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Common fixed point theorem for a sequence of Mappings in G-metric spaces

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Abstract:

In the present paper we have established a Common fixed point theorem for a sequence of mappings on a closed subset of Complete G-metric spaces. Also we introduce a example to support the usability of our result.

Kew words : Common fixed point , G-metric space

1. Introduction .

Throughout this paper , unless otherwise stated S is a closed subset of Complete G -metric space (X, G). In 1991 Koparde and Waghmode[3] have proved common fixed point theorem for the sequence $\{T_n\}$ of mappings . After that Pendhare and Waghmode[6], Veerapandi and kumar[9], Badshah and Meena[1] , proved many results for sequence of mappings in Hilbert spaces.

In 2005, Mustafa and Sims introduced a new class of generalized metric spaces(see[4,5]),which are called G -metric space, as generalization of a metric space (X, d). Subsequently, many fixed point results on such spaces appeared (see, for example[7,8]).

Here we present the necessary definitions and results in G-metric spaces , which will be useful for the rest of the paper.

Definition1.1. Let X be a nonempty set. Suppose that $G : X \times X \times X \to R_+$ is a function satisfying the following conditions :

(1) $G(x, y, z) = 0$ if and only if $x = y = z$;

(2) $0 < G(x, x, y)$ for all $x, y \epsilon X$ with $x \neq y$;

(3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y , z \epsilon X$ with $y \neq z$;

(4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables) ;

(5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$.

Then G is called a G-metric on X and (X,G) is called a G-metric space.

Definition1.2.A G-metric space (X,G) is said to be symmetric if $G(x,y,y) = G(y,x,x)$ for all $x,y \in X$.
Definition 1.3 Let (X,G) be a G-metric space. We say that $\{x_n\}$ is

(1) a G-cauchy sequence if, for any $\varepsilon > 0$, there is $M \in N$ (the set of all positive integers) such that for all

$n,m,l \geq M$, $G(x_n, x_m, x_l) < \varepsilon$.

(2) a G-convergent sequence to $x \in X$ if, for any $\varepsilon > 0$,there is $M \in N$ (the set of all positive integers) such

that for all $n,m \geq M$, $G(x, x_n, x_m) < \varepsilon$.

A G-metric space (X,G) is said to be complete if every G-Cauchy sequence in X is G-convergent in X.

Proposition 1.1 Let (X,G) be a G-metric space . The following are equivalent :

(1) $\{x_n\}$ is G-convergent to x ;

(2) $G(x_n, x_n, x) \to 0$ as $n \to +\infty$;

(3) $G(x_n, x, x) \to 0$ as $n \to +\infty$;

(4) $G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$.

Proposition 1.2 Let (X,G) be a G-metric space . The following are equivalent :

(1) the sequence $\{x_n\}$ is G-Cauchy ;

(2) $G(x_n, x_m, x_m) \to 0$ as $n,m \to +\infty$.

In this paper we illustrate a common fixed point theorem for sequence of mappings in complete G-metric spaces also give an example to support the usability of our result. Our result is a generalization of the results in Hilbert spaces to G-metric spaces . To present this work we also see Jaleli and Samet[2].

2.Main Result.

Theorem 2.1. Let S be a closed subset of Complete G-metric space (X,G) and $\{T_n\}$: S → S be a sequence of mappings which satisfies the condition :

for all $x,y,z \in S$

$$G(T_i x, T_j y, T_k z) \leq aG(x, T_i x, T_i x) + bG(y, T_j y, T_j y) + cG(z, T_k z, T_k z) + dG(x,y,z)$$

where a,b,c,d are positive Constants such that $a+b+c+d < 1$. Then $\{T_n\}$ has a unique Common fixed point.

Proof. Let $x_0 \in S$ be any arbitrary point . Defined a Sequence $\{x_n\}$ in S as $x_{n+1} = T_{n+1} x_n$ for $n =$ 0,1,2,3 ...

Now we shall prove that $\{x_n\}$ is a Cauchy sequence , so consider

$$G(x_n, x_{n+1}, x_{n+2}) = G(T_n x_{n-1}, T_{n+1} x_n, T_{n+2} x_{n+1})$$

$$\leq aG(x_{n-1}, T_n x_{n-1}, T_n x_{n-1}) + bG(x_n, T_{n+1} x_n, T_{n+1} x_n) + cG(x_{n+1}, T_{n+2} x_{n+1}, T_{n+2} x_{n+1}) + dG(x_{n-1}, x_n, x_{n+1})$$

$$\leq aG(x_{n-1}, x_n, x_n) + bG(x_n, x_{n+1}, x_{n+1})$$

$$+cG(x_{n+1}, x_{n+2}, x_{n+2}) + dG(x_{n-1}, x_n, x_{n+1})$$

$$\leq aG(x_{n-1}, x_n, x_{n+1}) + bG(x_n, x_{n+1}, x_{n+2})$$

$$+cG(x_n, x_{n+1}, x_{n+2}) + dG(x_{n-1}, x_n, x_{n+1})$$

$$\leq (a+d)G(x_{n-1}, x_n, x_{n+1}) + (b+c)G(x_n, x_{n+1}, x_{n+2})$$

$$G(x_n, x_{n+1}, x_{n+2}) \leq \frac{(a+d)}{1-(b+c)} G(x_{n-1}, x_n, x_{n+1})$$

i.e $\quad G(x_n, x_{n+1}, x_{n+2}) \leq kG(x_{n-1}, x_n, x_{n+1}) \quad$ where $k = \frac{(a+d)}{1-(b+c)} < 1$

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$$G(x_n, x_{n+1}, x_{n+2}) \leq k^n G(x_0, x_1, x_2) \quad \text{for all n}$$

Now for any positive integers $l \geq m \geq n \geq 1$. We Consider

$$G(x_n, x_m, x_l) \leq G(x_n, x_{n+1}, x_{n+2}) + G(x_{n+1}, x_{n+2}, x_{n+3}) + \cdots + G(x_{m-1}, x_m, x_{m+1}) +$$

$$+G(x_m, x_{m+1}, x_{m+2}) + G(x_{m+1}, x_{m+2}, x_{m+3}) + \cdots + G(x_{l-2}, x_{l-1}, x_l)$$

$$\leq k^n G(x_0, x_1, x_2) + k^{n+1} G(x_0, x_1, x_2) + \ldots + k^{m-1} G(x_0, x_1, x_2) +$$

$$+ k^m G(x_0, x_1, x_2) + k^{m+1} G(x_0, x_1, x_2) + \ldots + k^{l-2} G(x_0, x_1, x_2)$$

$$G(x_n, x_m, x_l) \leq \frac{k^n}{1-k} G(x_0, x_1, x_2) \rightarrow 0 \quad as \quad n \rightarrow \infty.$$

i.e. $\{x_n\}$ is a G-Cauchy sequence . Since S is a closed subset of Complete G-metric space X , so $\{x_n\}$ converges to a point u in S.

Now we shall prove that u is a common fixed point of the sequence $\{T_n\}$ of mappings from S into itself. Let $T_n u \neq u$ for all n , and Consider,

$$G(u, T_n u, T_m u) \leq G(u, x_{n-1}, x_{n-1}) + G(x_{n-1}, T_n u, T_m u)$$

$$\leq G(u, x_{n-1}, x_{n-1}) + G(T_{n-1} x_{n-2}, T_n u, T_m u)$$

$$\leq G(u, x_{n-1}, x_{n-1}) + aG(x_{n-2}, T_{n-1} x_{n-2}, T_{n-1} x_{n-2}) + bG(u, T_n u, T_n u)$$

$$+cG(u, T_m u, T_m u) + dG(x_{n-2}, u, u)$$

$$\leq G(u, x_{n-1}, x_{n-1}) + aG(x_{n-2}, x_{n-1}, x_{n-1}) + bG(u, T_m u, T_n u)$$

$$+cG(u, T_n u, T_m u) + dG(x_{n-2}, u, u)$$

$$G(u, T_n u, T_m u) \leq (b + c) G(u, T_n u, T_m u)$$

i.e. $G(u, T_n u, T_m u) < G(u, T_n u, T_m u)$, which is a contradiction .

Thus $T_n u = T_m u = u$, for all m, n .

Hence u is a Common fixed point of the sequence $\{T_n\}$ of mappings.

Uniqueness:

Suppose $u \neq v$ such that $T_n v = v$ for all n .

Consider,

$$G(u, u, v) = G\left(T_n u, T_m u, T_p v\right)$$

$$\leq aG(u, T_n u, T_n u) + bG(u, T_m u, T_m u) + cG\left(v, T_p v, T_p v\right) + dG(u, u, v)$$

i.e. $G(u, u, v) < G(u, u, v)$, which is again a contradiction .

Thus $u = v$. Hence u is a unique common fixed point of sequence $\{T_n\}$ of mappings.

Example.2.1. Let $X = [0,1]$, and $G = X \times X \times X \to R_+$ be defined by :

$$G(x, y, z) = \begin{cases} 0 , & \text{if } x = y = z \\ \text{Max}\{x, y, z\} & ,\text{otherwise} \end{cases}$$

Then (X, G) is a Complete G-metric space . Let $\{T_n\}: X \to X$ be a sequence of mappings defined by $T_n x = \frac{x}{4^n}$ for all $x \in X$, then $\{T_n\}$ satisfies the inequality of theorem 2.1. Hence the sequence $\{T_n\}$of mappings has a unique common fixed point in X.

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