $(\mathcal{U}, \mu_{(\Re,v)}, \varrho_{(\Re,\mathcal{U})}, \zeta_{(\Re,\mathcal{F})}, \otimes, \boxdot)$ be neutrosophic normed spaces. If there exists $r > 0$ and $0 < \varepsilon < 1$, for all Δ_{s_k} where $\mu_{(\Re,v)}\left(\Delta_{s_k}, r\right) \leq 1 - \varepsilon, \varrho_{(\Re,\mathcal{U}),}\left(\left(\Delta_{s_k}, r\right) \geq \varepsilon, \zeta_{(\Re,\mathcal{F})}\left(\left(\Delta_{s_k}, r\right) \geq \varepsilon\right.\right.$, then (s_k) is called $\Delta-$ bounded sequences in $(\mathcal{U}, \mu_{(\Re,v)}, \varrho_{(\Re,\mathcal{U})}, \zeta_{(\Re,\mathcal{F})}, \otimes, \boxdot)$.

Definition 2.3.5 (Nazmiye, 2022): Let $(\mathcal{U}, \mu_{(\Re,v)}, \varrho_{(\Re,\mathcal{U})}, \zeta_{(\Re,\mathcal{F})}, \otimes, \boxdot)$ be neutrosophic normed spaces, (s_k) is called to be $\Delta-$ Cauchy sequence if , for every $\varepsilon \in (0,1)$ and $r > 0$, there exists a $k_0 \in \mathbb{N}$ such that, for every $k, p \geq k_0$, $\mu_{(\Re,v)}\left(\Delta_{s_k} - \Delta_{s_p}, r\right) \leq 1 - \varepsilon, \varrho_{(\Re,\mathcal{U}),}\left(\left(\Delta_{s_k} - \Delta_{s_p}, r\right) \geq \varepsilon, \zeta_{(\Re,\mathcal{F})}\left(\left(\Delta_{s_k} - \Delta_{s_p}, r\right) \geq \varepsilon\right.\right.$

3. MAIN RESULTS

Now, we examine the $\Delta -$ Statistical Convergence of Double sequences via Neutrosophic Normed Spaces

3.1 $\Delta_{kl} -$ Statistical Convergence via Neutrosophic Normed Spaces

Definition 3.1 Let $(\mathcal{U}, \mu_{(\mathfrak{R},v)}, \varrho_{(\mathfrak{R},\mathcal{U})}, \zeta_{(\mathfrak{R},\mathcal{F})}, \otimes, \boxdot)$ be neutrosophic normed spaces; (s_{kl}) is called to be $\Delta_{kl} -$ statistical convergence with respect to $(\mu_{(\mathfrak{R},v)}, \varrho_{(\mathfrak{R},\mathcal{U})}, \zeta_{(\mathfrak{R},\mathcal{F})})$ if , for every $\varepsilon \in (0,1)$ and $r > 0$, there exists s such that $\{k \le n, l \le m : \mu_{(\mathfrak{R},v)}(\Delta s_{kl} - s, r) \le 1 - \varepsilon \text{ or } \varrho_{(\mathfrak{R},\mathcal{U})},((\Delta s_{kl} - s, r) \ge \varepsilon, \zeta_{(\mathfrak{R},\mathcal{F})}((\Delta s_{kl} - s, r) \ge \varepsilon\}$, has natural density zero, i.e.,

$$\lim_{n,m} \frac{1}{nm} \left|\{k \le n, l \le m : \mu_{(\mathfrak{R},v)}(\Delta s_{kl} - s, r) \le 1 - \varepsilon \text{ or } \varrho_{(\mathfrak{R},\mathcal{U})},((\Delta s_{kl} - s, r) \ge \varepsilon, \zeta_{(\mathfrak{R},\mathcal{F})}((\Delta s_{kl} - s, r) \ge \varepsilon\}\right| = 0.$$

Therefore, it will be denoted as $st^{\mathfrak{R}}_{\Delta_{kl}} - \lim s_{kl} = s$ or $s_{kl} - s(S^{\mathfrak{R}}_{\Delta_{kl}})$, where $k \to \infty$. $S^{\mathfrak{R}}_{\Delta_{kl}}$ denote set of all $\Delta_{kl} -$ statistical convergence sequences.

Lemma 3.1 Let $(\mathcal{U}, \mu_{(\mathfrak{R},v)}, \varrho_{(\mathfrak{R},\mathcal{U})}, \zeta_{(\mathfrak{R},\mathcal{F})}, \otimes, \boxdot)$ be neutrosophic normed spaces and if (s_{kl}) is $\Delta_{kl} -$ statistically convergent in this case, $st^{\mathfrak{R}}_{\Delta_{kl}} - \lim s_{kl}$ is unique.

Definition 3.2 Let $(\mathcal{U}, \mu_{(\mathfrak{R},v)}, \varrho_{(\mathfrak{R},\mathcal{U})}, \zeta_{(\mathfrak{R},\mathcal{F})}, \otimes, \boxdot)$ be neutrosophic normed spaces, (s_{kl}) is called to be Δ_{kl} – statistical Cauchy sequences if , for every $\varepsilon \in (0,1)$ and $r > 0$, there exists a $j, p \in \mathbb{N}$ such that, $\delta_{(nm)}(\{k \le n, l \le m : \mu_{(\mathfrak{R},v)}(\Delta s_{kl} - \Delta s_{jp}, r) \le 1 - \varepsilon, \varrho_{(\mathfrak{R},\mathcal{U})},((\Delta s_{kl} - \Delta s_{jp}, r) \ge \varepsilon, \zeta_{(\mathfrak{R},\mathcal{F})}((\Delta s_{kl} - \Delta s_{jp}, r) \ge \varepsilon\}) = 0$

Lemma 3.2 Let $(\mathcal{U}, \mu_{(\mathfrak{R},v)}, \varrho_{(\mathfrak{R},\mathcal{U})}, \zeta_{(\mathfrak{R},\mathcal{F})}, \otimes, \boxdot)$ be neutrosophic normed spaces and (s_{kl}) be a $\Delta_{kl} -$ statistical convergence sequences. Then, for each $\varepsilon > 0, r > 0$, the next properties are equivalent:

(i) $st^{\mathfrak{R}}_{\Delta_{kl}} - \lim s_{kl} = s$

(ii) $$\lim_{n,m} \left(\frac{1}{nm}\right)\left|\{k \le n, l \le m : \mu_{(\mathfrak{R},v)}(\Delta s_{kl} - s, r) > 1 - \varepsilon, \varrho_{(\mathfrak{R},\mathcal{U})}((\Delta s_{kl} - s, r) < \varepsilon, \zeta_{(\mathfrak{R},\mathcal{F})}((\Delta s_{kl} - s, r) < \varepsilon\}\right| = 1.$$

(iii) $\lim_{n,m\to\infty} \left(\frac{1}{nm}\right) \left|\{k \le n, l \le m : \mu_{(\mathfrak{R},v)}(\Delta s_{kl} - s, r) \le 1 - \varepsilon\}\right| = 0,$

$\lim_{n,m\to\infty} \left(\frac{1}{nm}\right) \left|\{k \le n, l \le m : \varrho_{(\mathfrak{R},\mathcal{U})}(\Delta s_{kl} - s, r) \ge \varepsilon\}\right| = 0$, and

$$\lim_{n\to\infty} \left(\frac{1}{nm}\right) \left|\{k \le n, l \le m : \zeta_{(\mathfrak{R},\mathcal{F})}(\Delta s_{kl} - s, r) \ge \varepsilon\}\right| = 0,$$

(iv) $\lim_{n,m\to\infty} \left(\frac{1}{nm}\right) \left|\{k \le n, l \le m : \mu_{(\mathfrak{R},v)}(\Delta s_{kl} - s, r) > 1 - \varepsilon\}\right| = 1,$

$\lim_{n,m\to\infty} \left(\frac{1}{nm}\right) \left|\{k \le n, l \le m : \varrho_{(\mathfrak{R},\mathcal{U})}(\Delta s_{kl} - s, r) < \varepsilon\}\right| = 1$, and

$$\lim_{n,m\to\infty} \left(\frac{1}{nm}\right) \left|\{k \le n, l \le m : \zeta_{(\mathfrak{R},\mathcal{F})}(\Delta s_{kl} - s, r) < \varepsilon\}\right| = 1$$

(v) $st^{\mathfrak{R}}_{\Delta_{kl}} - \lim_{n,m\to\infty} \mu_{(\mathfrak{R},v)}(\Delta s_{kl} - s, r) = 1$, $st^{\mathfrak{R}}_{\Delta_{kl}} - \lim_{n,m\to\infty} \varrho_{(\mathfrak{R},\mathcal{U})}(\Delta s_{kl} - s, r) = 0$,

$st^{\mathfrak{R}}_{\Delta_{kl}} - \lim_{n,m\to\infty} \zeta_{(\mathfrak{R},\mathcal{F})}(\Delta s_{kl} - s, r) = 0.$

Lemma 3.3 Let $(\mathcal{U}, \mu_{(\mathfrak{R},v)}, \varrho_{(\mathfrak{R},\mathcal{U})}, \zeta_{(\mathfrak{R},\mathcal{F})}, \otimes, \boxdot)$ be neutrosophic normed spaces. Every $\Delta_{kl} -$ statistical convergence sequences is $\Delta_{kl} -$ statistical Cauchy sequences.

Proof: let (s_{kl}) be a $\Delta_{kl} -$ statistical convergence sequences in $(\mathcal{U}, \mu_{(\mathfrak{R},v)}, \varrho_{(\mathfrak{R},\mathcal{U})}, \zeta_{(\mathfrak{R},\mathcal{F})}, \otimes, \boxdot)$ and $st^{\mathfrak{R}}_{\Delta_{kl}} - \lim s_{kl} = s$. For a given $\varepsilon \in (0,1)$, *choose* $\vartheta > 0$ *such that* $(1 - \varepsilon) \otimes (1 - \varepsilon) > (1 - \mu)$ and $\varepsilon \boxdot \varepsilon < \vartheta$. For $r > 0$,

$\delta_{(n,m)}(G_{n,m}) := \{k \le n, l \le m : \mu_{(\mathfrak{R},v)}\left(\Delta s_{kl} - s, \frac{1}{2}\right) \le 1 - \varepsilon \text{ or } \varrho_{(\mathfrak{R},\mathcal{U})},\left(\left(\Delta s_{kl} - s, \frac{1}{2}\right) \ge \varepsilon, \zeta_{(\mathfrak{R},\mathcal{F})}\left(\left(\Delta s_{kl} - s, \frac{1}{2}\right) \ge \varepsilon\right\} = 0.$

Can be written, so $\delta_{(n,m)}(G^c_{n,m}) = 1$. For $j, p \in G^c_{n,m}$, $\mu_{(\mathfrak{R},v)}(\Delta s_{jp} - s, r) > 1 - \varepsilon$ *and* $\varrho_{(\mathfrak{R},\mathcal{U})}(\Delta s_{jp} - s, r) < \varepsilon$, $\zeta_{(\mathfrak{R},\mathcal{F})}(\Delta s_{jp} - s, r) < \varepsilon$.

Let

$H_{(n,m)} = \{k \le n, l \le m : \mu_{(\mathfrak{R},v)}(\Delta s_{kl} - \Delta s_{jp}, r) \le 1 - \vartheta \text{ or } \varrho_{(\mathfrak{R},\mathcal{U})}((\Delta s_{kl} - \Delta s_{jp}, r) \ge \vartheta, \zeta_{(\mathfrak{R},\mathcal{F})}\left((\Delta s_{kl} - \Delta s_{jp}, r) \ge \vartheta\right\}.$

It is necessary to show that $H_{(n,m)} \subset G_{(n,m)}$. So, to show this, let $u, w \in (H \cap G^c)$. In this case,

$\mu_{(\mathfrak{R},v)}(\Delta s_{uw} - \Delta s_{jp}, r) \le 1 - \vartheta$ and $\mu_{(\mathfrak{R},v)}\left(\Delta s_{uw} - s, \frac{r}{2}\right) > 1 - \vartheta$, especially $\mu_{(\mathfrak{R},v)}(\Delta s_{jp} - s, r) > 1 - \vartheta$. So,

$1 - \vartheta \ge \mu_{(\mathfrak{R},v)}(\Delta s_{uw} - \Delta s_{jp}, r) \ge \mu_{(\mathfrak{R},v)}\left(\Delta s_{uw} - s, \frac{r}{2}\right) \otimes \mu_{(\mathfrak{R},v)}\left(\Delta s_{jp} - s, \frac{r}{2}\right) > (1 - \varepsilon) \otimes (1 - \varepsilon) > 1 - \vartheta.$

However, this is not possible. Moreover,

$\varrho_{(\mathfrak{R},\mathcal{U})}(\Delta s_{uw} - \Delta s_{jp}, r) \ge \vartheta$ and $\varrho_{(\mathfrak{R},\mathcal{U})}\left(\Delta s_{uw} - s, \frac{r}{2}\right) < \vartheta$, especially $\varrho_{(\mathfrak{R},\mathcal{U})}\left(\Delta s_{jp} - s, \frac{r}{2}\right) < \vartheta$. Hence,

$\vartheta \le \varrho_{(\mathfrak{R},\mathcal{U})}(\Delta s_{uw} - \Delta s_{jp}, r) \le \varrho_{(\mathfrak{R},\mathcal{U})}\left(\Delta s_{uw} - s, \frac{r}{2}\right) \boxdot \varrho_{(\mathfrak{R},\mathcal{U})}\left(\Delta s_{jp} - s, \frac{r}{2}\right) < \varepsilon \boxdot \varepsilon < \vartheta$, which is impossible. With a similar technique, we can apply for $\zeta_{(\mathfrak{R},\mathcal{F})}(\Delta s_{uw} - \Delta s_{jp}, r)$. So, $H_{(n,m)} \subset G_{(n,m)}$ and $\delta_{(n,m)}(G_{n,m}) = 0$. Then, $\Delta_{kl} -$ statistical Cauchy convergence sequences in $(\mathcal{U}, \mu_{(\mathfrak{R},v)}, \varrho_{(\mathfrak{R},\mathcal{U})}, \zeta_{(\mathfrak{R},\mathcal{F})}, \otimes, \boxdot)$.

Definition 3.3 Let $(\mathcal{U}, \mu_{(\mathfrak{R},v)}, \varrho_{(\mathfrak{R},\mathcal{U})}, \zeta_{(\mathfrak{R},\mathcal{F})}, \otimes, \boxdot)$ be neutrosophic normed spaces. If every $\Delta_{kl} -$ statistical cauchy sequences is $\Delta_{kl} -$ statistical convergence in $(\mathcal{U}, \mu_{(\mathfrak{R},v)}, \varrho_{(\mathfrak{R},\mathcal{U})}, \zeta_{(\mathfrak{R},\mathcal{F})}, \otimes, \boxdot)$, then this spaces is called complete.

Theorem 3.3 Let $(\mathcal{U}, \mu_{(\Re,\nu)}, \varrho_{(\Re,\mathcal{U})}, \zeta_{(\Re,\mathcal{F})}, \otimes, \boxdot)$ be neutrosophic normed spaces. Then, every $\Delta_{kl}-$ statistical Cauchy sequences is $\Delta_{kl}-$ statistical convergence in this spaces.

Proof: Let (s_{kl}) be $\Delta_{kl}-$ statistical Cauchy but not $\Delta-$ statistical convergent on $(\mathcal{U}, \mu_{(\Re,\nu)}, \varrho_{(\Re,\mathcal{U})}, \zeta_{(\Re,\mathcal{F})}, \otimes, \boxdot)$. For a given $\varepsilon \in (0,1), choose\ \vartheta > 0\ such\ that\ (1-\varepsilon)\otimes(1-\varepsilon) > (1-\mu)$ and $\varepsilon \boxdot \varepsilon < \vartheta$. Then,

$\mu_{(\Re,\nu)}(\Delta s_{kl} - \Delta s_{jp}, r) \geq \mu_{(\Re,\nu)}\left(\Delta s_{kl} - s, \frac{r}{2}\right) \otimes \mu_{(\Re,\nu)}\left(\Delta s_{kl} - s, \frac{r}{2}\right) > (1-\varepsilon)\otimes(1-\varepsilon) > 1-\vartheta.$

$\varrho_{(\Re,\mathcal{U})}\left(\Delta s_{kl} - \Delta_{s_{jp}}, r\right) \leq \varrho_{(\Re,\mathcal{U})}\left(\Delta s_{kl} - s, \frac{r}{2}\right) \boxdot$
$\varrho_{(\Re,\mathcal{U})}\left(\Delta s_{jp} - s, \frac{r}{2}\right) < \varepsilon \boxdot \varepsilon < \vartheta,$

$\zeta_{(\Re,\mathcal{F})}(\Delta s_{kl} - \Delta s_{jp}, r) \leq \zeta_{(\Re,\mathcal{F})}\left(\Delta s_{kl} - s, \frac{r}{2}\right) \boxdot$
$\zeta_{(\Re,\mathcal{F})}\left(\Delta s_{jp} - s, \frac{r}{2}\right) < \varepsilon \boxdot \varepsilon < \vartheta.$

So, for

$I_{(n,m)} = \{k \leq n, l \leq m :\ \mu_{(\Re,\nu)}(\Delta s_k - \Delta s_p, r) \leq 1 - \vartheta\ or\ \varrho_{(\Re,\mathcal{U})}((\Delta s_k - \Delta s_p, r) \geq \vartheta,\ \zeta_{(\Re,\mathcal{F})}\left((\Delta s_k - \Delta s_p, r\right) \geq \vartheta\right\}$, and $\delta_{(n,m)}(I^c_{n,m}) = 0$; also, $\delta_{(n,m)}(I_{nm}) = 1$. Since (s_{kl}) $is\ \Delta-$ statistical Cauchy, this is impossible. Thus, $(s_k)\ is$ $\Delta-$ statistical convergent in $(\mathcal{U}, \mu_{(\Re,\nu)}, \varrho_{(\Re,\mathcal{U})}, \zeta_{(\Re,\mathcal{F})}, \otimes, \boxdot)$.

Result 1. If $(\mathcal{U}, \mu_{(\Re,\nu)}, \varrho_{(\Re,\mathcal{U})}, \zeta_{(\Re,\mathcal{F})}, \otimes, \boxdot)$ is neutrosophic normed spaces, then this spaces is complete.

Result 2. Let $(\mathcal{U}, \mu_{(\Re,\nu)}, \varrho_{(\Re,\mathcal{U})}, \zeta_{(\Re,\mathcal{F})}, \otimes, \boxdot)$ be neutrosophic normed spaces and (s_{kl}) be a double sequences in this spaces. Then, (s_{kl}) is a $\Delta_{kl}-$ statistical convergence sequences, (s_{kl}) is a $\Delta_{kl}-$ statistical Cauchy sequences, and $\Leftrightarrow(\mathcal{U}, \mu_{(\Re,\nu)}, \varrho_{(\Re,\mathcal{U})}, \zeta_{(\Re,\mathcal{F})}, \otimes, \boxdot)$ is complete neutrosophic normed spaces.

REFERENCES

1. Aiyub (2015). Strongly $\left(V^{\lambda}, A, \Delta^{n}_{(vm)},\ p, q\right)$-summable sequence spaces defined by modulus function and statistical convergence. *Proyecciones Journal of Mathematics*, *34*(2), 191-203.
2. Aizpuru, Listan-Garcia and Rambla-Barreno, F. (2014). Density by moduli and Statistical convergence. *Quaestiones Mathematicae*, *37*(4), 525-530.
3. Alotaibi and Alroqi (2012). Statistical convergence in a paranormed space. *Journal of Inequalities and Applications*, 39.
4. Altinok and Kasap (2016). $f-$Statistical Convergence of order β for Sequences of Fuzzy Numbers. *Math, F. A.* 1-10.
5. Altinok Altin and Isik (2018). Statistical convergence of order $\beta$for double sequences of fuzzy numbers defined by a modulus function. AIP Conference Proceedings, 1-6.
6. Belen and Yildirim (2015). Generalized statistical convergence and some sequence spaces in 2-normed spaces. *Hacettepe Journal of Mathematics and Statistics Volume 44 (3),* 513 – 519.
7. Bhardwaj and Dhawan (2017). Density by moduli and Wijsman lacunary statistical convergence of sequences of sets. *Journal of Inequalities and Applications*, 25, 1-20.
8. Brono and Ali (2016). λ_2 −Statistical Convergence in 2n-normed Spaces. *Far East Journal of Mathematical Science*, *99*(10), 1551-1569.
9. Brono and Ali (2016). A Matrix Characterisation of Cesáro $C_{1.1}-$ Statistically Convergent Double Sequences.*Far East Journal of Mathematical Sciences*, *99*(11), 1693-1702.
10. Brono and Ali (2016). I-Cesaro Statistical Core of Double Sequences. *IOSR Journal of Mathematics. IOSR Journal of Mathematics*, *12*(2), 102-108.
11. Brono and Ali (2016). On Statistical Convergence of Double Sequences and Statistical Monotonicity. *IOSR Journal of Mathematics*, *12*(1), 45-51
12. Burgin (2000). Theory of fuzzy limits, fuzzy sets and systems 115, 433-443
13. Bera and Mahapatra (2017). Neutrosophic soft normed linear spaces. *Neutrosophic Sets and Systems, 23(2018), 52–71.*
14. Bera and Mahapatra (2017). Neutrosophic soft linear spaces. *Fuzzy Information and Engineering, 9, 299–324.*
15. Barros Bassanezi and Tonelli (2000). Fuzzy modelling in population dynamics, Ecol Model, 128, 27–33.
16. Cakalli and Kaplan (2016). A variation on strongly lacunary ward continuity. *Journal of Mathematical Analysis*, *7*(3), 13-20.
17. Cakalli and Kaplan (2017). A variation on lacunary statistical quasi Cauchy sequences, Cummun. *Fac. Sci. Univ. Ank. Series A1*, *66*(2), 71-79.
18. Cakan and Altin (2015). Some classes of statistically convergent sequences of fuzzy numbers generated by a modulus function. *Iranian Journal of Fuzzy Systems*, *12*(3), 47-55.
19. Connor and Kline (1996). On statistical limit points and the consistency of statistical convergence, *Journal of Mathematical Anal Application.197,* 393-399.
20. Connor and Swardson (1993). Strong integral summability and stone-chech compactification of the half-line. *Pacific Journal of Mathematics.157, 201-224.*
21. Connor and Ganichev and Kadets (2000). A characterization of Banach spaces with separable duals via week statistical convergence, *Journal of Mathematical Analysis and Application. 244*(1), 251-261.

22. Granados and Dhital (2021). Statistical Convergence of Double Sequences in Neutrosophic Normed Spaces. *Neutrosophic Sets and Systems, Vol. 42, 344*
23. Das and Savas (2014). On I-statistical and I-lacunary statistical convergence of order α. *Bulletin of Iranian Mathematical Society, 40*(2), 459-472.
24. Das and Savas, E. (2014). On I-statistically pre-Cauchy sequences. *Taiwanese J. Math., 18*(1), 115-126.
25. Deepmala and Mishra (2016). The $\int \Gamma^{3\lambda I}$statistical Convergence of pre-Cauchy over the p-Metric Space Defined by Musielak Orlicz Function.
26. Duman, Khan and Orhan (2003). A −Statistical convergence of approximating operators, *Math. Inequal. Appl. 6*(4), 689-699.
27. Dutta (2013). A New Class of Strongly Summable and Statistical Convergence Sequences of Fuzzy Numbers *An International Journal of Applied Mathematics and Information Science,7*(6), 2369-2372.
28. Erkus and Dumam (2003). *A*-Statistical extension of the Korovkin type approximation Theorem Proc. *Indian Academic Science (Mathematical Sciences) 115(4), 499–507.*
29. Fast (1951). Sur la convergence statistique. Colloquium Mathematicum, 2 (3–4), 241–244.
30. Gumus (2015). Lacunary Weak I-Statistical Convergence. *Gen. Math. Notes, 28*(1), 50-58.
31. Gurdal and Ozgur (2015). A generalized statistical convergence via moduli. *Electronic Journal of Mathematical Analysis and Applications,3*(2),173-178.
32. George and Veeramani (1994). On some results in fuzzy metric spaces, Fuzzy Sets and Systems, 64, 395–399.
33. George and Veeramani (1997). On some results of analysis for fuzzy metric spaces, Fuzzy Sets and Systems, 90, 365–368.
34. Giles (1980). Computer program for fuzzy reasoning, Fuzzy Sets Syst., 4, 221–234.
35. Hazarika and Savas (2013). λ-statistical convergence in n-normed spaces.*An. St. Univ. Ovidius Constanta. 21*(2): 141-153.
36. Hong and Sun (2006). Bifurcations of fuzzy nonlinear dynamical systems, Commun Nonlinear Sci Numer Simul 1, 1–12.
37. Indu (2011). On Weak Statistical Convergence of Sequence of Functionals. *International Journal of Pure and Applied Mathematics.70 (5): 647-653*
38. Ilkhan and Kara (2018). A new type of Statistical Cauchy sequence and its relation to Bourkaki completeness. *Cogent Mathematics & Statistics, 5, 1-9.*
39. Kaya, Kucukaslan and Wagner (2013). On Statistical Convergence and Statistical Monotonicity.*Annales Univ. Sci. Budapest., Sect. Comp. 39:* 257-270.
40. Khan, Shafiq and Rababah (2015). On *I*-convergent sequence spaces defined by a compact operator and a modulus function. *Cogent Mathematics2,* 1-13.
41. Kukul (2014). $\alpha\beta$ −Statistical convergence. *Unpublished M. Sc. Dissertation, Eastern Mediterrenean University, Gazimagusa, North Cyprus.*
42. Kaleva and Seikkala (1984). On fuzzy metric spaces, Fuzzy Sets and Systems, 12, 215– 229.
43. Kirisci and Simsek (2020). Neutrosophic normed spaces and statistical convergence. *The Journal of Analysis, 28, 1059–1073.*
44. Mursaleen and Edely (2003). Statistical convergence of double sequences, *J. Math. Anal. Appl. 288 223–231.*
45. Maddox (1988). Statistical convergence in a locally convex space, *Math. Proc. Cambridge Phil. Soc. 104,* 141-145.
46. Malik and Ghosh (2017). On I-statistical cluster point of double sequences. *Functional Analysis,* 205-212.
47. Miller (1995). A measure theoretical subsequence characterization of statistical convergence, *Trans, Amer. Math. Soc. 347,* 1811-1819.
48. Madore (1992). Fuzzy physics, Ann Phys 219, 187–98.
49. Menger (1942). Statistical metrics. Proceedings of the National Academy of Sciences 28: 535– 537.
50. Nuray (2000). Generalized Statistical Convergence and Convergence Free Spaces. *Journal of Mathematical Analysis and Applications 245,*513-527.
51. Niven, Zuckerman and Montgomery (1991). *An introduction to the theory of numbers (5th ed.). New York, NY: Wiley.*
52. Nazmiye (2022). Hybrid Δ − statistical convergence for neutrosophic normed space. *Hindawi Journal of Mathematics, Volume 22, 10 Pages*
53. Pancaroglu and Nuray (2014). Invariant Statistical Convergence of Sequences of Sets with respect to a Modulus Function. *Hindawi Journal of Abstract and Applied Analysis.*
54. Park (2004). Intuitionistic fuzzy metric spaces, *Chaos, Solitons and Fractals, 22, 1039–1046.*
55. Raj and Sharma (2014). Some spaces of double difference sequences of fuzzy numbers. *Matematiqki Vesnik, 66*(1), 91–100.
56. Schoenberg (1959). The integrability of certain functions and related summability methods. *American Mathematics. Monthly 66:*361-375.

57. Steinhauss (1951). Sur la convergence orndinaireet la convergence asymptotique, *Colloq. Mathematics, 2:72-73.*
58. Steinhauss (1951). Sur la convergence ordinate et al convergence asymptotique. *Colloquium Mathematicum,* 2, 73–84.
59. Smarandache (2005). Neutrosophic set, a Generalisation of the Intuitionistic Fuzzy Sets, *International Journal of Pure and Applied Mathematics, 24, 287–297.*
60. Smarandache (1998). Neutrosophy, Neutrosophic Probability, Set, and Logic, *ProQuest Information and Learning. Ann Arbor, Michigan, USA.*
61. Smarandache (2016). Degree of Dependence and Independence of the (sub) Components of Fuzzy Set and Neutrosophic Set. *Neutrosophic Sets and Systems, Vol. 11, pp. 95-97*
62. Sariana and Dauda (2017). A Review on Neutrosophic Set and Its Development. *Menemui Matematik (Discovering Mathematics) Vol. 39, No. 2: 61-69*
63. Vakeel et al (2021). Space of Neutrosophic $\lambda -$ Statistical Convergence Sequences and their Properties. *Journal of Mathematics and Computer Science,* 23, 1-9
64. Vinod and Rani (2012). Weak ideal convergence in *lp* Spaces. *International Journal of Pure and Applied Mathematics. 75*(2), 247-256.
65. Yamanci and Gurdal (2014). I-statistically pre-Cauchy double sequences. *Global Journal of Mathematical Analysis, 2*(4), 297-303.
66. Zygmund (1979). Trigonometric series, second edition. *Cambridge University Press, Londo*
67. Zadeh (1965) Fuzzy sets, *Information and Control, 8(3), 338–353.*