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Solution of differential equation using by Sumudu transform

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ABSTRACT
In this paper we have discussed Infinite Sumudu transform as well as Laplace transform was applied to solve linear ordinary differential equations with constant coefficients with convolution terms.
Keywords: Integral transforms, Sumudu transform, Differential equation.

1. INTRODUCTION

In the literature there are several works on the theory and applications of integral transforms such as Laplace, Fourier, Mellin, Hankel, to name a few, but very little on the power series transformation such as Sumudu transform, probably because it is little known and not widely used yet. The Sumudu transform was proposed originally by Watugala (1993) to solve differential equations and control engineering problems. In Watugula (2002), the Sumudu transform was applied for functions of two variables. Some of the properties were established by

Weerakoon (1994, 1998). In Asiru (2002), further fundamental properties of this transform were also established. Similarly, this transform was applied to the one-dimensional neutron transport equation in Kadem (2005). In fact it was shown that there is strong relationship between Sumudu and other integral transform, see Kilicman *et al.* (2011). In particular the relation between Sumudu transform and Laplace transforms was proved in Kilicman (2011). Further, in Eltayeb *et al.* (2010), the Sumudu transform was extended to the distributions and some of their properties were also studied in Kilicman and Eltayeb (2010).

Recently, this transform is applied to solve the system of differential equations, see Kilicman *et al.* (2010). Note that a very interesting fact about Sumudu transform is that the

Original function and its Sumudu transform have the same Taylor coefficients except the factor

n, See Zhang (2007). Thus if $f(t)=\sum_{n=-\infty}^{\infty} a_n t^n$ then $F(u)=\sum_{n=-\infty}^{\infty} n!\, a_n u^n$, see Kilicman et al. (2011). Similarly, the Sumudu transform sends combinations, $C(m,n)$, into permutations, $P(m,n)$ and hence it will be useful in the discrete systems.

The Sumudu transform is defined by the formula

$$F(u)=s[f(t);u]$$

$$= \frac{1}{u} \int_{-\infty}^{\infty} e^{-\frac{t}{u}} f(t) dt,$$

$$u \epsilon (-\tau_1, \tau_2).$$

Over the set of

$$A = \left\{ f(t) \middle|_{if\ t \epsilon (-1)_j X(-\infty,\infty)}^{\exists M, and\ or/, \tau_2 > 0, such\ that\ |f(t)| < Me^{\frac{|t|}{\tau_j}}} \right\}$$

Our purpose in this study is to show the applicability of this interesting new transform and its efficiency in solving the linear ordinary differential equations with constant and non constant coefficients having the non homogenous term as convolutions.

Throughout this paper we need the following theorem which was given by Belgacem (2007), where they discussed the Sumudu transform of the derivatives:

Theorem 1:

Let $n \geq 1$ and let $G_n(u)$ be the Sumudu transforms of the $f^n(t)$. Then

$$G_n(u) = \frac{G(u)}{u^n} - \sum_{k=0}^{n-1} \frac{f^k(0)}{u^{n-k}}$$

for more details, see Belgacem (2007).

Since the transform is defined as improper integral therefore we need to discuss the existence and the uniqueness.

Theorem 2:

Let $f\ (t)$ and $g\ (t)$ be continuous functions defined for $t \geq 0$ and have Sumudu transforms, $F(u)$ and $G(u)$, respectively. If $F(u) = G(u)$ then $f(t) = g(t)$ where u is complex number.

Proof: If α are sufficiently large, then the integral representation of f by

$$f(t) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{\frac{t}{u}} F(u) du$$

Since $F(u) = G(u)$ almost everywhere then we have

$$f(t) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{\frac{t}{u}} G(u) du.$$

By using Laplace transform of the function $f\ (t\ )$ denoted by

$$F(s) = \int_{-\infty}^{\infty} e^{-st} f(t) dt,$$

Can be rewritten after a change of variable, $w = st$ with $dw = sdt$

$$F(s) = \int_{-\infty}^{\infty} e^{-w} f\left(\frac{w}{s}\right) \frac{dw}{s},$$

From above relation and replace u by $\frac{1}{s}$ we obtain the inverse Sumudu transform as follow

$$f(t) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{st} G\left(\frac{1}{s}\right) \frac{ds}{s} = g(t).$$

and the theorem is proven.

In the next theorem we study the existence of Sumudu transform as follows.

Theorem 3: (Existence of the Sumudu transform) If f is of exponential order, then its Sumudu transform $S\,[\,f\,(t); u] = F(u)$ is given by

$$F(u) = \frac{1}{u}\int_{-\infty}^{\infty} e^{-\frac{t}{u}} f(t)dt,$$

Where $\frac{1}{u} = \frac{1}{\eta} + \frac{i}{\tau}$. The defining integral for F exists at points $\frac{1}{u} = \frac{1}{\eta} + \frac{i}{\tau}$ in right half plane $\frac{1}{\eta} > \frac{1}{k}$ and $\frac{1}{\zeta} > \frac{1}{L}$.

Proof:

Using $\frac{1}{u} = \frac{1}{\eta} + \frac{i}{\tau}$ and we can express $F(u)$ as

$$F(u) = \int_{-\infty}^{\infty} f(t)\cos\left(\frac{t}{\tau}\right) e^{-\frac{t}{\eta}} dt$$

$$- i\int_{-\infty}^{\infty} f(t)\sin\left(\frac{t}{\tau}\right) e^{-\frac{t}{\eta}} dt.$$

Then for values of $\frac{1}{\eta} > \frac{1}{k}$, we have

$$\int_{-\infty}^{\infty} |f(t)| \left|\cos\left(\frac{t}{\tau}\right)\right| e^{-\frac{t}{\eta}} dt \le M\int_{-\infty}^{\infty} e^{\left(\frac{1}{K}-\frac{1}{\eta}\right)t} f(t)dt$$

$$\le \left(\frac{M\eta K}{\eta - K}\right)$$

And

$$\int_{-\infty}^{\infty} |f(t)| \left|\sin\left(\frac{t}{\tau}\right)\right| e^{-\frac{t}{\eta}} dt \le M\int_{-\infty}^{\infty} e^{\left(\frac{1}{K}-\frac{1}{\eta}\right)t} f(t)dt$$

$$\le \left(\frac{M\eta K}{\eta - K}\right).$$

Which imply that the integrals defining the real and imaginary parts of F exist for value of $Re\left(\frac{1}{u}\right) > \frac{1}{k}$, the derivation is complete.

Note: The function f on R is said to vanish below if there is a constant $c\epsilon R$ such that $f\,(t\,) = 0$ for $t < c$. The set of functions that are locally integrable and vanish below will be denoted by loc_- . Most of the functions we shall be concerned in this paper vanish for $t < 0$.

Theorem 4:

Let $\lambda > -1$ then

(1) If $f = loc_-$ and $\lim_{t\to\infty}\left[\frac{f(t)}{t^{\lambda}}\right]$ exists, so does $\lim_{\frac{1}{u}\to 0_-}\left[\frac{S[f(t);u]}{u^{\lambda+1}}\right]$. and we have $\lim_{t\to\infty}\left[\frac{f(t)}{t^{\lambda}}\right] =$ *1Γλ+1*lim*1u→0−Sft;uuλ+1.*

(2) If f is Sumudu transformable and satisfies $f\,(t\,) = 0$ for $t < 0$ and if $\lim_{t\to 0_-}\left[\frac{f(t)}{t^{\lambda}}\right]$ Exists, so does $\lim_{\frac{1}{u}\to\infty}\left[\frac{S[f(t);u]}{u^{\lambda+1}}\right]$. and we have $\lim_{t\to 0_-}\left[\frac{f(t)}{t^{\lambda}}\right] = \frac{1}{\Gamma(\lambda+1)}\lim_{\frac{1}{u}\to\infty}\left[\frac{S[f(t);u]}{u^{\lambda+1}}\right]$.

Proof:

(1) Let $\frac{f(t)}{t} \to \alpha\ as\ t \to \infty$. this implies that there are constants $A\ and\ \rho > 0.$ such that $\frac{|f(t)|}{t^{\lambda}} \le A\ for\ t > \rho$. this further implies that $e^{-\frac{t}{u}} f(t)$ is integrable for all $\frac{1}{u} > 0$ so that we may write, if $f(t) = 0\ \ for\ t < -\infty,$

$$S[f(t);u] = \int_{-\infty}^{\infty} e^{-\frac{1}{u}} f(t)dt$$

$$= \int_{-\infty}^{\rho} e^{-\frac{1}{u}} f(t)dt + \int_{\rho}^{\infty} e^{-\frac{1}{u}} f(t)dt. \qquad (1)$$

It is easy to see that $\frac{1}{u^{\lambda+1}}$ time the first term on the right of equation (1) tends to zero as $\frac{1}{u} \to 0_-$.

Then $\frac{1}{u^{\lambda+1}}$ times the second term on the right side of equation (1) may be written as follows

$$\frac{1}{u^{\lambda}} \int_{\frac{\rho}{u}}^{\infty} e^{-x} f(ux)dx$$

$$= \int_{\frac{\rho}{u}}^{\infty} x^{\lambda} e^{-x} \frac{f(ux)}{(ux)^{\lambda}} dx \; as \; \frac{1}{u} \to 0_-,$$

$$\frac{f(ux)}{(ux)^{\lambda}} \; tends \; to \; \alpha$$

And since it is bounded in the range of integration by the constant A , we may apply the dominated convergence theorem and conclude that

$$\frac{1}{u^{\lambda+1}} S[f(t);u] \to \int_{\frac{\rho}{u}}^{\infty} x^{\lambda} e^{-x} \alpha dx \; as \; \frac{1}{u} \to 0_-.$$

(2) Let $\frac{f(t)}{t^{\lambda}} \to \beta \; as \; t \to -\infty$. Since this function bounded in a neighborhood of zero then there are constants B and $\sigma > 0$ such that $\frac{|f(t)|}{t^{\lambda}} \leq B \; for \; -\infty < t < \sigma$. using a method similar to that in the proof of theorem (3) we let $f_0 = f(t)[1 - Ht-\sigma$ and $f1=ftHt-\sigma$.

Then

$$S[f(t);u] = \int_{-\infty}^{\sigma} e^{-\frac{t}{u}} f(t)\,dt + S[f_1(t);u] \qquad (2)$$

For $\frac{1}{u} \in domS[f]$ and apply $|S[f(t);u]| \leq Ae^{-\frac{c}{u}}$ then we have $|S[f_1(t);u]| \leq Ke^{-\frac{\sigma}{u}}$ for some constant K and $\frac{1}{u}$ sufficiently large, further $\frac{1}{u^{\lambda+1}} S[f_1(t);u] \to 0 \; as \; \frac{1}{u} \to \infty$. Also, by a similar argument to that was used in (1) $\frac{1}{u^{\lambda+1}}$ times the first term on the right hand side of equation (2) tends to $\beta\Gamma(\lambda+1) \; as \; \frac{1}{u} \to \infty$.

Now we let, f be a locally integrable function on R. We shall say that f is convergent (rather than integrable) if there is a constant k such that, for each, $\omega \epsilon\, D, \lim_{\lambda\to\infty} \left(\int \omega\left(\frac{t}{\lambda}\right) f(t)\,dt\right)$ exists and equal $k\omega(0)$ (where λ tends to infinity through) real values greater than zero. The constant k we shall denote by $\int f(t)\,dt$.

Other notations might also be used, for example if $f(t) = 0$ for $t < 0$, $\int f(t)\,dt$ will also be written as $\int_{-\infty}^{\infty} f(t)\,dt$.

That means $\int f(t)\,dt = \lim_{\lambda\to\infty} \left(\int \omega\left(\frac{t}{\lambda}\right) f(t)\,dt\right)$ for any $\omega \epsilon\, D$ such that, $\omega(0) = 1$.

Proposition 1: (Sumudu transform of derivative)

(1) Let f be differentiable on $(-\infty,\infty)$ and let $f(t)=0\ for\ t<0$. suppose that $f'\in L_{loc}$. then $f'\in L_{loc}, dom(Sf)\subset dom(f')$ and $S(f')=\frac{1}{u}S[f(t);u]-\frac{1}{u}f(0_-)$.

(2) For $u\in dom\,S(f)$. More generally, if f is differentiable on$(-\infty,\infty)$, the function $f(t)=0\ for\ t<0\ and\ f'\in L_{loc}\ then\ S[f'(t);u]=\frac{1}{u}S[f(t);u]-\frac{1}{u}e^{-\frac{c}{u}}f(0_-)\ for\ u\epsilon dom\,S(f)$.

Proof

We start by (2) as follow; the local integrability implies that $f(c+)$ exists since if $x>c$,

$$f(x) = f(c+1) - \int_x^{c+1} f'(t)\,dt \quad as\ x\to c_+ .$$

Let $u\in(dom\,S[f(t);u])$. if $\omega\epsilon\,D$ on using the integration by parts we have

$$\frac{1}{u}\int\omega\left(\frac{t}{\lambda}\right)e^{-\frac{t}{u}}f'(t)\,dt$$

$$=\frac{1}{u}\int_c^\infty\omega\left(\frac{t}{\lambda}\right)e^{-\frac{t}{u}}f'(t)dt$$

$$=\lim_{x\to c_+}\left[\frac{1}{u}\int_c^\infty\omega\left(\frac{t}{\lambda}\right)e^{-\frac{t}{u}}f'(t)dt\right]$$

$$=\lim_{x\to c_+}\left[\frac{1}{u}\omega\left(\frac{t}{\lambda}\right)e^{-\frac{t}{u}}f(t)dt\right]_{\boldsymbol{c}}^{\infty} - \lim_{x\to c_+}\left(\frac{1}{u}\int_c^\infty e^{-\frac{t}{u}}\left[\frac{t}{\lambda}\omega'\left(\frac{t}{\lambda}\right) - \frac{1}{u}\omega\left(\frac{t}{\lambda}\right)\right]f(t)dt\right)$$

$$=\lim_{x\to c_+}\left[-\frac{1}{u}\omega\left(\frac{x}{\lambda}\right)e^{-\frac{x}{u}}f(x)\right] - \left(\frac{1}{u}\int_c^\infty e^{-\frac{t}{u}}\left[\frac{t}{\lambda}\omega'\left(\frac{t}{\lambda}\right) - \frac{1}{u}\omega\left(\frac{t}{\lambda}\right)\right]f(t)dt\right).$$

He first term on the right hand side is given by $-\frac{1}{u}\omega\left(\frac{c}{\lambda}\right)e^{-\frac{c}{u}}f(c_+)$ which tends to $-\frac{1}{u}\omega(0)e^{-\frac{c}{u}}f(c_+)\ as\ \lambda\to\infty$. Then the second term is given by

$$\left(\frac{1}{u}\int_c^\infty e^{-\frac{t}{u}}\left[\frac{t}{\lambda}\omega'\left(\frac{t}{\lambda}\right) - \frac{1}{u}\omega\left(\frac{t}{\lambda}\right)\right]f(t)dt\right).$$

This tends to $0_+\frac{1}{u}\omega(0)S(f)\ as\ \lambda\to\infty$.

We have this proved for any $\omega\epsilon\,D$,

$$\lim_{x\to c_+}\left[\frac{1}{u}\int\omega\left(\frac{t}{\lambda}\right)e^{-\frac{t}{u}}f'(t)\,dt\right] = \frac{\omega(0)}{u}[S(f) - f(c_+)].$$

This implies that $e^{-\frac{t}{u}}f'$ is convergent, that is, $u\in dom[S(f(t);u)]$, and that

$$S[f'(t);u] = \frac{1}{u}[S(f(t);u)] - \frac{1}{u}e^{-\frac{c}{u}}f(c_+)$$

In order to prove part (1), we just replace c by zero.

In general case, if f be differentiable on $(a,b)\ with\ a<b\ and\ f(t)=0\ for\ t<a\ or\ t>b\ and\ f'\in L_{loc}$ then, for all u

$$S[f'(t);u] = \frac{1}{u}[S(f(t);u)] - \frac{1}{u}e^{-\frac{a}{u}}f(a_+) + \frac{1}{u}e^{-\frac{b}{u}}f(b_-).$$

In the following example we use differential equation and Sumudu transform.

Example: Let $g(t) = \cosh(\sqrt{t})$. the Sumudu transform of $f = g$ is required.

We first obtain a differential equation satisfied by g. we have

$$g'(t) = -\frac{\sinh(\sqrt{t})}{2\sqrt{t}}.$$

Hence

$$tg'(t) = -\frac{\sqrt{t}\sinh(\sqrt{t})}{2}.$$

Therefore

$$\frac{d[tg'(t)]}{dt} = -\frac{1}{4}\cosh(\sqrt{t}) - \frac{\sinh(\sqrt{t})}{2(2\sqrt{t})}. = -\frac{g(t)}{4} + \frac{g'(t)}{2}, \quad t > 0.$$

If we now write $f = g$ we shall have, for any $t \neq 0$,

$$\frac{d[tf'(t)]}{dt} = -\frac{f(t)}{4} + \frac{f'(t)}{2}. \qquad (3)$$

Now by taking Sumudu transform of equation (3) we have

$$\frac{1}{u}S[tf'(t);u] - k = -\frac{S[f(t);u]}{4} + \frac{S[f'(t);u]}{2}.$$

Where

$k = \lim_{t\to 0_+}(tf'(t))$. $clearly\ f(0) = 0\ and\ k = \left[-\frac{\sqrt{t}\sinh(\sqrt{t})}{2}\right] = 0$,

Follows that

$$\frac{1}{u}S[tf'(t);u] = -\frac{S[f(t);u]}{4} + \frac{S[f(t);u]}{2u}. \qquad (4)$$

Now on using the proposition (1) the left hand side of equation (4) becomes

$$u\frac{d}{du}S[f'(t);u] = -\frac{S[f(t);u]}{4} + \frac{S[f(t);u]}{2u}.$$

By simplification we have

$$\frac{F'(u)}{F(u)} = -\frac{1}{4} + \frac{3}{2u}.$$

i.e.

$$F(u) = C'e^{\frac{1}{4}u}\sqrt{(u^3)}.$$

Now by replacing $u\ by\ \frac{1}{s}$ and in order to find the value of C' we apply the formula

$$\lim_{t\to 0_+}\frac{f(t)}{\sqrt{t}} = \lim_{t\to 0_+}\frac{\cosh(\sqrt{t})}{\sqrt{t}} = 1$$

Therefore

$$F\left(\frac{1}{s}\right) = C'e^{\frac{1}{4s}}\sqrt{\left(\left(\frac{1}{s}\right)^3\right)}.$$

Then

$$\lim_{s\to\infty}\frac{(s)^{\frac{3}{2}}F\left(\frac{1}{s}\right)}{\Gamma\left(\frac{3}{2}\right)} = 1$$

Since

$$\frac{(s)^{\frac{3}{2}}F\left(\frac{1}{s}\right)}{\Gamma\left(\frac{3}{2}\right)} = \frac{C' e^{\frac{1}{4s}}}{\Gamma\left(\frac{3}{2}\right)} \to \frac{C'}{\Gamma\left(\frac{3}{2}\right)}$$

As $s \to \infty$ and we have $C' = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$, thus finally we obtain

$$S\left[-\frac{\cosh(\sqrt{t})}{(\sqrt{t})}; u\right] = \frac{\sqrt{\pi} e^{\frac{1}{4s}}}{2s}.$$

Before we extend proposition (1) to higher derivatives, we introduce the following notation: Let $P(x) = \sum_{k=0}^{n} \frac{a_k}{x^k}$ be a polynomial in x, where $n \geq 0$ *and* $a_n \neq 0$. We define $M_p(x)$ to be the $1 \times n$ matrix of polynomials which is given by the following matrix product:

$$M_p(x) = \left(\frac{1}{x}\frac{1}{x^2}\frac{1}{x^3}\cdots\frac{1}{x^{n-1}}\right)\begin{pmatrix} a_1 & a_2 & .\ . & . & a_n \\ a_2 & a_3 & .\ . & a_n & 0 \\ . & . & .a_n & 0 & 0 \end{pmatrix} \quad (5)$$

For each complex number $M_p(x), x$ define a linear mapping of c^n into c in obvious way.

CONCLUSION

In this study the applications of the Sumudu (Laplace transform also) transform to the solution of differential equations with constant and non-constant coefficients have been demonstrated.

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