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ANNIHILATORS AND MAXIMISORS IN ADL'S

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Abstract

The notion of maximisor of a subset of an Almost Distributive Lattice (ADL) is introduced and certain properties of these are discussed analogous to those of annihilators. Mainly, Almost Boolean algebras (ABA's) are characterized interms of their annihilators and maximisors.

Keywords: Boolean algebra, Almost Distributive Lattice (ADL), Almost Boolean algebra (ABA), associate elements, annihilator and maximisor.

1 Introduction

It is well-known that a complemented distributive lattice is called a Boolean algebra and that a ring with unity, in which every element is an idempotent,

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is called a Boolean ring. Swamy, and Rao [5] have introduced the notion of an Almost Distributive Lattice (ADL) as a common abstraction of several lattice theoretic and ring theoretic generalizations of Boolean algebras (Boolean rings). In this paper, we introduce the concept of maximisor of a subset of an ADL and prove certain properties of these analogous to those of annihilator ideals.

An Almost Boolean algebra [5, 6] is an ADL $(A, \wedge, \vee, 0)$ with a maximal element satisfying the condition that for any $a, b \in A$, there exists $x \in A$ such that $a \wedge x = 0$ and $a \vee x = a \vee b$ and this is equivalent to the condition that to each $a \in A$, there exists $b \in A$ such that $a \wedge b = 0$ and $a \vee b$ is maximal; here b is called a complement of a and note that b is not unique.
We characterize ABA's interms of their annihilators and maximisors. Also, we define the set $c(a)$ of all complements of an element a in an ABA A and it is proved that the set $\{c(a) : a \in A\}$ is a Boolean algebra under the induced operations.

2 Preliminaries

Definition 2.1. An algebra $A = (A, \wedge, \vee, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (abbreviated as ADL) if it satisfies the following identities

(1). $0 \wedge a = 0$

(2). $a \vee 0 = a$

(3). $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(4). $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

(5). $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(6). $(a \vee b) \wedge b = b$.

Any distributive lattice bounded below is an ADL, where 0 is the smallest element. Also, a commutative regular ring $(R, +, ., 0, 1)$ with unity can be made into an ADL by defining the operations $\wedge$ and $\vee$ on R by

$$a \wedge b = a_0 b \text{ and } a \vee b = a + b - a_0 b,$$

where, for any $a \in R$, a_0 is the unique idempotent in R such that $aR = a_0 R$ and 0 is the additive identity in R. Further any nonempty set X can be made into an ADL by fixing an arbitrarily choosen element 0 in X and by defining the operations $\wedge$ and $\vee$ on X by

$$a \wedge b = \begin{cases} 0, & \text{if } a = 0 \\ b, & \text{if } a \neq 0 \end{cases} \quad \text{and} \quad a \vee b = \begin{cases} b, & \text{if } a = 0 \\ a, & \text{if } a \neq 0. \end{cases}$$

This ADL $(X, \wedge, \vee, 0)$ is called the discrete ADL. An ADL A is said to be an associative ADL if the operation $\vee$ on A is associative. Throughout this paper, by an ADL we always mean an associative ADL only.

Definition 2.2. Let $A = (A, \wedge, \vee, 0)$ be an ADL. For any a and $b \in A$, define

$$a \leq b \text{ if and only if } a = a \wedge b \ (\Leftrightarrow a \vee b = b).$$

Then $\leq$ is a partial order on A.

Theorem 2.3. *The following hold for any a, b and c in an ADL A.*

(1) 0 *is the zero element for the operation* $\wedge$ *(that is,* $a \wedge 0 = 0 = 0 \wedge a$*)*

(2) 0 *is the identity for the operation* $\vee$ *(that is,* $a \vee 0 = a = 0 \vee a$*)*

(3) $a \wedge a = a = a \vee a$

(4) $a \wedge b \leq b \leq b \vee a$

(5) $a \wedge b = a \Leftrightarrow a \vee b = b$ *and* $a \wedge b = b \Leftrightarrow a \vee b = a$

(6) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ *(i.e.,* $\wedge$ *is associative)*

(7) $a \vee (b \vee a) = a \vee b$

(8) $a \leq b \Rightarrow a \wedge b = a = b \wedge a$ *and* $a \vee b = b = b \vee a$

(9) $(a \wedge b) \wedge c = (b \wedge a) \wedge c$ *and* $(a \vee b) \wedge c = (b \vee a) \wedge c$

(10) $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$ *whenever* $a \wedge b = 0$

(11) $a \wedge b = b \wedge a \Leftrightarrow a \vee b = b \vee a \Leftrightarrow a \wedge b = \inf\{a, b\} \Leftrightarrow a \vee b = \sup\{a, b\}$.

An element $m \in A$ is said to be maximal if $m \leq x$ implies $m = x$.

Theorem 2.4. *Let A be an ADL and $\leq$ be the induced partial order. Then the following are equivalent*

(1) m *is a maximal element in* $(A, \leq)$

(2) $m \wedge a = a$ *for all* $a \in A$

(3) $m \vee a = m$ *for all* $a \in A$

(4) $a \vee m$ *is maximal for all* $a \in A$.

Definition 2.5. Let A be an ADL.

(1) A non empty subset I of A is called an ideal of A if a and $b \in I \Rightarrow a \vee b \in I$ and $a \wedge x \in I$ for all $x \in A$.

(2) A non empty subset F of A is called a filter of A if a and $b \in F \Rightarrow a \wedge b \in F$ and $x \vee a \in F$ for all $x \in A$.

It follows as a consequence that, for any ideal I of A, $x \wedge a \in I$ for all $a \in I$ and $x \in A$ and, for any filter F of A, $a \vee x \in F$ for all $a \in F$ and $x \in A$. For any $X \subseteq A$, we denote the ideal generated by X (the smallest ideal of A containing X) by $(X]$ and the filter generated by X by $[X)$. We have

$$(X] = \Big\{ \big(\bigvee_{i=1}^{n} x_i \big) \wedge a \mid n \geq 0,\ x_i \in X \text{ and } a \in A \Big\}$$

$$\text{and} \quad [X) = \Big\{ a \vee \big(\bigwedge_{i=1}^{n} x_i \big) \mid n \geq 0,\ x_i \in X \text{ and } a \in A \Big\}.$$

In particular, when $X = \{x\}$, we have

$$(X] = (x] = \{x \wedge a \mid a \in A\} \text{ and } [X) = [x) = \{a \vee x \mid a \in A\}.$$

and these are called the principal ideal (filter respectively) generated by x in A.

3 Maximisors

In this section, we define the concept of maximisor (in other words, dual annihilator) of a subset of an ADL and prove certain properties analogous to those of annihilator ideals [3].

Definition 3.1. Let A be an ADL with a maximal element. For any $X \subseteq A$, define

$$X^{+} = \{a \in A : a \vee x \text{ is maximal for all } x \in X\}$$

X^{+} is called the **maximisor** of X in A. If X is a singleton set $\{x\}$, we simply write x^{+} for $\{x\}^{+}$.

Note that, for any elements a and b of an ADL A, $a \vee b$ is maximal if and only if $b \vee a$ is maximal (since $(a \vee b) \wedge x = (b \vee a) \wedge x$ for all $x \in A$). Therefore, $X^+ = \{a \in A : x \vee a \text{ is maximal for all } x \in X\}$.

Theorem 3.2. *An ADL A has maximal elements if and only if S^+ is a filter of A for all $S \subseteq A$.*

Corollary 3.3. *Let A be an ADL with maximal elements. Then the following hold good.*

(1) $A^+ =$ The set of all maximal elements in A

(2) For any $S \subseteq A$, S^+ contains all the maximal elements of A

(3) $\phi^+ = A$

(4) If M is any set of maximal elements, then $M^+ = A$

(5) The set of all maximal elements is the smallest filter of A.

Theorem 3.4. *Let A be an ADL with maximal elements. Then the following hold good for any subsets S and T of A.*

(1) $S \cap S^+ \subseteq M$, the set of maximal elements of A.

(2) $(S \cup T)^+ = S^+ \cap T^+$

(3) $S \subseteq S^{++}$

(4) $S \subseteq T \Rightarrow T^+ \subseteq S^+$

(5) $S^{+++} = S^+$

(6) $S^+ = [S)^+$.

Proof. (1): $m \in S \cap S^+ \Rightarrow m \in S$ and $s \vee m$ is maximal for all $s \in S$
$\Rightarrow m \vee m (= m)$ is maximal since $m \in S$.
Thus $S \cap S^+ \subseteq M$.

(2) is trivial

(3): $s \in S \Rightarrow s \vee x$ is maximal for all $x \in S^+ \Rightarrow s \in S^{++}$.
Therefore $S \subseteq S^{++}$.

(4): follows from (2).

(5): follows from (3) and (4).

(6): By (4), $[S)^+ \subseteq S^+$. On the other hand, let $x \in S^+$. Then, $s \vee x$ is maximal for all $s \in S$. Let $y \in [S)$. Then by 2.5, $y = a \vee (\bigwedge_{i=1}^{n} s_i)$,

$n > 0, s_i \in S$ and $a \in A$. For any $z \in A$, $(y \vee x) \wedge z = (a \vee (\bigwedge_{i=1}^{n} s_i) \vee x) \wedge z = \bigwedge_{i=1}^{n} (a \vee s_i \vee x) \wedge z = z$ since each $a \vee s_i \vee x$ is maximal. Therefore $y \vee x$ is maximal for all $y \in [S)$ and hence $x \in [S>^+$. This implies that $S^+ \subseteq [S>^+$. Thus $S^+ = [S>^+$. □

Definition 3.5. A filter F of an ADL A is called a maximizing filter if $F = S^+$ for some $S \subseteq A$. We denote the set of all maximizing filters of A by $\mathcal{M}(A)$.

The following is an important elementary property of maximisors.

Theorem 3.6. *For any filters F and G of an ADL A in which M is the smallest filter,*

$$F \cap G = M \Leftrightarrow G \subseteq F^+ \Leftrightarrow F \subseteq G^+.$$

Proof. Suppose that $F \cap G = M$. Then for any $x \in G$ and $a \in F$, $a \vee x \in F \cap G = M$ and hence $a \vee x$ is maximal. Therefore $G \subseteq F^+$. Similarly, $F \subseteq G^+$. Conversely, suppose that $F \subseteq G^+$. Then

$$x \in F \cap G \Rightarrow x = x \vee x \text{ is maximal } \Rightarrow x \in M.$$

Therefore $F \cap G \subseteq M$ and hence $F \cap G = M$. □

Recall that $F \in \mathcal{M}(A)$ if and only if $F = S^+$ for some $S \subseteq A$. In the following we prove that the set $\mathcal{M}(A)$ is a complete Boolean algebra that is, it is a complete lattice which is distributive and complemented.

Theorem 3.7. *Let A be an ADL with maximal elements. Then the set $\mathcal{M}(A)$ of all maximizing filters of A forms a complete Boolean algebra in which the lattice operations are as follows: If $\{F_\alpha\}_{\alpha\in\Delta} \subseteq \mathcal{M}(A)$, then*

$$inf\{F_\alpha\}_{\alpha\in\Delta} = \bigcap_{\alpha\in\Delta} F_\alpha \text{ and } sup\{F_\alpha\}_{\alpha\in\Delta} = (\bigcup_{\alpha\in\Delta} F_\alpha)^{++}.$$

Proof. First we observe that M and A are the smallest and the greatest elements respectively in $\mathcal{M}(A)$ since $M = A^+$ and $A = M^+$. Therefore $(\mathcal{M}(A), \subseteq)$ is a bounded poset. Let $\{F_\alpha\}_{\alpha\in\Delta} \subseteq \mathcal{M}(A)$ where $F_\alpha = S_\alpha^+$ for some $S_\alpha \subseteq A$. Then

$$\bigcap_{\alpha\in\Delta} F_\alpha = \bigcap_{\alpha\in\Delta} S_\alpha^+ = (\bigcup_{\alpha\in\Delta} S_\alpha)^+ \in \mathcal{M}(A)$$

and hence $\mathcal{M}(A)$ is closed under arbitrary intersections. Therefore $inf\{F_\alpha\}_{\alpha\in\Delta} = \bigcap_{\alpha\in\Delta} F_\alpha$. Now, $sup\{F_\alpha\}_{\alpha\in\Delta} = \cap\{F \in \mathcal{M}(A) : F_\alpha \subseteq F \text{ for all } \alpha \in \Delta\} =$

$\cap\{F \in \mathcal{M}(A) : \bigcup_{\alpha\in\Delta} F_\alpha \subseteq F\} = (\bigcup_{\alpha\in\Delta} F_\alpha)^{++}$. Thus $(\mathcal{M}(A), \subseteq)$ is a complete lattice. The distributivity of maximisors can be easily proved by using the lattice operations on $\mathcal{M}(A)$. Further, for any $F \in \mathcal{M}(A)$, we have $F^+ \in \mathcal{M}(A)$ such that $F \cap F^+ = M$ and $F \vee F^+ = (F \cup F^+)^{++} = M^+ = A$. Therefore F^+ is the complement of F in $\mathcal{M}(A)$. Thus $\mathcal{M}(A)$ is a complete Boolean algebra. □

4 Annihilators and Maximisors

Let us recall from [5, 6] that a non- trivial ADL A is said to be an Almost Boolean algebra (ABA) if it has a maximal element and satisfies the condition that, for any $a, b \in A$, there exist $x \in A$ such that

$$a \wedge x = 0 \text{ and } a \vee x = a \vee b.$$

In this section, we characterize Almost Boolean algebras interms of its annihilators (ideals) and maximisors (filters).
As defined in [3], for any element a in an ADL A, the annihilator a^* is defined as

$$a^* = \{x \in A : a \wedge x = 0\}$$

and recall that the maximisor a^+ is defined as

$$a^+ = \{x \in A : x \vee a \text{ is maximal in } A\}$$

The following is a characterization of Almost Boolean algebra interms of their complemented ideals.

Theorem 4.1. *Let A be an ADL with a maximal element. Then A is an Almost Boolean algebra if and only if every principal ideal of A is complemented in the lattice of ideals of A.*

Proof. Suppose that A is an ABA. Let I be a principal ideal of A. Then $I =< x]$ for some $x \in A$. Now we prove that $< x] \cap x^* = \{0\}$ and $< x] \ \vee \ x^* = A$. For, $a \in < x] \ \cap \ x^* \Rightarrow a = x \wedge a$ and $x \wedge a = 0 \Rightarrow a = 0$.
Therefore $< x] \cap x^* = \{0\}$. Let $y \in A$. Then there exists $a \in A$ such that $x \wedge a = 0$ and $x \vee a = x \vee y$ and hence $y = (x \vee y) \wedge y = (x \vee a) \wedge y = (x \wedge y) \vee (a \wedge y)$. Now, $x \wedge y \in < x]$ and $a \wedge y \in x^*$ (since $x \wedge a \wedge y = 0 \wedge y = 0$). Therefore $y \in < x] \vee x^*$. This implies $< x] \vee x^* = A$. Therefore x^* is a complement of $< x]$ and hence $< x]$ is complemented.
Conversely suppose that every principal ideal is complemented. $a, b \in A$. Then there exist an ideal J of A such that $< a] \cap J = \{0\}$ and $< a] \vee J = A$.

Now $a \vee b \in A$ and hence $a \vee b = (a \wedge x) \vee y$ for some $x \in A$ and $y \in J$. Then $a \wedge y \in (a] \cap J = \{0\}$ and therefore $a \wedge y = 0$ and $a \vee y = a \vee (a \wedge x) \vee y = a \vee a \vee b = a \vee b$. Thus A is an ABA. □

In the above, every principal ideal is complemented in any ABA. Infact, the converse holds good in any ADL with a maximal element.

Theorem 4.2. *Let A be an ADL with a maximal element m. Then every complemented ideal of A is a principal ideal.*

Proof. Let I be a complemented ideal of A. Then $I \cap J = \{0\}$ and $I \vee J = A$ for some ideal J of A. Now $m \in A = I \vee J$ and hence $m = a \vee b$ for some $a \in I$ and $b \in J$. Now we prove that $I =< a]$. Clearly $< a] \subseteq I$. On the other hand, $x \in I \Rightarrow x = m \wedge x$ (since m is maximal)

$\Rightarrow x = (a \vee b) \wedge x = (a \wedge x) \vee (b \wedge x)$

$\Rightarrow x = a \wedge x$ since $b \wedge x = 0$

$\Rightarrow x \in< a]$.

Thus $I =< a]$ and hence I is a principal ideal. □

Corollary 4.3. *Let A be an ABA and $I \in \mathcal{I}(A)$. Then I is complemented if and only if I is a principal ideal.*

The following is a characterization of Almost Boolean algebra in terms of it's maximisors (filters).

Theorem 4.4. *Let A be an ADL with a maximal element. Then the following are equivalent.*

(1) A is an Almost Boolean algebra.

(2) For any $a \in A$, $[a > \vee\ a^+ = A$

(3) Every principal filter in A is complemented.

Proof. (1) $\Rightarrow$ (2) : Let $a \in A$ and m be a maximal element in A.
Then by (1), there exists $x \in A$ such that $a \wedge x = 0$ and $a \vee x = a \vee m$. Since m is maximal, so is $a \vee m$ and hence $a \vee x$ is maximal. Therefore $x \in a^+$ and $a \wedge x = 0$. This implies that $0 \in [a > \vee a^+$ and hence $[a > \vee a^+ = A$.
(2) $\Rightarrow$ (3) : is trivial since $[a > \cap\ a^+ = M$, the smallest filter of A.
(3) $\Rightarrow$ (1) : Let $a, b \in A$. Then, by (3), there exists a filter F of A such that $[a > \cap\ F = M$ and $[a > \vee\ F = A$. Since $0 \in A$, we get $0 = (y \vee a) \wedge z$ for some $y \in A$ and $z \in F$. Put $x = z \wedge (a \vee b)$. Then $a \vee z \in [a > \cap F = M$.
Now, $\quad a \wedge x = a \wedge z \wedge (a \vee b)$

$= (y \vee a) \wedge a \wedge z \wedge (a \vee b)$

$$= (y \vee a) \wedge z \wedge a \wedge (a \vee b)$$
$$= 0 \wedge a = 0$$
and $$a \vee x = a \vee (z \wedge (a \vee b))$$
$$= (a \vee z) \wedge (a \vee a \vee b)$$
$$= (a \vee z) \wedge (a \vee b)$$
$$= a \vee b.$$
Thus A is an ABA. □

Corollary 4.5. *Let A be an ABA and $a \in A$. Then a^+ and $[a >$ are complements to each other in the lattice $\mathcal{F}(A)$ of filters of A.*

In the above, we have proved that any principal filter in an ABA is complemented. The converse of this is true in any ADL with maximal elements.

Theorem 4.6. *Let A be an ADL with maximal elements and F and G are filters in A such that $F \cap G = M$ and $F \vee G = A$. Then $F = [a >$ and $G = a^+$ for some $a \in A$.*

Proof. Since $0 \in A$, we get that $0 = a \wedge b$ for some $a \in F$ and $b \in G$. Now, we prove that $F = [a >$ and $G = a^+$. Clearly $[a > \subseteq F$. On the other hand,
$$x \in F \Rightarrow x = x \vee (b \wedge a) \quad (\text{since } a \wedge b = 0 = b \wedge a)$$
$$= (x \vee b) \wedge (x \vee a)$$
$$= x \vee a \quad (\text{since } x \vee b \text{ is maximal, because } x \vee b \in F \cap G = M).$$
and hence $x \in [a >$. Thus $F = [a >$. Similarly, $G = [b >$. Since $F \cap G = M$, we get that $G \subseteq F^+ = a^+$.
Now, $$x \in a^+ \Rightarrow x \vee a \quad \text{is maximal, } x \vee a \in F \cap G$$
$$\Rightarrow (x \vee a) \wedge b \in G$$
$$\Rightarrow x \wedge b \in G \quad (\text{since } a \wedge b = 0)$$
$$\Rightarrow x = x \vee (x \wedge b) \in G.$$
Thus $G = a^+$ and $F = [a >$. □

Corollary 4.7. *Let A be an ABA and F a filter of A. Then F is complemented in the lattice $\mathcal{F}(A)$ if and only if F is a principal filter of A.*

Corollary 4.8. *Let A be an ABA and $a \in A$. Then there exists $b \in A$ such that $a^+ = [b >$.*

Theorem 4.9. *Let A be an ADL with a maximal element. Then A is an Almost Boolean algebra if and only if $a^* \cap a^+$ is non-empty for all $a \in A$.*

Proof. Suppose that A is an ABA. Let m be a maximal element in A. Then, for any $a \in A$, there exists $x \in A$ such that $a \wedge x = 0$ and $a \vee x = a \vee m$. Since m is maximal so is $a \vee m$ and hence $a \vee x$ is maximal. This implies $x \in a^* \cap a^+$ and hence $a^* \cap a^+ \neq \phi$. Conversely, suppose that $a^* \cap a^+ \neq \phi$ for all $a \in A$. Let

m be a maximal element in A. Then clearly $[0, m]$ is a bounded distributive lattice. Let $a \in [0, m]$ and choose $x \in a^* \cap a^+$. Then $a \wedge x = 0$ and $x \vee a$ is maximal. Put $b = x \wedge m$. Then $b \in [0, m]$ and $a \wedge b = a \wedge x \wedge m = 0 \wedge m = 0$ and $a \vee b = a \vee (x \wedge m) = (a \vee x) \wedge (a \vee m) = a \vee m = m$. Therefore b is the complement of a in $[0, m]$. Thus $[0, m]$ is a Boolean algebra for all maximal m in A and hence A is a Boolean algebra. □

5 The Complements

For any element a in a Boolean algebra it well known that the annihilator a^* is precisely the ideal generated by the complement a' of a and the maximisor a^+ is precisely the filter generated by a'. These imply that $a^* \cap a^+ = \{a'\}$. This may not hold good in case of an Almost Boolean algebra. However, in this section we prove that $a^* \cap a^+$ is a congruence class corresponding to the associativity relation $\sim$ [1].

Definition 5.1. Let A be an ADL and for any $a \in A$, define the set $c(a)$ by

$$c(a) = a^* \cap a^+$$

and $c(a)$ is called the set of complements of a.

Theorem 4.9 says that an ADL A with a maximal element is an ABA if and only if $c(a)$ is non-empty for all $a \in A$. Infact $c(a)$ is an associative class, as proved below. First, let us recall from [1], that two elements a and b in an ADL A are said to be associates to each other if $a \wedge b = b$ and $b \wedge a = a$; in this case we write $a \sim b$. Also the relation $\sim$ is a congruence relation on A such that the quotient $A/\sim$ is a lattice.

Theorem 5.2. *Let A be an ABA and $a \in A$. Then $c(a)$ is an associate class.*

Proof. For any $x, y \in c(a)$ we have $a \wedge x = 0 = a \wedge y$ and $x \vee a$ and $y \vee a$ are maximal and hence $x \wedge y = (x \wedge y) \vee 0 = (x \wedge y) \vee (a \wedge y) = (x \vee a) \wedge y = y$. Similarly, $y \wedge x = x$. Therefore $x \sim y$ for all $x, y \in c(a)$. Now, by the theorem 4.9, $c(a) \neq \phi$ and hence we can choose $b \in c(a)$. Then, by the above argument $x \sim b$ for all $x \in c(a)$. Therefore $c(a) \subseteq\ \sim(b)$.
Let $x \in\ \sim(b)$. Then $x \sim b$ and hence $x \wedge b = b$ and $b \wedge x = x$. Now, $a \wedge x = a \wedge (b \wedge x) = (a \wedge b) \wedge x = 0 \wedge x = 0$. For any $y \in A$, consider

$$\begin{aligned}(x \vee a) \wedge y &= ((x \vee b) \vee a) \wedge y \\ &= (a \vee b \vee x) \wedge y \\ &= (a \vee b) \wedge y \qquad (\text{since } b \wedge x = x,\ b \vee x = b) \\ &= y \qquad (\text{since } b \in c(a),\ a \vee b \text{ is maximal }).\end{aligned}$$

Therefore $a \wedge x = 0$ and $x \vee a$ is maximal for all $x \in \sim(b)$. This implies $\sim(b) \subseteq c(a)$. Thus $c(a) = \sim(b)$ for any $b \in \sim(a)$. □

Finally we prove that the class $\{c(a) : a \in A\}$ forms a Boolean algebra. First we have the following.

Theorem 5.3. *Let A be an ADL with a maximal element m and $a, b \in A$. Then*

(1) $(a \vee b)^* = a^* \cap b^* = (b \vee a)^*$

(2) $0^* = A$ *and* $m^* = \{0\}$

(3) $(a \wedge b)^+ = a^+ \cap b^+ = (b \wedge a)^+$

(4) $m^+ = A$ *and* $0^+ = M$*, the set of maximal elements in* A.

Proof. (1). $x \wedge a = 0 = x \wedge b \Rightarrow x \wedge (a \vee b) = (x \wedge a) \vee (x \wedge b) = 0 \vee 0 = 0$. This implies $a^* \cap b^* \subseteq (a \vee b)^*$. On the other hand,
$x \wedge (a \vee b) = 0 \Rightarrow (x \wedge a) \vee (x \wedge b) = 0 \Rightarrow x \wedge a = 0 = x \wedge b$. Therefore $(a \vee b)^* \subseteq a^* \cap b^*$. Thus $(a \vee b)^* = a^* \cap b^* = (b \vee a)^*$.
(2). It is clear.
(3).
$$\begin{aligned} x \in a^+ \cap b^+ &\Rightarrow x \vee a \text{ and } x \vee b \text{ are maximal} \\ &\Rightarrow (x \vee a) \wedge (x \vee b) \text{ is maximal} \\ &\Rightarrow x \vee (a \wedge b) \text{ is maximal} \\ &\Rightarrow x \in (a \wedge b)^+. \end{aligned}$$
Therefore $a^+ \cap b^+ \subseteq (a \wedge b)^+$. On the other hand suppose that $x \vee (a \wedge b)$ is maximal. Then, it can be easily observed that $x \vee (b \wedge a)$ is maximal and hence $x \vee a$ and $x \vee b$ are maximal. Therefore $x \in a^+ \cap b^+$. This implies $(a \wedge b)^+ \subseteq a^+ \cap b^+$. Thus $a^+ \cap b^+ = (a \wedge b)^+$.
(4). is trivial. □

Theorem 5.4. *Let A be an ABA and $a, b \in A$. Then the following are equivalent.*

(1) $a \sim b$

(2) $a^* = b^*$ *and* $a^+ = b^+$

(3) $c(a) = c(b)$.

Proof. (1) $\Rightarrow$ (2) : Suppose $a \wedge b = b$ and $b \wedge a = a$. Then $a \vee b = a$ and $b \vee a = b$. Therefore, $a^* = (a \vee b)^* = (b \vee a)^* = b^*$ and
$a^+ = (b \wedge a)^+ = (a \wedge b)^+ = b^+$.
(2) $\Rightarrow$ (3) is trivial since $c(x) = x^* \cap x^+$ for all $x \in A$.

$(3) \Rightarrow (1)$: By theorem 5.2, $c(a) = \sim(x)$ for some $x \in A$.
Now, $x \in \sim(x) = c(a) = c(b)$. Therefore $x \wedge a = 0 = x \wedge b$ and $x \vee a$ and $x \vee b$ are maximal. Now,
$a = (x \vee b) \wedge a = (x \wedge a) \vee (b \wedge a) = 0 \vee (b \wedge a) = b \wedge a$ and
$b = (x \vee a) \wedge b = (x \wedge b) \vee (a \wedge b) = 0 \vee (a \wedge b) = a \wedge b$. Therefore $a \sim b$. $\square$

The following is the converse of the theorem 5.2.

Theorem 5.5. *Let A be an ABA. Then any associate class in A is of the form $c(x)$ for some $x \in A$.*

Proof. Let $a \in A$ and consider $\sim(a)$. Let $x \in c(a)$. Then $a \wedge x = 0 = x \wedge a$ and $x \vee a$ and hence $a \vee x$ are maximal. Therefore $a \in x^* \cap x^+ = c(x)$. Since $c(x)$ is an associate class (by theorem 5.2), it follows that
$\sim(a) \subseteq c(x) = \sim(b)$ for some $b \in A$ and hence $\sim(a) = \sim(b) = c(x)$ (since $\sim$ is an equivalence relation). $\square$

The following are consequences of theorems 5.2 and 5.5.

Corollary 5.6. *For any ABA A, $A/\sim = \{c(a) : a \in A\}$.*

Proof. For any ABA A, it can be proved that $A/\sim$ is a Boolean algebra under the induced operations defined by

$$\sim(a) \wedge \sim(b) = \sim(a \wedge b) \text{ and } \sim(a) \vee \sim(b) = \sim(a \vee b)$$

and the complement $\sim(a)' = \sim(x)$ for some $x \in c(a)$. $\square$

Corollary 5.7. *For any ABA A, the set $\{c(a) : a \in A\}$ is a Boolean algebra.*

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