

Generalized Area and Arc Length of Sine Function in Polar Coordinate

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ARTICLE INFO	ABSTRACT
Published Online: 22 October 2026	Polar coordinate is formed by the radius and rotation angle. The radius indicates distance of a point from pole, while the rotation angle is angle from which starting point moves to ending point relative to the axis-x. Because polar coordinate is affected by two variables, polar coordinate is defined on a plane. This research aims to find a general formula for determining area and length of a curve. The curve equation is a sine function, which is circular and cardioid. The graph of circle is affected by constant and positive or negative sign on the sine function. The constant determines the size of the graph, while positive or negative sign indicates position of the graph in the quadrant. Similarly, the graph of cardioid is affected by constant and positive or negative sign on the sine function. The constant determines the size of the graph, while positive or negative sign indicates the direction of the cardioid. The area of circle is directly proportional to the square of constant. The arc length of circle is directly proportional to the constant. The area of cardioid is directly proportional to the square of constant. The arc length of cardioid is equal to eight times the constant. The general formula for the area and arc length of the curve is used as a predictive tool.
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I. INTRODUCTION

Polar coordinate is a pair of radius and rotation angle. The rotation angle rotates completely by 360° . The angle is independent variable and radius is dependent variable, so the curve equation is written as radius is function of rotation angle [3]. Because there are two variables, polar coordinates are presented in a plane. The angle will be processed by trigonometric functions, resulting in a regular graph. The resulting figure can be a circle or a cardioid [7]. Polar coordinate be useful for solving definite integral problems that cannot or be difficult to solve using Cartesian coordinate. By using polar coordinate, definite integral problem can be solved easily and quickly. In addition, the discovery of polar coordinate can produce notation or constant that is very important in mathematics, for example notation π [1]. Polar coordinate in statistics is useful for proving the property of the probability density function of normal distribution. The development of polar coordinate is in the form of cylindrical coordinate and spherical coordinate. These coordinates are useful for determining the volume of cylinder or sphere [6].

Several researchs on polar coordinate have been conducted, in [2] discusses the arc length of a lemniscate. A

lemniscate is a curve whose equation is the product of a constant and the square root of cosine or sine functions. The lemniscate's shape resembles a rose. In [4] discusses calculating the area and arc length of sine and cosine functions about simple functions. In [5] discusses generalization of determining area and art length of cosine function.

This research is generalized area and arc length of sine function in polar coordinate. The general formula is useful for simplifying determination of area and arc length of the curves. Furthermore, the general formula is useful for predicting their values for any positive and negative real number. The general formula is applied by conducting simulations on several both positive and negative integers.

II. LITERATURE REVIEW

Cartesian and polar coordinates are coordinates that each have their own characteristics. But they have an interrelated relationship. In Cartesian coordinate, there is a plane area, whereas in polar coordinate, there is also a plane area. In Cartesian coordinate, there is a curve length, likewise the arc length is also found in polar coordinate [3]. Therefore, in this article we will discuss the area and arc

length in polar coordinate, which of course they are based on Cartesian coordinate.

2.1. Relation between Cartesian and Polar Coordinates

There is a reciprocal relation between Cartesian and polar coordinates. Cartesian coordinate and polar coordinate are two coordinate systems for determining the position of a point on a two-dimensional plane. In Cartesian coordinates, the abscissa is the perpendicular distance from the point to the axis-y and the ordinate is the perpendicular distance from the point to the axis-x. Meanwhile, in polar coordinates, the radius is the distance from the pole to the point and the angle is the rotation angle from the positive axis-x to the point. Both coordinate systems are related through trigonometric functions [6]. The relationship between Cartesian and polar coordinates can be tabulated in Table 1.

Table 1. Relation Cartesian and polar coordinates

Cartesian Coordinate	Polar Coordinate
x	r
y	θ
(x, y)	(r, θ)
$P(x, y)$	$P(r, \theta)$
$x = r \cos \theta$	$r = \sqrt{x^2 + y^2}$
$y = r \sin \theta$	$\theta = \tan^{-1} \left(\frac{y}{x} \right)$

2.2. The Area in Polar Coordinate

We need to know the basic formula for area. The basic formula for area is the sector. A sector is a portion of a circle [7]. We need to know the area of a sector, as stated in Proposition 1 below.

Proposition 1.

Let there is a sector, then the area of a sector is $A = \frac{1}{2} r^2 \theta$, where r is the radius of the circle, and θ is the rotation angle of the sector.

Proof:

If we define A as the area of a sector, B as the area of a circle, C as the rotation angle of a sector, and D as the rotation angle of a circle. Next, we use this comparison, namely

$$\frac{A}{B} = \frac{C}{D} \Leftrightarrow A = \frac{C}{D} B = \frac{\theta}{2\pi} \pi r^2 = \frac{1}{2} r^2 \theta. \quad (1)$$

Therefore, the area of a sector is $A = \frac{1}{2} r^2 \theta$.

In polar coordinate, the independent variable is θ , while the dependent variable is r . Therefore, the relationship between variables be written as $r = f(\theta)$. The notation states that the radius is function of the rotation angle [1]. Based on Proposition 1, we can find a method for calculating the area of a sector or region based on the

function $r = f(\theta)$. The method for calculating the area of a sector is presented in Theorem 1.

Theorem 1.

Let θ is the rotation angle of the point, whereas r is the distance from the pole to the point. If $r = f(\theta)$, where $\alpha \leq \theta \leq \beta$, then the sector area A be obtained by

$$A = \frac{1}{2} \int_{\theta=\alpha}^{\beta} r^2 d\theta = \frac{1}{2} \int_{\theta=\alpha}^{\beta} [f(\theta)]^2 d\theta. \quad (2)$$

Proof:

Based on Equation 1 that $A = \frac{1}{2} r^2 \theta$, then for $r = f(\theta)$ with the rotation angle $\alpha \leq \theta \leq \beta$, the sector area is

$$A = \frac{1}{2} \int_{\theta=\alpha}^{\beta} r^2 d\theta = \frac{1}{2} \int_{\theta=\alpha}^{\beta} [f(\theta)]^2 d\theta.$$

2.3. The Arc Length of Curve in Polar Coordinate

The basis for determining arc length in polar coordinate is the same as in Cartesian coordinate. The arc length of a curve in polar coordinates is as in Theorem 2.

Theorem 2.

Let $r = f(\theta)$ be a function about θ . It has derivative and continuous on interval $\alpha \leq \theta \leq \beta$. The arc length of the curve $r = f(\theta)$, with the rotation angle $\alpha \leq \theta \leq \beta$ is

$$L = \int_{\theta=\alpha}^{\beta} \sqrt{r^2 + (r')^2} d\theta. \quad (3)$$

Proof:

Let $r = f(\theta)$, where $\alpha \leq \theta \leq \beta$, then

$x = r \cos \theta = f(\theta) \cos \theta$ and $y = r \sin \theta = f(\theta) \sin \theta$, so

$x' = f'(\theta) \cos \theta - f(\theta) \sin \theta$ and

$y' = f'(\theta) \sin \theta + f(\theta) \cos \theta$.

The arc length of the curve $r = f(\theta)$ is

$$\begin{aligned} L &= \int_{\theta=\alpha}^{\beta} \sqrt{(x')^2 + (y')^2} d\theta \\ &= \int_{\theta=\alpha}^{\beta} \sqrt{(f'(\theta) \cos \theta - f(\theta) \sin \theta)^2 + (f'(\theta) \sin \theta + f(\theta) \cos \theta)^2} d\theta \\ &= \int_{\theta=\alpha}^{\beta} \sqrt{(f'(\theta))^2 + (f(\theta))^2} d\theta = \int_{\theta=\alpha}^{\beta} \sqrt{r^2 + (r')^2} d\theta \end{aligned}$$

where $\cos^2 \theta + \sin^2 \theta = 1$. Therefore, if $\alpha \leq \theta \leq \beta$, then

$$L = \int_{\theta=\alpha}^{\beta} \sqrt{r^2 + (r')^2} d\theta.$$

III. RESULT AND DISCUSSION

The following discussion applies area and arc length to curves in the shape of a circle and a cardioid. The result is presented starting with a simple problem, then a simulation is carried out until a general formula is obtained. The general formula is presented in the form of a theorem. The theorem is useful as a tool for predicting the area and arc length of the curve.

3.1. The Area and Arc Length of a Circle

A Circle in Quadrants One and Two

If given a curve $r = a \sin \theta$, where a is a positive real number and $0 \leq \theta \leq 2\pi$. The resulting graph is a circle

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located in quadrants one and two. The circle has center at $(0, \frac{1}{2}a)$ and radius is half a .

Problem 1.

Suppose there is a curve $r = \sin \theta$, where the rotation angle is $0 \leq \theta \leq 2\pi$. Determine the area and arc length of this graph.

Solution:

We choose $r_{\max} = 1$, $1 = \sin \theta$, $\theta = \frac{\pi}{2}$. If we choose $r_{\min} = 0$, $0 = \sin \theta$, $\theta = 0$. Therefore, the rotation angle is $0 \leq \theta \leq \frac{\pi}{2}$.

The graph $r = \sin \theta$ be presented at Figure 1 below.

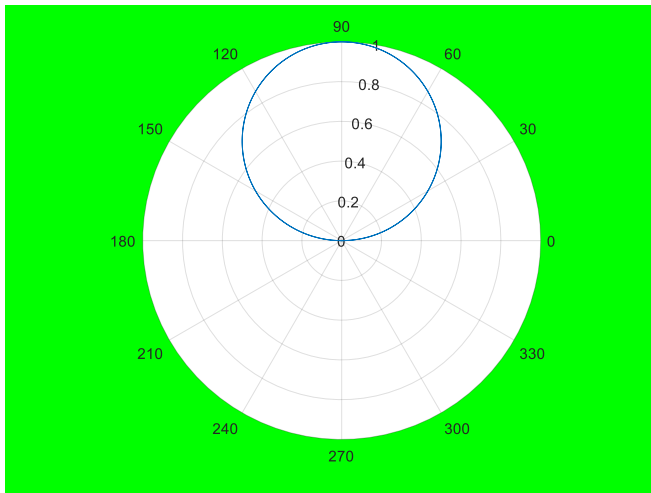


Figure 1. The graph $r = \sin \theta$

The area be obtained by the curve $r = \sin \theta$ is

$$A = \int_{\theta=0}^{\pi/2} \sin^2 \theta \, d\theta = \frac{1}{2} \int_{\theta=0}^{\pi/2} [1 - \cos 2\theta] \, d\theta$$

$$= \frac{1}{2} \left(\theta - \frac{1}{2} \sin 2\theta \right) \Big|_{\theta=0}^{\pi/2} = \left[\frac{1}{2} \left(\frac{\pi}{2} \right) \right] = \frac{\pi}{4}$$

If $r = \sin \theta$, then $r' = \cos \theta$, so that the arc length of the graph r is

$$L = 2 \int_{\theta=0}^{\pi/2} \sqrt{\sin^2 \theta + \cos^2 \theta} \, d\theta = 2\theta \Big|_{\theta=0}^{\pi/2} = \pi.$$

Problem 2.

Let there is the curve $r = 2 \sin \theta$, where the rotation angle is $0 \leq \theta \leq 2\pi$. Determine the area and arc length of this graph.

Solution:

If we choose $r_{\max} = 2$, $2 = 2 \sin \theta$, $\theta = \frac{\pi}{2}$. If we choose $r_{\min} = 0$, $0 = 2 \sin \theta$, $\theta = 0$. Therefore, the rotation angle is $0 \leq \theta \leq \frac{\pi}{2}$.

The graph $r = 2 \sin \theta$ can be presented as Figure 2.

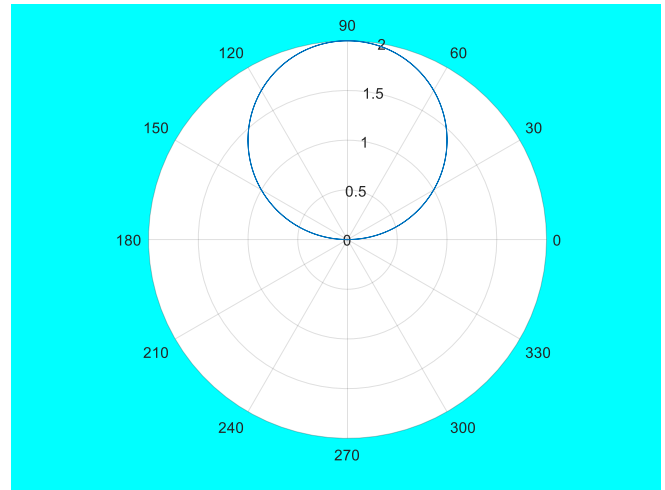


Figure 2. The graph $r = 2 \sin \theta$

The area be made by the curve $r = 2 \sin \theta$ is

$$A = 4 \int_{\theta=0}^{\pi/2} \sin^2 \theta \, d\theta = 2 \int_{\theta=0}^{\pi/2} [1 - \cos 2\theta] \, d\theta$$

$$= 2 \left(\theta - \frac{1}{2} \sin 2\theta \right) \Big|_{\theta=0}^{\pi/2} = \left[2 \left(\frac{\pi}{2} \right) \right] = \pi$$

If we take $r = 2 \sin \theta$, then $r' = 2 \cos \theta$ and the arc length of the graph r is

$$L = 2 \int_{\theta=0}^{\pi/2} \sqrt{4 \sin^2 \theta + 4 \cos^2 \theta} \, d\theta = 4\theta \Big|_{\theta=0}^{\pi/2} = 2\pi.$$

Another expansion of the multiplication of positive integers by the sine function is written in Theorem 3. This theorem is a generalization of the area and arc length of a curve.

Theorem 3.

If there is a curve $r = a \sin \theta$, where a is positive real number, and $0 \leq \theta \leq 2\pi$. Then the area be bounded by the curve r is $\frac{a^2}{4} \pi$, and the arc length of the graph r is $a \pi$.

Proof:

We choose $r_{\max} = a$, $a = a \sin \theta$, $\theta = \frac{\pi}{2}$. If we choose $r_{\min} = 0$, $0 = a \sin \theta$, $\theta = 0$. Therefore, the rotation angle is $0 \leq \theta \leq \frac{\pi}{2}$. The area of the graph $r = a \sin \theta$ is

$$A = a^2 \int_{\theta=0}^{\pi/2} \sin^2 \theta \, d\theta = \frac{a^2}{2} \int_{\theta=0}^{\pi/2} [1 - \cos 2\theta] \, d\theta$$

$$= \frac{a^2}{2} \left(\theta - \frac{1}{2} \sin 2\theta \right) \Big|_{\theta=0}^{\pi/2} = \left[\frac{a^2}{2} \left(\frac{\pi}{2} \right) \right] = \frac{a^2}{4} \pi$$

If the graph $r = a \sin \theta$, then $r' = a \cos \theta$. Therefore, the arc length of the graph $r = a \sin \theta$ is

$$L = 2 \int_{\theta=0}^{\pi/2} \sqrt{a^2 \sin^2 \theta + a^2 \cos^2 \theta} \, d\theta = 2a \theta \Big|_{\theta=0}^{\pi/2} = a \pi.$$

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If we take $a = 1, 2, \dots, 10$, based on Theorem 3, we can determine the area and arc length formed by the curve $r = a \sin \theta$ as the Table 2.

Table 2. Simulation value of curve $r = a \sin \theta$

a	$r = a \sin \theta$	A	L
1	$r = \sin \theta$	$\frac{\pi}{4}$	π
2	$r = 2 \sin \theta$	π	2π
3	$r = 3 \sin \theta$	$\frac{9}{4}\pi$	3π
$\vdots$	$\vdots$	$\vdots$	$\vdots$
9	$r = 9 \sin \theta$	$20\frac{1}{4}\pi$	9π
10	$r = 10 \sin \theta$	25π	10π

A Circle in Quadrants Three and Four

If given a curve $r = -a \sin \theta$, where $-a$ is a negative real number and $0 \leq \theta \leq 2\pi$. The resulting graph is a circle located in quadrants three and four. The circle has center at $(0, -\frac{1}{2}a)$ and radius is half a .

Problem 3.

If there is the curve $r = -\sin \theta$, where the rotation angle is $0 \leq \theta \leq 2\pi$. Find the area and arc length produced by the graph.

Solution:

We choose $r_{\max} = 1$, $1 = -\sin \theta$, $\theta = -\frac{\pi}{2}$. If we choose $r_{\min} = 0$, $0 = -\sin \theta$, $\theta = 0$. Therefore, the rotation angle is $-\frac{\pi}{2} \leq \theta \leq 0$.

The graph of circle $r = -\sin \theta$ be shown by Figure 3 as follow.

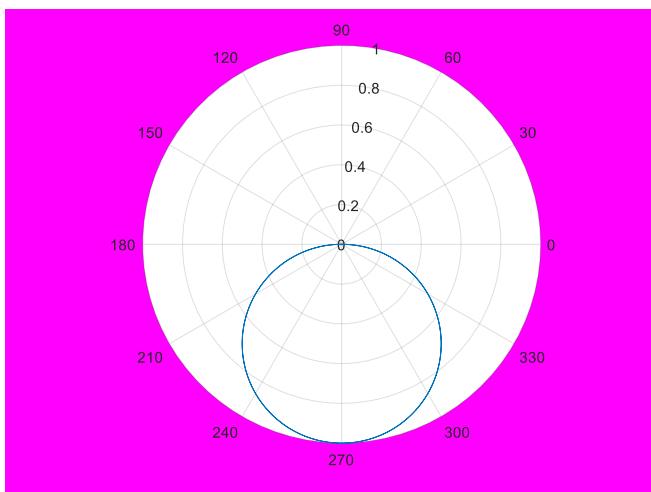


Figure 3. The graph $r = -\sin \theta$

The area of the curve $r = -\sin \theta$ is

$$A = \int_{\theta=-\pi/2}^0 \sin^2 \theta d\theta = \frac{1}{2} \int_{\theta=-\pi/2}^0 [1 - \cos 2\theta] d\theta$$

$$= \frac{1}{2} \left(\theta - \frac{1}{2} \sin 2\theta \right) \Big|_{\theta=-\pi/2}^0 = \left[\frac{1}{2} \left(\frac{\pi}{2} \right) \right] = \frac{\pi}{4}$$

If $r = -\sin \theta$, then $r' = -\cos \theta$. Therefore, the arc length of the graph is

$$L = 2 \int_{\theta=-\pi/2}^0 \sqrt{\sin^2 \theta + \cos^2 \theta} d\theta = 2\theta \Big|_{\theta=-\pi/2}^0 = \pi.$$

Problem 4.

If there is the curve $r = -2 \sin \theta$, where the rotation angle is $0 \leq \theta \leq 2\pi$. Find the area and arc length be made by that graph.

Solution:

If we choose $r_{\max} = 2$, $2 = -2 \sin \theta$, $\theta = -\frac{\pi}{2}$. If we choose $r_{\min} = 0$, $0 = -2 \sin \theta$, $\theta = 0$. Therefore, the rotation angle is $-\frac{\pi}{2} \leq \theta \leq 0$.

The graph $r = -2 \sin \theta$ is presented at Figure 4.

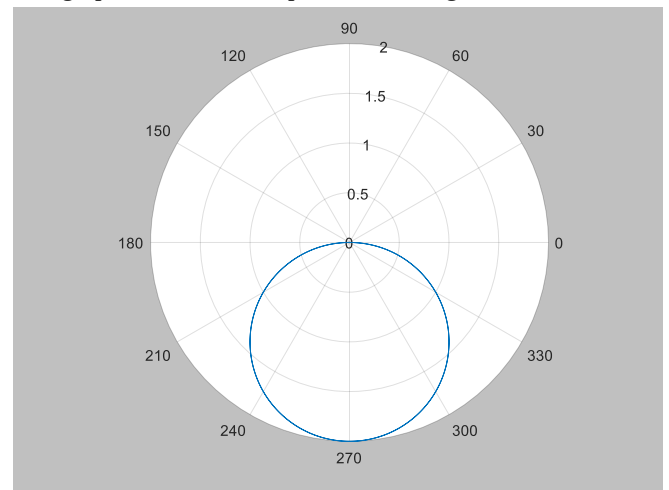


Figure 4. The graph $r = -2 \sin \theta$

The area be created by the curve $r = -2 \sin \theta$ is

$$A = 4 \int_{\theta=-\pi/2}^0 \sin^2 \theta d\theta = 2 \int_{\theta=-\pi/2}^0 [1 - \cos 2\theta] d\theta$$

$$= 2 \left(\theta - \frac{1}{2} \sin 2\theta \right) \Big|_{\theta=-\pi/2}^0 = \left[2 \left(\frac{\pi}{2} \right) \right] = \pi$$

If we put on $r = -2 \sin \theta$, then $r' = -2 \cos \theta$. Therefore, the arc length of the graph is

$$L = 2 \int_{\theta=-\pi/2}^0 \sqrt{4 \sin^2 \theta + 4 \cos^2 \theta} d\theta = 4\theta \Big|_{\theta=-\pi/2}^0 = 2\pi.$$

Another expansion of the multiplication of negative integers by the sine function is written in Theorem 4. This theorem is a generalization of the area and arc length of a graph.

Theorem 4.

If there is a curve $r = -a \sin \theta$, where $-a$ is negative real number, $0 \leq \theta \leq 2\pi$. Then the area be bounded by the curve is $\frac{a^2}{4}\pi$, and the arc length of the graph is $a\pi$.

Proof:

We choose $r_{\max} = a$, $a = -a \sin \theta$, $\theta = -\frac{\pi}{2}$. If we choose $r_{\min} = 0$, $0 = a \sin \theta$, $\theta = 0$. Therefore, the rotation angle is $-\frac{\pi}{2} \leq \theta \leq 0$. Therefore, the rotation angle is $-\frac{\pi}{2} \leq \theta \leq 0$.

The area of the curve $r = -a \sin \theta$ is

$$\begin{aligned} A &= a^2 \int_{\theta=-\pi/2}^0 \sin^2 \theta d\theta = \frac{a^2}{2} \int_{\theta=-\pi/2}^0 [1 - \cos 2\theta] d\theta \\ &= \frac{a^2}{2} \left(\theta - \frac{1}{2} \sin 2\theta \right) \Big|_{\theta=-\pi/2}^0 = \left[\frac{a^2}{2} \left(\frac{\pi}{2} \right) \right] = \frac{a^2}{4} \pi \end{aligned}$$

If the graph $r = -a \sin \theta$, then $r' = -a \cos \theta$. Therefore, the arc length of the graph $r = -a \sin \theta$ is

$$L = 2 \int_{\theta=-\pi/2}^0 \sqrt{a^2 \sin^2 \theta + a^2 \cos^2 \theta} d\theta = 2a \theta \Big|_{\theta=-\pi/2}^0 = a\pi.$$

If we take $a = 1, 2, \dots, 10$, based on Theorem 4, we could find both area and arc length made by the graph $r = -a \sin \theta$ as shown in Table 3 below.

Table 3. Simulation value of curve $r = -a \sin \theta$

a	$r = -a \sin \theta$	A	L
1	$r = -\sin \theta$	$\frac{\pi}{4}$	π
2	$r = -2 \sin \theta$	π	2π
3	$r = -3 \sin \theta$	$2\frac{1}{4}\pi$	3π
$\vdots$	$\vdots$	$\vdots$	$\vdots$
9	$r = -9 \sin \theta$	$20\frac{1}{4}\pi$	9π
10	$r = -10 \sin \theta$	25π	10π

3.2. The Area and Arc Length of a Cardioid A Cardioid with Direction to the Up

If given a curve $r = \pm a + a \sin \theta$, where a is a positive real number and $0 \leq \theta \leq 2\pi$. The resulting graph is a cardioid with direction to the up.

Problem 5.

Let there is a curve $r = \pm 1 + \sin \theta$, where the rotation angle is $0 \leq \theta \leq 2\pi$. Find the area and arc length be shown by a curve $r = \pm 1 + \sin \theta$.

Solution:

We choose $r_{\max} = 2$, $2 = 1 + \sin \theta$, $\theta = \frac{\pi}{2}$. If we choose $r_{\min} = 0$, $0 = 1 + \sin \theta$, $\theta = -\frac{\pi}{2}$. Therefore, the rotation angle is $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

The graph of the cardioid is drawn below.

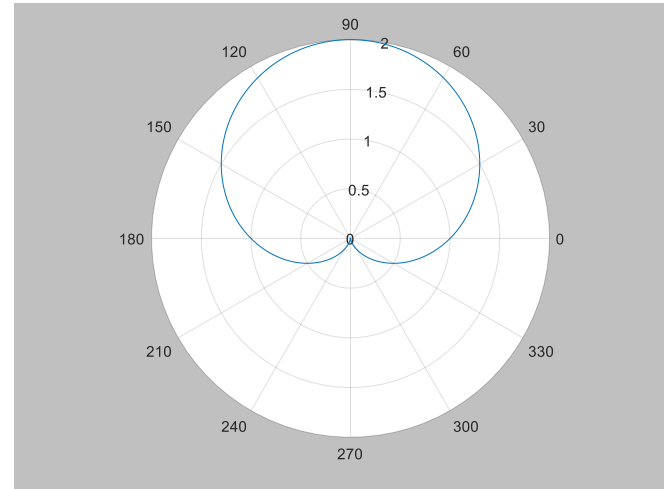


Figure 5. The graph $r = \pm 1 + \sin \theta$

The area be generated by graph $r = 1 + \sin \theta$ is

$$\begin{aligned} A &= \int_{\theta=-\pi/2}^{\pi/2} (1 + 2 \sin \theta + \sin^2 \theta) d\theta \\ &= \int_{\theta=-\pi/2}^{\pi/2} \left[1 + 2 \sin \theta + \frac{1}{2}(1 - \cos 2\theta) \right] d\theta \\ &= \left(\theta - 2 \cos \theta + \frac{1}{2} \left(\theta - \frac{1}{2} \sin 2\theta \right) \right) \Big|_{\theta=-\pi/2}^{\pi/2} = \left(\frac{3}{2} \pi \right) \end{aligned}$$

The area be generated by graph $r = -1 + \sin \theta$ is

$$\begin{aligned} A &= \int_{\theta=-\pi/2}^{\pi/2} (1 - 2 \sin \theta + \sin^2 \theta) d\theta \\ &= \int_{\theta=-\pi/2}^{\pi/2} \left[1 - 2 \sin \theta + \frac{1}{2}(1 - \cos 2\theta) \right] d\theta \\ &= \left(\theta + \cos \theta + \frac{1}{2} \left(\theta - \frac{1}{2} \sin 2\theta \right) \right) \Big|_{\theta=-\pi/2}^{\pi/2} = \left(\frac{3}{2} \pi \right) \end{aligned}$$

For the curve $r = 1 + \sin \theta$, then $r' = \cos \theta$, so the arc length be generated is

$$\begin{aligned} L &= 2 \int_{\theta=-\pi/2}^{\pi/2} \sqrt{1 + 2 \sin \theta + \sin^2 \theta + \cos^2 \theta} d\theta \\ &= 2 \int_{\theta=-\pi/2}^{\pi/2} \sqrt{2(1 + \sin \theta)} d\theta = 2\sqrt{2} \int_{\theta=-\pi/2}^{\pi/2} \left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2} \right) d\theta \\ &= 4\sqrt{2} \left(\sin \frac{\theta}{2} - \cos \frac{\theta}{2} \right) \Big|_{\theta=-\pi/2}^{\pi/2} = 4\sqrt{2} (\sqrt{2}) = 8 \end{aligned}$$

For the curve $r = -1 + \sin \theta$, then $r' = \cos \theta$, so the arc length be generated is

$$\begin{aligned} L &= 2 \int_{\theta=-\pi/2}^{\pi/2} \sqrt{1 - 2 \sin \theta + \sin^2 \theta + \cos^2 \theta} d\theta \\ &= 2 \int_{\theta=-\pi/2}^{\pi/2} \sqrt{2(1 - \sin \theta)} d\theta = 2\sqrt{2} \int_{\theta=-\pi/2}^{\pi/2} \left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2} \right) d\theta \\ &= 4\sqrt{2} \left(\sin \frac{\theta}{2} + \cos \frac{\theta}{2} \right) \Big|_{\theta=-\pi/2}^{\pi/2} = 4\sqrt{2} (\sqrt{2}) = 8 \end{aligned}$$

Problem 6.

Suppose there is a curve $r = \pm 2 + 2 \sin \theta$, where the rotation angle is $0 \leq \theta \leq 2\pi$. Determine the area and the arc length be made the graph.

Solution:

We choose $r_{\max} = 4$, $4 = 2 + 2\sin\theta$, $\theta = \frac{\pi}{2}$. If we choose $r_{\min} = 0$, $0 = 2 + 2\sin\theta$, $\theta = -\frac{\pi}{2}$. Therefore, the rotation angle is $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

The graph of the cardioid be presented as follow.

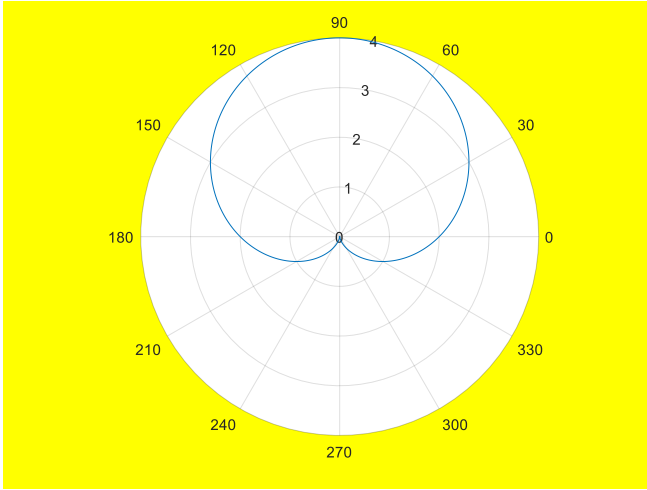


Figure 6. The graph $r = \pm 2 + 2\sin\theta$

The resulting area of curve $r = 2 + 2\sin\theta$ is

$$\begin{aligned} A &= \int_{\theta=-\pi/2}^{\pi/2} (4 + 8\sin\theta + 4\sin^2\theta) d\theta \\ &= \int_{\theta=-\pi/2}^{\pi/2} [4 + 8\sin\theta + 2(1 - \cos 2\theta)] d\theta \\ &= (4\theta - 8\cos\theta + 2(\theta - \frac{1}{2}\sin 2\theta)) \Big|_{\theta=-\pi/2}^{\pi/2} = 6\pi \end{aligned}$$

The resulting area of curve $r = -2 + 2\sin\theta$ is

$$\begin{aligned} A &= \int_{\theta=-\pi/2}^{\pi/2} (4 - 8\sin\theta + 4\sin^2\theta) d\theta \\ &= \int_{\theta=-\pi/2}^{\pi/2} [4 - 8\sin\theta + 2(1 - \cos 2\theta)] d\theta \\ &= (4\theta + 8\cos\theta + 2(\theta - \frac{1}{2}\sin 2\theta)) \Big|_{\theta=-\pi/2}^{\pi/2} = 6\pi \end{aligned}$$

For the curve $r = 2 + 2\sin\theta$, then $r' = 2\cos\theta$, so that the resulting arc length be

$$\begin{aligned} L &= 2 \int_{\theta=-\pi/2}^{\pi/2} \sqrt{4 + 8\sin\theta + 4\sin^2\theta + 4\cos^2\theta} d\theta \\ &= 2 \int_{\theta=-\pi/2}^{\pi/2} \sqrt{8(1 + \sin\theta)} d\theta = 4\sqrt{2} \int_{\theta=-\pi/2}^{\pi/2} (\cos \frac{\theta}{2} + \sin \frac{\theta}{2}) d\theta \\ &= 8\sqrt{2} (\sin \frac{\theta}{2} - \cos \frac{\theta}{2}) \Big|_{\theta=-\pi/2}^{\pi/2} = 8\sqrt{2} (\sqrt{2}) = 16 \end{aligned}$$

For the curve $r = -2 + 2\sin\theta$, then $r' = 2\cos\theta$, so that the resulting arc length be

$$\begin{aligned} L &= 2 \int_{\theta=-\pi/2}^{\pi/2} \sqrt{4 - 8\sin\theta + 4\sin^2\theta + \cos^2\theta} d\theta \\ &= 2 \int_{\theta=-\pi/2}^{\pi/2} \sqrt{8(1 - \sin\theta)} d\theta = 4\sqrt{2} \int_{\theta=-\pi/2}^{\pi/2} (\cos \frac{\theta}{2} - \sin \frac{\theta}{2}) d\theta \\ &= 8\sqrt{2} (\sin \frac{\theta}{2} + \cos \frac{\theta}{2}) \Big|_{\theta=-\pi/2}^{\pi/2} = 8\sqrt{2} (\sqrt{2}) = 16 \end{aligned}$$

Another expansion of the multiplication of positive integers with sine function to find area and arc length is written in Theorem 5. The theorem is generalization in

determining area and length of a curve $r = \pm a + a\sin\theta$, for any positive real number a .

Theorem 5.

If there is a curve $r = \pm a + a\sin\theta$, where a is any positive real number, $0 \leq \theta \leq 2\pi$. Then the area be bounded by that curve is $\frac{3}{2}a^2\pi$, and the arch length of that curve is $8a$.

Proof:

We choose $r_{\max} = 2a$, $2a = a + a\sin\theta$, $\theta = \frac{\pi}{2}$. If we choose $r_{\min} = 0$, $0 = a + a\sin\theta$, $\theta = -\frac{\pi}{2}$. Therefore, the rotation angle is $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

The area produced by the curve $r = a + a\sin\theta$ is

$$\begin{aligned} A &= \int_{\theta=-\pi/2}^{\pi/2} (a^2 + 2a^2\sin\theta + a^2\sin^2\theta) d\theta \\ &= \int_{\theta=-\pi/2}^{\pi/2} [a^2 + 2a^2\sin\theta + \frac{1}{2}a^2(1 - \cos 2\theta)] d\theta \\ &= (a^2\theta - 2a^2\cos\theta + \frac{1}{2}a^2(\theta - \frac{1}{2}\sin 2\theta)) \Big|_{\theta=-\pi/2}^{\pi/2} = \frac{3}{2}a^2\pi \end{aligned}$$

The area produced by the curve $r = -a + a\sin\theta$ is

$$\begin{aligned} A &= \int_{\theta=-\pi/2}^{\pi/2} (a^2 - 2a^2\sin\theta + a^2\sin^2\theta) d\theta \\ &= \int_{\theta=-\pi/2}^{\pi/2} [a^2 - 2a^2\sin\theta + \frac{1}{2}a^2(1 - \cos 2\theta)] d\theta \\ &= (a^2\theta + 2a^2\cos\theta + \frac{1}{2}a^2(\theta - \frac{1}{2}\sin 2\theta)) \Big|_{\theta=-\pi/2}^{\pi/2} = \frac{3}{2}a^2\pi \end{aligned}$$

For the curve $r = a + a\sin\theta$, then $r' = a\cos\theta$, so that the resulting arc length is

$$\begin{aligned} L &= 2 \int_{\theta=-\pi/2}^{\pi/2} \sqrt{a^2 + 2a^2\sin\theta + a^2\sin^2\theta + a^2\cos^2\theta} d\theta \\ &= 2 \int_{\theta=-\pi/2}^{\pi/2} \sqrt{2a^2(1 + \sin\theta)} d\theta \\ &= 2\sqrt{2}a \int_{\theta=-\pi/2}^{\pi/2} (\cos \frac{\theta}{2} + \sin \frac{\theta}{2}) d\theta \\ &= 4\sqrt{2}a (\sin \frac{\theta}{2} - \cos \frac{\theta}{2}) \Big|_{\theta=-\pi/2}^{\pi/2} = 4\sqrt{2}a (\sqrt{2}) = 8a \end{aligned}$$

For the curve $r = -a + a\sin\theta$, then $r' = a\cos\theta$, so that the resulting arc length is

$$\begin{aligned} L &= 2 \int_{\theta=-\pi/2}^{\pi/2} \sqrt{a^2 - 2a^2\sin\theta + a^2\sin^2\theta + a^2\cos^2\theta} d\theta \\ &= 2 \int_{\theta=-\pi/2}^{\pi/2} \sqrt{2a^2(1 - \sin\theta)} d\theta \\ &= 2\sqrt{2}a \int_{\theta=-\pi/2}^{\pi/2} (\cos \frac{\theta}{2} - \sin \frac{\theta}{2}) d\theta \\ &= 4\sqrt{2}a (\sin \frac{\theta}{2} + \cos \frac{\theta}{2}) \Big|_{\theta=-\pi/2}^{\pi/2} = 4\sqrt{2}a (\sqrt{2}) = 8a \end{aligned}$$

If we choose $a = 1, 2, \dots, 10$, based on Theorem 5, we can determine the area and arc length formed by the graph $r = \pm a + a\sin\theta$ as in Table 4 below.

Table 4. Simulation value of curve $r = \pm a + a \sin \theta$

a	$r = \pm a + a \sin \theta$	A	L
1	$r = \pm 1 + \sin \theta$	$1\frac{1}{2}\pi$	8
2	$r = \pm 2 + 2 \sin \theta$	6π	16
3	$r = \pm 3 + 3 \sin \theta$	$13\frac{1}{2}\pi$	24
$\vdots$	$\vdots$	$\vdots$	$\vdots$
9	$r = \pm 9 + 9 \sin \theta$	$121\frac{1}{2}\pi$	72
10	$r = \pm 10 + 10 \sin \theta$	150π	80

A Cardioid with Direction to the Down

A cardioid with direction to the down, it has equation $r = \pm a - a \sin \theta$, where $-a$ is any negative real number and the rotation angle is $0 \leq \theta \leq 2\pi$.

Problem 7.

Suppose there is a curve $r = \pm 1 - \sin \theta$, where the rotation angle is $0 \leq \theta \leq 2\pi$. Find the area and arc length be made by a curve $r = \pm 1 - \sin \theta$.

Solution:

We choose $r_{\max} = 2$, $2 = 1 - \sin \theta$, $\theta = -\frac{\pi}{2}$. If we choose $r_{\min} = 0$, $0 = 1 - \sin \theta$, $\theta = \frac{\pi}{2}$. Therefore, the rotation angle is $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

The graph of the cardioid is presented as Figure 7 below.

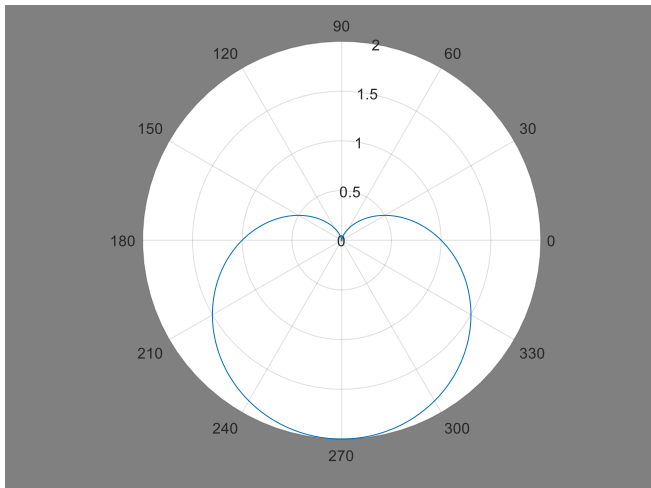


Figure 7. The graph $r = \pm 1 - \sin \theta$

The area obtained by graph $r = 1 - \sin \theta$ is

$$\begin{aligned}
 A &= \int_{\theta=-\pi/2}^{\pi/2} (1 - 2 \sin \theta + \sin^2 \theta) d\theta \\
 &= \int_{\theta=-\pi/2}^{\pi/2} [1 - 2 \sin \theta + \frac{1}{2}(1 - \cos 2\theta)] d\theta \\
 &= (\theta + 2 \cos \theta + \frac{1}{2}(\theta - \frac{1}{2} \sin 2\theta)) \Big|_{\theta=-\pi/2}^{\pi/2} = \frac{3}{2}\pi
 \end{aligned}$$

The area obtained by graph $r = -1 - \sin \theta$ is

$$\begin{aligned}
 A &= \int_{\theta=-\pi/2}^{\pi/2} (1 + 2 \sin \theta + \sin^2 \theta) d\theta \\
 &= \int_{\theta=-\pi/2}^{\pi/2} [1 + 2 \sin \theta + \frac{1}{2}(1 - \cos 2\theta)] d\theta \\
 &= (\theta - 2 \cos \theta + \frac{1}{2}(\theta - \frac{1}{2} \sin 2\theta)) \Big|_{\theta=-\pi/2}^{\pi/2} = \frac{3}{2}\pi
 \end{aligned}$$

For the curve $r = 1 - \sin \theta$, then $r' = -\cos \theta$, so that the resulting arc length is

$$\begin{aligned}
 L &= 2 \int_{\theta=-\pi/2}^{\pi/2} \sqrt{1 - 2 \sin \theta + \sin^2 \theta + \cos^2 \theta} d\theta \\
 &= 2 \int_{\theta=-\pi/2}^{\pi/2} \sqrt{2(1 - \sin \theta)} d\theta = 2\sqrt{2} \int_{\theta=-\pi/2}^{\pi/2} (\cos \frac{\theta}{2} - \sin \frac{\theta}{2}) d\theta \\
 &= 4\sqrt{2} (\sin \frac{\theta}{2} + \cos \frac{\theta}{2}) \Big|_{\theta=-\pi/2}^{\pi/2} = 4\sqrt{2} (\sqrt{2}) = 8
 \end{aligned}$$

For the curve $r = -1 - \sin \theta$, then $r' = -\cos \theta$, so that the resulting arc length is

$$\begin{aligned}
 L &= 2 \int_{\theta=-\pi/2}^{\pi/2} \sqrt{1 + 2 \sin \theta + \sin^2 \theta + \cos^2 \theta} d\theta \\
 &= 2 \int_{\theta=-\pi/2}^{\pi/2} \sqrt{2(1 + \sin \theta)} d\theta = 2\sqrt{2} \int_{\theta=-\pi/2}^{\pi/2} (\cos \frac{\theta}{2} + \sin \frac{\theta}{2}) d\theta \\
 &= 4\sqrt{2} (\sin \frac{\theta}{2} - \cos \frac{\theta}{2}) \Big|_{\theta=-\pi/2}^{\pi/2} = 4\sqrt{2} (\sqrt{2}) = 8
 \end{aligned}$$

Problem 8.

Let there is a cardioid $r = \pm 2 - 2 \sin \theta$, where the rotation angle is $0 \leq \theta \leq 2\pi$. Determine the area and arc length be made by that cardioid.

Solution:

We choose $r_{\max} = 4$, $4 = 2 - 2 \sin \theta$, $\theta = -\frac{\pi}{2}$. If we choose $r_{\min} = 0$, $0 = 2 - 2 \sin \theta$, $\theta = \frac{\pi}{2}$. Therefore, the rotation angle is $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

The graph of the cardioid be preseted as follow.

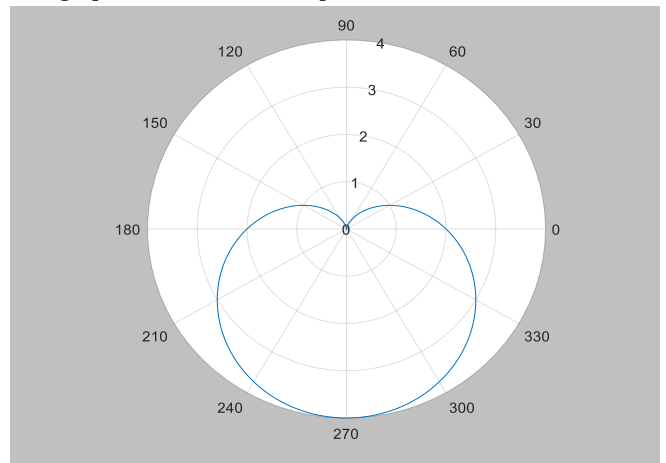


Figure 8. The cardioid $r = \pm 2 - 2 \sin \theta$

The area produced by cardioid $r = 2 - 2 \sin \theta$ is

$$\begin{aligned}
 A &= \int_{\theta=-\pi/2}^{\pi/2} (4 - 8 \sin \theta + 4 \sin^2 \theta) d\theta \\
 &= \int_{\theta=-\pi/2}^{\pi/2} [4 - 8 \sin \theta + 2(1 - \cos 2\theta)] d\theta \\
 &= (4\theta + 8 \cos \theta + 2(\theta - \frac{1}{2} \sin 2\theta)) \Big|_{\theta=-\pi/2}^{\pi/2} = 6\pi
 \end{aligned}$$

The area produced by cardioid $r = -2 - 2\sin \theta$ is

$$\begin{aligned} A &= \int_{\theta=-\pi/2}^{\pi/2} (4 + 8\sin \theta + 4\sin^2 \theta) d\theta \\ &= \int_{\theta=-\pi/2}^{\pi/2} [4 + 8\sin \theta + 2(1 - \cos 2\theta)] d\theta \\ &= (4\theta - 8\cos \theta + 2(\theta - \frac{1}{2}\sin 2\theta)) \Big|_{\theta=-\pi/2}^{\pi/2} = 6\pi \end{aligned}$$

For curve $r = 2 - 2\sin \theta$, then $r' = -2\cos \theta$, so that the resulting arc length is

$$\begin{aligned} L &= 2 \int_{\theta=-\pi/2}^{\pi/2} \sqrt{4 - 8\sin \theta + 4\sin^2 \theta + 4\cos^2 \theta} d\theta \\ &= 2 \int_{\theta=-\pi/2}^{\pi/2} \sqrt{8(1 - \sin \theta)} d\theta = 4\sqrt{2} \int_{\theta=-\pi/2}^{\pi/2} (\cos \frac{\theta}{2} - \sin \frac{\theta}{2}) d\theta \\ &= 8\sqrt{2} (\sin \frac{\theta}{2} + \cos \frac{\theta}{2}) \Big|_{\theta=-\pi/2}^{\pi/2} = 8\sqrt{2} (\sqrt{2}) = 16 \end{aligned}$$

For curve $r = -2 - 2\sin \theta$, then $r' = -2\cos \theta$, so that the resulting arc length is

$$\begin{aligned} L &= 2 \int_{\theta=-\pi/2}^{\pi/2} \sqrt{4 + 8\sin \theta + 4\sin^2 \theta + 4\cos^2 \theta} d\theta \\ &= 2 \int_{\theta=-\pi/2}^{\pi/2} \sqrt{8(1 + \sin \theta)} d\theta = 4\sqrt{2} \int_{\theta=-\pi/2}^{\pi/2} (\cos \frac{\theta}{2} + \sin \frac{\theta}{2}) d\theta \\ &= 8\sqrt{2} (\sin \frac{\theta}{2} - \cos \frac{\theta}{2}) \Big|_{\theta=-\pi/2}^{\pi/2} = 8\sqrt{2} (\sqrt{2}) = 16 \end{aligned}$$

Another expansion of the multiplication of any negative real number with sine function to find the area and arc length is written in Theorem 6. The theorem is a generalization in determining the area and length of a curve $r = \pm a - a\sin \theta$, for any negative real number $-a$.

Theorem 6.

If there is a curve $r = \pm a - a\sin \theta$, where $-a$ is any negative real number, $0 \leq \theta \leq 2\pi$. Then the area be bounded by the graph is $\frac{3}{2}a^2\pi$, and arch length of the graph is $8a$.

Proof:

We choose $r_{\max} = 2a$, $2a = a - a\sin \theta$, $\theta = -\frac{\pi}{2}$. If we choose $r_{\min} = 0$, $0 = a - a\sin \theta$, $\theta = \frac{\pi}{2}$. Therefore, the rotation angle is $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

The area produced by curve $r = a - a\sin \theta$ is

$$\begin{aligned} A &= \int_{\theta=-\pi/2}^{\pi/2} (a^2 - 2a^2\sin \theta + a^2\sin^2 \theta) d\theta \\ &= \int_{\theta=-\pi/2}^{\pi/2} [a^2 - 2a^2\sin \theta + \frac{1}{2}a^2(1 - \cos 2\theta)] d\theta \\ &= (a^2\theta + 2a^2\cos \theta + \frac{1}{2}a^2(\theta - \frac{1}{2}\sin 2\theta)) \Big|_{\theta=-\pi/2}^{\pi/2} = \frac{3}{2}a^2\pi \end{aligned}$$

The area produced by curve $r = -a - a\sin \theta$ is

$$\begin{aligned} A &= \int_{\theta=-\pi/2}^{\pi/2} (a^2 + 2a^2\sin \theta + a^2\sin^2 \theta) d\theta \\ &= \int_{\theta=-\pi/2}^{\pi/2} [a^2 + 2a^2\sin \theta + \frac{1}{2}a^2(1 - \cos 2\theta)] d\theta \\ &= (a^2\theta - 2a^2\cos \theta + \frac{1}{2}a^2(\theta - \frac{1}{2}\sin 2\theta)) \Big|_{\theta=-\pi/2}^{\pi/2} = \frac{3}{2}a^2\pi \end{aligned}$$

For the curve $r = a - a\sin \theta$, then $r' = -a\cos \theta$, so that the resulting arc length is

$$\begin{aligned} L &= 2 \int_{\theta=-\pi/2}^{\pi/2} \sqrt{a^2 - 2a^2\sin \theta + a^2\sin^2 \theta + a^2\cos^2 \theta} d\theta \\ &= 2 \int_{\theta=-\pi/2}^{\pi/2} \sqrt{2a^2(1 - \sin \theta)} d\theta \\ &= 2\sqrt{2}a \int_{\theta=-\pi/2}^{\pi/2} (\cos \frac{\theta}{2} - \sin \frac{\theta}{2}) d\theta \\ &= 4\sqrt{2}a (\sin \frac{\theta}{2} + \cos \frac{\theta}{2}) \Big|_{\theta=-\pi/2}^{\pi/2} = 4\sqrt{2}a (\sqrt{2}) = 8a \end{aligned}$$

For the curve $r = -a - a\sin \theta$, then $r' = -a\cos \theta$, so that the resulting arc length is

$$\begin{aligned} L &= 2 \int_{\theta=-\pi/2}^{\pi/2} \sqrt{a^2 + 2a^2\sin \theta + a^2\sin^2 \theta + a^2\cos^2 \theta} d\theta \\ &= 2 \int_{\theta=-\pi/2}^{\pi/2} \sqrt{2a^2(1 + \sin \theta)} d\theta \\ &= 2\sqrt{2}a \int_{\theta=-\pi/2}^{\pi/2} (\cos \frac{\theta}{2} + \sin \frac{\theta}{2}) d\theta \\ &= 4\sqrt{2}a (\sin \frac{\theta}{2} - \cos \frac{\theta}{2}) \Big|_{\theta=-\pi/2}^{\pi/2} = 4\sqrt{2}a (\sqrt{2}) = 8a \end{aligned}$$

If we take $a = 1, 2, \dots, 10$, based on Theorem 6, we could find area and arc length formed by the curve $r = \pm a - a\sin \theta$ as in Table 5 below.

Table 5. Simulation value of curve $r = \pm a - a\sin \theta$

a	$r = \pm a - a\sin \theta$	A	L
1	$r = \pm 1 - \sin \theta$	$1\frac{1}{2}\pi$	8
2	$r = \pm 2 - 2\sin \theta$	6π	16
3	$r = \pm 3 - 3\sin \theta$	$13\frac{1}{2}\pi$	24
$\vdots$	$\vdots$	$\vdots$	$\vdots$
9	$r = \pm 9 - 9\sin \theta$	$121\frac{1}{2}\pi$	72
10	$r = \pm 10 - 10\sin \theta$	150π	80

IV. CONCLUSION

A point in polar coordinate is a pair of the radius r and rotation angle θ . Polar coordinate point notation is written as (r, θ) . A set of points will form the curve $r = f(\theta)$.

If given a curve $r = a\sin \theta$, where a is a positive real number and $0 \leq \theta \leq 2\pi$. The resulting graph is a circle located in quadrants one and two. The circle has center at $(0, \frac{1}{2}a)$ and radius is half a . Whereas if given a curve $r = -a\sin \theta$, where $-a$ is a negative real number and $0 \leq \theta \leq 2\pi$. The resulting graph is a circle located in quadrants three and four. The circle has center at $(0, -\frac{1}{2}a)$ and radius is half a . The area obtained by the graph is $\frac{a^2}{4}\pi$, and the arc length of the graph is $a\pi$.

If there is a curve $r = \pm a + a\sin \theta$, where a is a positive real number and $0 \leq \theta \leq 2\pi$. The resulting graph is

a cardioid with direction to the up. While if there is a curve $r = \pm a - a \sin \theta$, where $-a$ is any negative real number and $0 \leq \theta \leq 2\pi$. The resulting graph is a cardioid with direction to the down. The area enclosed by the graph is $\frac{3}{2}a^2\pi$, and the arch length of that graph is $8a$. The general formulas for the area and arc length of this sine function serve as tools for predicting their values, if the value a is known.

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