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Kiefer Bound in Truncated Distributions

D.B. Jadhav, A.Shanubhogue

D.P.Bhosale College, Koregaon (Satara)-415501

Maharashtra, India.

dbjp1955@gmail.com

Sardar Patel University Vallabh Vidyanagar-388120

Gujarat, India.

a_shanubhogue@yahoo.com

Abstract

We consider the densities of truncated distributions in their natural form. In the estimation of parametric functions involved in these densities and their r^{th} powers, necessary prior densities are identified and attainable Kiefer bounds on variance of unbiased estimators are computed. The problem of estimation of these Kiefer bounds [which is same as estimating variance of UMVU estimators] is considered. It is shown that the variances of UMVU estimators of these Kiefer bounds have attainable Kiefer bounds. Results are illustrated by examples.

Key words: Kiefer bound; variance bound; minimum variance unbiased estimator; non regular and truncated distribution; parametric function; ideal estimation equation.

1 Introduction

In the non- regular family of distributions Cramér-Rao bound cannot be computed as the support depends upon the parameter and the regularity conditions are violated. Therefore, authors like Chapman and Robbins(1951), Fraser and Guttman(1952),Hammersley(1950),Kiefer (1952), Vincze(1979) have provided the lower bounds on the variance of estimators in non-regular situations. Amongst them the bound due to Kiefer only is attained by the variance of UMVU estimators of parameter θ. Blischke et.el. (1965-69), Polfeldt (1970), Akahira and Ohyauchi (2007) extended Kiefer's results for asymptotic situations. Bartlett (1982) extended them for parameters of some more probability distributions. Jadhav and Prasad (1986-87) extended those for some parametric functions in a family of distributions and gave a necessary and sufficient condition for the attainment of Kiefer bound. Now, we intend to extend the results for left as well as right truncated families of distributions and for some parametric functions.

2 Kiefer Bound

Let X be a r.v. having p.d.f. f(x,θ) with x∈$\mathfrak{X}$ and θ ∈**Θ**. Let ($\mathfrak{X}$, $\mathbb{F}$, μ) be the measure space with a σ -finite measure μ. An estimator T of θ or its function m(θ) is a measurable function T: $\mathfrak{X}$⟶ℝ: the real line. Let **Θ** be an interval of ℝ. For each θ ∈**Θ**, let $\boldsymbol{\Theta}_\theta$ = {h; (θ + h) ∈**Θ**}. For a fixed θ, let G_1 ,G_2 be any two probability measures ∈ G={G_i ; E_i(h)= expectation of h w.r.t. G_i exists}. Then Kiefer (1952) has proved, for the variance of estimator T of θ, that,

$$Var(T) \geq sup_{G_1,G_2 \in G} K(G_1,G_2,\theta) = K(\theta), \tag{2.1}$$

where,

$$K(G_1,G_2,\theta) = \frac{\{E_1(h)-E_2(h)\}^2}{\int_{\mathfrak{X}}\left[\frac{\left\{\int_{\Theta_\theta} f(x;\theta+h)d[G_1(h)-G_2(h)]\right\}^2}{f(x;\theta)}\right]d\mu(x)}$$

Inequality (2.1) is called Kiefer inequality and its R.H.S. K (θ) is called Kiefer bound. Identification of proper prior distributions G_1 and G_2 is the most important part to compute Kiefer bound. Kiefer (1952) and Bartlett (1982) have identified a few such prior distributions and computed Kiefer bound for variance of estimator of θ. We intend to generalize their results for some families of distributions and for variance of estimators of functions of θ.

Bartlett (1982) provides an ideal estimation equation involving parameter, estimator and its variance attaining Kiefer bound. We use it to ascertain attainment of Kiefer bound. For this the generalized difference of the p.d.f. w.r.t. prior distributions G_1 and G_2 is taken. The generalized difference of f(x,θ) w.r.t. G_1 and G_2 , is defined by

$$\Delta_2 f(x,\theta) = \int f(x,\theta+h)dG_1(h) - \int f(x,\theta+h)dG_2(h) \tag{2.2}$$

If $G_2(h)$ is concentrated at 0, generalized difference is denoted by $\boldsymbol{\Delta}_1$. Thus,

$$\Delta_1 f(x,\theta) = \int f(x,\theta+h)dG_1(h) - f(x,\theta), \qquad \Delta_1\theta = \int h dG_1(h) \tag{2.3}$$

With a proper choice of G_1, G_2, the ideal estimation equation takes the form:

$$\frac{\Delta_1 f(x,\theta)}{f(x,\theta)\Delta_1\theta} = \frac{1}{var(T)}(T-\theta) \tag{2.4}$$

From (2.4) we infer that T is UMVUE of θ with its variance = K(θ) - the Kiefer bound for the variance of unbiased estimator of θ. We describe this like: T is uniformly minimum variance unbiased estimator whose variance attains Kiefer bound (UMVUKBE).Then, attainable Kiefer bound on the variance of unbiased estimators of m (θ) = (a_n θ ± b_n), is given by,

$$K\big(m(\theta)\big) = (a_n)^2\, K(\theta) \tag{2.5}$$

Here, a_n, b_n are real constants which may depend upon the sample size n.

3 Kiefer Bound in Left Truncated Distributions

In this section we extend results due to Kiefer (1952) and Jadhav and Prasad (1986-87) to left truncated distributions. Let q(·)be a positive real valued function such that the p.d.f. of left truncated random variable becomes,

$$f_1(x;\theta) = \frac{q(x)}{Q(b) - Q(\theta)}, \qquad -\infty < a < \theta < x < b < \infty \tag{3.1}$$

where $Q(x) = \int_a^x q(y)dy$. The expression for p.d.f. in (3.1) is considered to be natural, scientific and generalized form of left truncated p.d.f.. Then cumulative distribution function is given by

$$F(x) = \int_\theta^x q(t)dt = \frac{Q(x)-Q(\theta)}{Q(b)-Q(\theta)} \text{and } 1 - F(x) = \frac{Q(b)-Q(x)}{Q(b)-Q(\theta)} \tag{3.2}$$

Theorem 3.1

Let the p.d.f. of a r. v. X be of the form (3.1). Then $T_r(Z) = \left(\frac{n+r}{n}\right)\left(Q(b) - Q(z)\right)^r$ is UMVUKBE of $\varphi = [Q(b)- Q(\theta)]^r, r > -\frac{n}{2}, so\ that\ Var\left(T_r(Z)\right)$ equals Kiefer bound

$$K(\varphi) = \frac{[r^2\varphi^2]}{[n(n+2r)]},$$

with the choice of priors as,

$$dG_1(h) = \frac{[(n/r)+1](\varphi + h)^{(n/r)}dh}{\varphi^{(n/r)+1}}, -\varphi < h < 0\ and\ G_2(h) = I_{\{0\}}(h) \tag{3.3}$$

Proof:

It can be seen from the structure of p.d.f.(3.1) that the minimum observation Z is complete sufficient statistic and so would be $Q(Z)$ .The p.d.f.s of Z and $Q(Z)$ are given by,

$$g_Z(z,\theta) = \frac{n[Q(b) - Q(z)]^{n-1}q(z)}{[Q(b)- Q(\theta)]^n}, \quad \theta < z < b \tag{3.4}$$

$$g_{Q(Z)}\left(Q(z), Q(\theta)\right) = \frac{n[Q(b) - Q(z)]^{n-1}}{[Q(b)- Q(\theta)]^n}, \qquad Q(\theta) < Q(z) < Q(b) \tag{3.5}$$

The p.d.f of $Q(Z)$ in terms of φ is,

$$g_{Q(Z)}(Q(z), \varphi) = \frac{n[Q(b) - Q(z)]^{n-1}}{\varphi^{\frac{n}{r}}}, \qquad 0 < [Q(b)- Q(z)]^r < \varphi \tag{3.6}$$

The parameter space of φ is $\Phi = \{\varphi;\ g_{Q(Z)}(Q(z), \varphi) > 0\} = \left(0, [Q(b)- Q(a)]^r\right)$. For each $\varphi \epsilon \Phi$, Let $\Phi\varphi = \{h; (\varphi + h)\epsilon\ \Phi\} = \left\{h: h\epsilon(-\varphi, [Q(b)- Q(a)]^r - \varphi)\right\}$.The prior distribution is defined on the subset $(-\varphi, 0)$ of $\Phi\varphi$, as defined in (3.3).

Then,

$$E_1(h) = \frac{-r\varphi}{n+2r} \quad \text{and} \qquad E_2(h) = 0 \tag{3.7}$$

If φ is increased to φ + h, from (3.6), we get $[(Q(b) - Q(z))^r - \varphi] < h < 0$.

Therefore, from (2.3),(3.3) and (3.6), we have,

$$\Delta_1\, g_{Q(Z)}(Q(z), \varphi)$$

$$= \int_{-\varphi}^{0} n \left[\frac{\{Q(b) - Q(z)\}^{n-1}}{(\varphi + h)^{\frac{n}{r}}}\right] \left[\frac{\left(\frac{n}{r} + 1\right)(\varphi + h)^{\frac{n}{r}}}{\varphi^{\left(\frac{n}{r}+1\right)}}\right] I_{\left([(Q(b)-Q(z))^r - \varphi],\ 0\right)}(h)\,dh\ .$$

$$-\ n \left[\frac{[Q(b) - Q(z)]^{n-1}}{\varphi^{\frac{n}{r}}}\right]$$

$$= n\{Q(b) - Q(z)\}^{n-1} \left[\frac{\left(\frac{n}{r} + 1\right)}{\varphi^{\left(\frac{n}{r}+1\right)}}\right] \int_{\{Q(b) - Q(z)\}^r - \varphi}^{0} dh$$

$$-\ n \left[\frac{[Q(b) - Q(z)]^{n-1}}{\varphi^{\frac{n}{r}}}\right]$$

$$= \frac{[Q(b) - Q(Z)]^{n-1}}{\varphi^{\frac{n}{r}}} \left\{\left[\frac{\left(\frac{n}{r} + 1\right)}{\varphi}\right] [\varphi - \{Q(b) - Q(z)\}^r] - 1\right\}$$

Thus,

$$\frac{\Delta_1 g_{Q(Z)}(Q(z),\varphi)}{g_{Q(Z)}(Q(z),\varphi)\Delta_1\varphi} = \left\{\left(\frac{n}{r} + 1\right)[\varphi - \{Q(b) - Q(z)\}^r] - \varphi\right\}\left\{\frac{n+2r}{-r\varphi^2}\right\}.$$

Therefore, the ideal estimation equation becomes

$$\frac{\Delta_1 g_{Q(Z)}(Q(z), \varphi)}{g_{Q(Z)}(Q(z), \varphi)\Delta_1\varphi} = \left(\frac{n(n + 2r)}{r^2\varphi^2}\right)\left\{\left(\frac{n + r}{n}\right)\{Q(b) - Q(z)\}^r - \varphi\right\} \tag{3.8}$$

To verify the validity of (3.8) as (2.4), consider,

$$E[Q(b) - Q(Z)]^k = \int_0^{Q(b)-Q(\theta)} [Q(b) - Q(z)]^k g_{Q(Z)}\big(Q(z), Q(\theta)\big)\,dQ(z)$$

$$= \left(\frac{n}{n + k}\right)\big(Q(b) - Q(\theta)\big)^k \int_0^{Q(b)-Q(\theta)} (n + k)\frac{\big(Q(b) - Q(z)\big)^{n+k-1} dQ(z)}{(Q(b) - Q(\theta))^{n+k}}$$

$$= \left(\frac{n}{n+k}\right)\left(Q(b) - Q(\theta)\right)^k \quad (3.9)$$

Replacing k by r and 2r, we have

$$Var[Q(b) - Q(Z)]^r = \left(\frac{n}{n+2r}\right)\left(Q(b) - Q(\theta)\right)^{2r} - \left(\frac{n}{n+r}\right)^2 \left(Q(b) - Q(\theta)\right)^{2r}$$

$$= \frac{nr^2\varphi^2}{(n+2r)(n+r)^2} \quad (3.10)$$

From (3.9), it is clear that

$$\mathrm{E}\left\{\left(\frac{n+k}{n}\right) E\left(Q(b) - Q(Z)\right)^k\right\} = (Q(b) - Q(\theta))^k \quad (3.11)$$

Therefore,

$$Var[T_r(Z)] = \left(\frac{n+r}{n}\right)^2 Var\left(Q(b) - Q(z)\right)^r$$

$$= \left(\frac{n+r}{n}\right)^2 \frac{nr^2\varphi^2}{(n+2r)(n+r)^2}$$

$$= \frac{r^2\varphi^2}{n(n+2r)} = K(\varphi) \quad (3.12)$$

Kiefer bound K(φ) is a parametric function giving variance of UMVUE under the above conditions. Therefore, we proceed to estimate K(φ) in the following:

Corollary 3.1:

The parametric function ψ = K(φ) in (3.12) has UMVUKBE

$$T = T_{K,r}(Z) = \left(\frac{r}{n}\right)^2 [Q(b) - Q(Z)]^{2r}, \quad (3.13)$$

with $$Var\left[T_{K,r}(Z)\right] = \frac{4r^2\psi^2}{n(n+4r)} = K(\psi), r > -\frac{n}{4} \quad .$$

Proof:

From equation (3.11),

$$E\left\{\left(\frac{n+2r}{n}\right)[Q(b) - Q(z)]^{2r}\right\} = [Q(b) - Q(\theta)]^{2r}$$

Therefore, using equation (3.9),

$$E\{T_{K,r}(Z)\} = \left(\frac{r}{n}\right)^2 \frac{n}{(n+2r)} [Q(b) - Q(\theta)]^{2r}$$

$$= \frac{r^2\varphi^2}{n(n+2r)} = \psi \tag{3.14}$$

Putting k=4r, in (3.9),we get,

$$E\{T_{K,r}(Z)^2\} = \left(\frac{r}{n}\right)^4 \frac{n\varphi^4}{(n+4r)}$$

$$Var\left\{\left(\frac{r}{n}\right)^2 [Q(b)-Q(Z)]^{2r}\right\} = \left(\frac{r}{n}\right)^4 \frac{n\varphi^4}{(n+4r)} - \frac{r^4\varphi^4}{n^2(n+2r)^2}$$

$$= \frac{4r^2\psi^2}{n(n+4r)}$$

$$= \frac{4r^6\varphi^4}{n^3(n+4r)(n+2r)^2} \tag{3.15}$$

From (3.13)

$$[Q(b)-Q(z)] = \left(\frac{\mathrm{n}}{\mathrm{r}}\right)^{\frac{1}{\mathrm{r}}} t^{\frac{1}{2r}}$$

Therefore,

$$\frac{dQ(z)}{dt} = -\left(\frac{n}{r}\right)^{\frac{1}{r}} \frac{1}{2r}\, t^{\frac{1}{2r}-1}$$

Since

$$Q(a) < Q(\theta) < Q(Z) < Q(b),$$

we have

$$0 < \left(\frac{\mathrm{r}}{\mathrm{n}}\right)^2 [Q(\theta)-Q(a)]^{2r} < \left(\frac{\mathrm{r}}{\mathrm{n}}\right)^2 [Q(Z)-Q(a)]^{2r} = t < \left(\frac{\mathrm{r}}{\mathrm{n}}\right)^2 [Q(b)-Q(a)]^{2r}.$$

Again, since

$$\psi = \left(\frac{r}{n}\right)^2 \frac{n}{(n+2r)} [Q(b)-Q(\theta)]^{2r} \quad ,$$

we have

$$\left(\frac{r}{n}\right)^2 [Q(b)-Q(\theta)]^{2r} = \frac{(n+2r)}{n}\psi \quad .$$

Therefore,

$$0 < t < \frac{(n+2r)}{n}\psi \Leftrightarrow 0 < \frac{n}{n+2r} t < \psi$$

From (3.5), the p.d.f. of T is given by,

$$g(t;\psi)=\frac{n}{2r}\left(\frac{n}{n+2r}\right)^{\frac{n}{2r}}\frac{t^{\frac{n}{2r}-1}}{\psi^{\frac{n}{2r}}},0<t<\frac{n}{n+2r}\psi \tag{3.16}$$

Consider the set $\Psi=\{\psi;g(t,\psi)>0\}=\left(0,\frac{r^2[Q(b)-Q(a)]^{2r}}{n(n+2r)}\right)$. For each $\psi\in\Psi$ let us define

$$\begin{aligned}\Psi_\psi&=\{h;(\psi+h)\epsilon\Psi\}\\&=\left\{h;0<\psi+h<\frac{r^2[Q(b)-Q(a)]^{2r}}{n(n+2r)}\right\}\\&=\left\{h;-\psi<h<\frac{r^2[Q(b)-Q(a)]^{2r}}{n(n+2r)}-\psi\right\}\quad.\end{aligned}$$

On the subset $(-\psi,0)$ of Ψ_ψ, let us define prior probability distributions as

$dG_1(h)=\frac{\left(\frac{n}{2r}+1\right)(\psi+h)^{\frac{n}{r}}}{\psi^{\frac{n}{2r}+1}},-\psi<h<0$ and $dG_2(h)=\mathrm{I}_{\{0\}}(\mathrm{h})$.

Then,

$$\begin{aligned}E_1(h)&=\int_{-\psi}^{0}h\frac{\left[\left(\frac{n}{2r}\right)+1\right](\psi+h)^{\left(\frac{n}{2r}\right)}dh}{\psi^{\left(\frac{n}{2r}\right)+1}}\\&=\frac{-2r\psi}{(n+4r)}\\E_2(h)&=0\end{aligned}$$

If ψ is incremented to $(\psi+h)$ then $0<\frac{nt}{n+2r}<(\psi+h)$.Then corresponding interval for h is $\frac{nt}{n+2r}-\psi<h<0$. Using this range for h we have,

$$\begin{aligned}\Delta_1 g(t,\psi)&=\int_{\frac{nt}{n+2r}-\psi}^{0}g(t,\psi+h)dG_1(h)-g(t,\psi)\\&=\frac{n}{2r}\left(\frac{n}{n+2r}\right)^{\left(\frac{n}{2r}\right)}\left(\frac{n}{2r}+1\right)\frac{t^{\frac{n}{2r}-1}}{\psi^{\frac{n}{2r}+1}}\int_{\frac{nt}{n+2r}-\psi}^{0}dh-g(t,\psi)\\&=\frac{n}{2r}\left(\frac{n}{n+2r}\right)^{\left(\frac{n}{2r}\right)}\frac{t^{\frac{n}{2r}-1}}{\psi^{\frac{n}{2r}+1}}\left\{\left(\frac{n}{2r}+1\right)\left(\psi-\frac{nt}{n+2r}\right)-\psi\right\}\\&=\frac{\left(\frac{n}{2r}\right)\left(\frac{n}{n+2r}\right)^{\frac{n}{2r}}t^{\frac{n}{2r}-1}}{\psi^{(n/2r)+1}}\left[\left(\frac{n}{2r}\right)\psi-\frac{nt}{2r}\right]\end{aligned}$$

Then the ideal estimation equation becomes,

$$\frac{\Delta_1 g(t;\psi)}{g(t;\psi)\Delta_1\psi} = \frac{(n+4r)}{\psi}\frac{\left[\left(\frac{n}{2r}\right)\psi - \frac{nt}{2r}\right]}{-2r\psi}$$

$$= \frac{n(n+4r)}{4r^2\psi^2}[t-\psi] \qquad (3.17)$$

Thus,

$$\hat{\psi} = \left(\frac{r}{n}\right)^2 [Q(y)-Q(a)]^{2r}$$

is UMVUKBE of

$$\psi = \frac{r^2\varphi^2}{[n(n+2r)]}$$

with its variance

$$Var(\hat{\psi}) = \frac{4r^2\psi^2}{n(n+4r)}$$

$$= \frac{4r^2}{n(n+4r)}\frac{r^4[Q(\theta)-Q(a)]^{4r}}{[n^2(n+2r)^2]} \qquad = \text{K}(\psi).$$

where, K(ψ) is the Kiefer bound on the variance of unbiased estimator of ψ .

***Corollary 3.2*:**

$\hat{\hat{\bar{F}}}(t) = \frac{(n-1)}{n}\frac{Q(b)-Q(t)}{(Q(b)-Q(Z))}$ is UMVUKBE of survival function of lifetime with left truncated distribution having p.d.f. (3.1) with its variance which equals Kiefer bound

$$K(\hat{\hat{\bar{F}}}(t)) = \frac{(Q(b)-Q(t))^2}{n(n-2)(Q(b)-Q(\theta))^2}.$$

Proof:

From (3.2) we have, $\bar{F}(t) = \frac{Q(b)-Q(t)}{Q(b)-Q(\theta)}$. Now, putting r = -1 in (3.9), we get

$$E\left(\frac{n-1}{n}\right)\left(Q(b)-Q(z)\right)^{-1} = \frac{1}{Q(b)-Q(\theta)}$$

Therefore,

$$(\hat{\hat{\bar{F}}}(t)) = \frac{Q(b)-Q(t)}{Q(b)-Q(\theta)} \quad .$$

Now using the results in (3.12) with r= -1, (2.5) and the relation $Var(\hat{\hat{F}}(t)) = Var(1-\hat{\hat{F}}(t))$ we get,

$$Var\left(\frac{(n-1)}{n}\frac{Q(b)-Q(t)}{\left(Q(b)-Q(z)\right)}\right) = \frac{\left(Q(b)-Q(t)\right)^2}{n(n-2)\left(Q(b)-Q(\theta)\right)^2}$$

$$= K\left(\bar{F}(t)\right) = K\left(F(t)\right),$$

Remark 3.1: $T_1(Z) = \left[\frac{n+1}{n}\right]Q(z) - \frac{Q(b)}{n}$ is UMVUKBE of Q(θ) with its variance

$\frac{[Q(b)-Q(\theta)]^2}{n(n+2)} =$ Kiefer bound, K(Q(θ)).

Example 3.1: Let $f_1(x;\theta) = e^{-(x-\theta)}; \theta < x < \infty$. For this density we have $q(x) = e^{-x}$, $Q(x)=-e^{-x}$, $Q(b=\infty)=0$. Further, $F(x) = 1 - e^{-(x-\theta)}$. Obviously, Z = minimum of the random sample of size n is complete sufficient for θ. Then the pdf of Z and Q(Z) are respectively given by

$$g_z(z,\theta) = ne^{-n(z-\theta)}, z > \theta,$$
$$g_{Q(z)}\left(Q(z),Q(\theta)\right) = ne^{-(n-1)z+n\theta}.$$

Then the UMVUKBE of $\varphi(\theta) = e^{-\theta}$ is given by $T_r(Z) = \frac{(n+r)}{n}e^{-rz}$ with its Kiefer bound $\frac{r^2}{n(n+2r)}e^{-2r\theta} = K\left(e^{-r\theta}\right) = r > -n/2.$

Further, $T_{K,r}(Z) = \left(\frac{r}{n}\right)^2 e^{-2rz}$ is UMVUKBE of $K\left(e^{-r\theta}\right) = \frac{r^2}{n(n+2r)}e^{-2r\theta}$

with its Kiefer Bound $= \frac{4r^6e^{-4r\theta}}{[n^3(n+4r)(n+2r)^2]} = \frac{4r^2\psi^2}{[n\ \ (n+4r)]} = K(\psi), r > -n/4$

Again , from Corollary 3.2., UMVUKBE of $\bar{F}(t) = e^{-(t-\theta)}$ is $\frac{(n-1)}{n}e^{-(t-z)}$ with its Kiefer Bound $K\left(\hat{\hat{F}}(t)\right) = \frac{e^{-2(t-\theta)}}{n(n-2)}$

Example 3.2: Let, $f_1(x;\theta) = (b-\theta)^{-1}, \theta < x < b$. Here, $q(x) = 1$ and $Q(\theta) = \theta$.

Then, $T_r(Z) = \frac{(n+r)}{n}(b-Z)^r$ is UMVUKBE of $(b-\theta)^r$ with its Kiefer Bound is given by $\frac{r^2(b-\theta)^{2r}}{n(n+2r)} = K((b-\theta)^r), r > -n/2$. Now, taking $r=1$, we have $T_1(Z) = \frac{(n+1)}{n}(b-Z)$ as UMVUKBE of $(b-\theta)$ with its Kiefer Bound $\frac{(b-\theta)^2}{n(n+2)} = K(b-\theta) = K(\theta)$.Therefore,

$\hat{\theta} = \left[\frac{n+1}{n}\right] Z - \frac{b}{n}$ is UMVUKBE of θ with $Var(\hat{\theta}) = \frac{(b-\theta)^2}{n(n+2)} = K(b-\theta)$.Also, UMVUKBE of Kiefer bound $K((b-\theta)^r) = \frac{r^2(b-\theta)^{2r}}{n(n+2r)} = \psi$ is, $T_{K,r}(Z) = \left(\frac{r}{n}\right)^2 (b-Z)^{2r}$ with variance, $\frac{4r^2\psi^2}{[n\,(n+4r)]} = K(\psi)$.

Example 3.3: Let $f_1(x;\theta) = \frac{e^{-x}}{[e^{-\theta}-e^{-1}]}$, $1 < \theta < x < b$.Here, $q(x) = e^{-x}$, $Q(x) = -e^{-x}$. Then, $T_r(Z) = \frac{(n+r)}{n}(e^{-z}-e^{-1})^r$ is UMVUKBE of $\left(e^{-\theta}-e^{-1}\right)^r$ with Kiefer Bound $K((e^{-\theta}-e^{-1})^r = \frac{r^2\left(e^{-\theta}-e^{-1}\right)^{2r}}{n(n+2r)}), r > -n/2$. Further, $T_1(Z) = \frac{(n+1)}{n}(e^{-z}-e^{-1})$ is UMVUKBE of $(e^{-\theta}-e^{-1})$ with its Kiefer Bound $K\left(e^{-\theta}-e^{-1}\right) = \frac{(e^{-\theta}-e^{-1})^2}{n(n+2)}$.Therefore $T_1(Z) + \frac{e^{-1}}{n}$ is UMVUKBE of $e^{-\theta}$ with it Kiefer Bound $K(e^{-\theta}-e^{-1}) = \mathrm{Var}[T_1(Z) + \frac{e^{-1}}{n}] = \mathrm{Var}(T_1(Z))$. Furthermore, $T_{K,r}(Z) = \left(\frac{r}{n}\right)^2 (e^{-Z}-e^{-1})^{2r}$ is UMVUKBE of
$K\left(\left(e^{-\theta}-e^{-1}\right)^r\right) = \frac{r^2\left(e^{-\theta}-e^{-1}\right)^{2r}}{n(n+2r)} = \psi$ with its Kiefer Bound $K(\psi) = \frac{4r^2\psi^2}{[n\,(n+4r)]}$.

4. Kiefer Bound in Right Truncated Distributions

In this section we extend results due to Kiefer(1952) and Jadhav and Prasad(1986-87) to right truncated distributions admitting maximum observation as sufficient statistic. Let the p.d.f. of right truncated r.v. be

$$f_2(x;\theta) = \frac{q(x)}{Q(\theta) - Q(a)}, \qquad -\infty < a < x < \theta < b < \infty \tag{4.1}$$

where , $\int_a^\theta q(x)dx = Q(\theta) - Q(a)$.

And the cumulative probability distribution function is

$$F(x) = \int_a^x u(t)dt = \frac{Q(x)-Q(a)}{Q(\theta)-Q(a)} \tag{4.2}$$

Theorem 4.1:

If X is a right truncated r.v. with p.d.f. of the form given in equation (4.1) and prior probability distributions,

$$dG_1(h) = \frac{[(n/r)+1](\varphi+h)^{(n/r)}dh}{\varphi^{(n/r)+1}}, -\varphi < h < 0 \text{ and } G_2(h) = I_{\{0\}}(h) \tag{4.3}$$

are used, then UMVUKBE of $\varphi = \left[Q(\theta) - Q(a)\right]^r$ is $T_r(Y) = \frac{(n+r)}{n}[Q(y) - Q(a)]^r$ with variance

$$Var[T_r(Y)] = \frac{r^2\varphi^2}{[n(n+2r)]} = K(\varphi) .$$

Proof:

It can be seen from the structure of p.d.f.(4.1) that maximum observation Y is complete sufficient statistic and so is $Q(Y)$ with respective p.d.f.s

$$g_Y(y,\theta) = \frac{n[Q(y)-Q(a)]^{n-1}q(y)}{[Q(\theta)-Q(a)]^{n}}; -\infty < a < y < \theta < b < \infty \tag{4.4}$$

$$g_{Q(Y)}\big(Q(y),Q(\theta)\big) = \frac{n[Q(y)-Q(a)]^{n-1}}{[Q(\theta)-Q(a)]^{n}}; -\infty < Q(a) < Q(y) < Q(\theta) < Q(b) < \infty \tag{4.5}$$

Let us write $g_{Q(Y)}\big(Q(y),Q(\theta)\big)$ in terms of $\varphi = [Q(\theta) - Q(a)]^r$ as,

$$g_{Q(Y)}(Q(y),\varphi) = \frac{n[Q(y)-Q(a)]^{n-1}}{\varphi^{\frac{n}{r}}}; -\infty < Q(a) < Q(y) < Q(\theta) < Q(b) < \infty \tag{4.6}$$

Let, $\Phi = \{\varphi; g_{Q(Y)}(Q(y),\varphi) > 0\} = (0,[Q(b)-Q(a)]^r)$. For each $\varphi \epsilon \Phi$, let $\Phi_\varphi = \{h;\ \varphi + h\ \epsilon\ \Phi\} = (-\varphi, [Q(b)-Q(a)]^r - \varphi)$. Further, Let us define the probability distributions (4.3) on the subset $(-\varphi, 0)$ of $\Phi\varphi$. Clearly, $E_1(h) = \frac{-r\varphi}{n+2r}$ and $E_2(h) = 0$

Then, from equation (4.5), we have,

$$0 < [Q(y) - Q(a)]^r < [Q(\theta) - Q(a)]^r = \varphi < [Q(b) - Q(a)]^r.$$

If φ is incremented to $\varphi + h$ then $0 < [Q(y) - Q(a)]^r - \varphi < h < [Q(b) - Q(a)]^r - \varphi$.

Using this fact we have

$$\int_{[Q(y)-Q(a)]^r-\varphi}^{0} g_{Q(Y)}(Q(y),\varphi+h)\,dG_1(h)$$

$$= \frac{n[Q(y)-Q(a)]^{n-1}}{\varphi^{\left(\frac{n}{r}\right)+1}}\left[\left(\frac{n}{r}\right)+1\right]\int_{[Q(y)-Q(a)]^r-\varphi}^{0} dh$$

$$= \frac{n[Q(y)-Q(a)]^{n-1}}{\varphi^{(n/r)+1}}[(n/r)+1]\{-([Q(y)-Q(a)]^r-\varphi) \tag{4.7}$$

Therefore,

$$\Delta_1 g_{Q(Y)}(Q(y),\varphi) = \frac{n[Q(y)-Q(a)]^{n-1}}{\varphi^{(n/r)}}\left\{\frac{\left[\left(\frac{n}{r}\right)+1\right]\{-([Q(y)-Q(a)]^r-\varphi)\}}{\varphi} - 1\right\} \tag{4.8}$$

Then, ideal estimation equation is given by,

$$\frac{\Delta_1 g_{Q(Y)}(Q(y),\varphi)}{g_{Q(Y)}(Q(y),\varphi)\Delta_1\varphi} = \left\{\frac{\left[\left(\frac{n}{r}\right)+1\right]\{-([Q(y)-Q(a)]^r-\varphi)\}-\varphi}{\varphi(-r\varphi)}\right\}(n+2r)$$

$$= \left\{\frac{n\,(n+2r)}{(r\varphi)^2}\right\}\left\{\frac{(n+r)}{n}[Q(y)-Q(a)]^r - \varphi\right\} \tag{4.9}$$

We have,

$$E[Q(y)-Q(a)]^k = \int_{Q(a)}^{Q(\theta)} [Q(y)-Q(a)]^k \frac{n[Q(y)-Q(a)]^{n-1}dQ(y)}{[Q(\theta)-Q(a)]^n}$$

$$= \left[\frac{n}{n+k}\right] [Q(\theta) - Q(a)]^k \tag{4.10}$$

Therefore,

$$E[T_k(Y)] = [Q(y) - Q(a)]^k \tag{4.11}$$

Therefore, putting k=r in (4.11), we have,

$$E[T_r(Y)] = [Q(\theta) - Q(a)]^r = \varphi \tag{4.12}$$

Putting k= r and 2r in (4.10)

$$Var[Q(y) - Q(a)]^r = \left[\frac{n}{n+2r}\right] \varphi^2 - \left[\frac{n}{n+r}\right]^2 \varphi^2$$

$$= \frac{nr^2\varphi^2}{[(n+r)^2(n+2r)]} \tag{4.13}$$

Therefore, using (4.13), we have,

$$Var[T_r(Y)] = Var\left\{\left[\frac{n+r}{n}\right][Q(Y) - Q(a)]^r\right\}$$

$$= \left[\frac{n+r}{n}\right]^2 \frac{nr^2\varphi^2}{[(n+r)^2(n+2r)]}$$

$$= \frac{r^2\varphi^2}{[n(n+2r)]} = K(\varphi). \tag{4.14}$$

Thus, conclusions follow from (4.9) as in (2.4).

Kiefer bound K(φ) is a parametric function giving variance of UMVUE under the above conditions. Therefore, estimating K(φ)is essential. We do it in the following:

Corollary 4.1:

The parametric function $K(\varphi) = \psi$, in equation (4.14) has UMVUKBE

$$T_{K,r}(Y) = \left(\frac{r}{n}\right)^2 [Q(y) - Q(a)]^{2r} \quad , \tag{4.15}$$

with variance

$$Var[T_{K,r}(Y)] = \frac{4r^2\psi^2}{n(n+4r)}$$
$$= K(\psi) \tag{4.16}$$

Proof:

Putting k=2r in equation (4.10), $E[Q(y) - Q(a)]^{2r} = \frac{n}{n+2r}\varphi^2$.Therefore, from equation (4.15),

$$Var[T_{K,r}(Y)] = \frac{4r^2\psi^2}{n(n+4r)}$$
$$= K(\psi) \tag{4.16}$$

$$E(\ T_{K,r}(Y))^2 = \left(\frac{r}{n}\right)^4 E[Q(Y) - Q(a)]^{4r}$$

$$= \frac{r^4\varphi^4}{n^3(n+4r)}$$

Therefore,

$$Var(T_{K,r}(Y)) = \frac{\left(\frac{r}{n}\right)^4 n\varphi^4}{[(n+4r)]} - \frac{\{r^2\varphi^2\}^2}{[n(n+2r)]^2}$$
$$= \frac{4r^6}{n^3}\frac{\varphi^4}{(n+4r)(n+2r)^2}$$
$$= \frac{4r^2\psi^2}{[n(n+4r)]} = K(\psi)$$

For Kiefer bound K(ψ) on variance of unbiased estimator of K(φ),consider p.d.f. of $T_{K,r}$(Y).

From $T = \left(\frac{r}{n}\right)^2 [Q(y) - Q(a)]^{2r}$ we have $[Q(y) - Q(a)] = [t\left(\frac{n}{r}\right)^2]^{\frac{1}{2r}}$. Then Jocobian of transformation is

$$\frac{dQ(y)}{dt} = \frac{1}{2r}\left(\frac{n}{r}\right)^2 [t\left(\frac{n}{r}\right)^2]^{\frac{1}{2r}-1}$$

Therefore the density of

$$\text{g}(\text{t}, \psi) = \frac{n}{2r}\left(\frac{n}{n+2r}\right)^{\frac{n}{2r}} \frac{t^{\frac{n}{2r}-1}}{\psi^{\frac{n}{2r}}},\ 0 < t < \frac{n}{n+2r}\psi \qquad (4.16)$$

From (3.17) and (4.16), it is clear from (3.15) that Kiefer bound K(ψ) on the variance of unbiased estimator of ψ = K(φ) is given by

$$K(\psi) = \frac{4r^2\psi^2}{[n(n+4r)]}$$
$$= Var(T_{K,r}(Y)) \qquad (4.17)$$

Corollary 4.2:

$\hat{\hat{F}}(t) = \frac{(Q(y)-Q(t))}{(Q(y)-Q(a))} + \frac{Q(t)-Q(a)}{n(Q(y)-Q(a))}$ is UMVUKBE of survival function of lifetime with right truncated distribution having p.d.f. (4.1) with its variance which equals Kiefer bound $K(\hat{\hat{F}}(t) = \frac{(Q(t)-Q(a))^2}{n(n-2)\left(Q(\theta)-Q(a)\right)^2}$.

Proof:

From (4.2), $F(t) = \frac{\text{Q(t)}-\text{Q(a)}}{\text{Q}(\theta)-\text{Q(a)}}$. Using r = -1 in (4.11), and (4.14) we have,

$\text{E}\left(\frac{\text{n}-1}{\text{n}}\right)\left(\text{Q(y)} - \text{Q(a)}\right)^{-1} = \frac{1}{(\text{Q(b)}-\text{Q}(\theta))}$ and $Var\left[\left(\frac{\text{n}-1}{\text{n}}\right)\left(\text{Q(y)} - \text{Q(a)}\right)^{-1}\right] = \frac{\varphi^{-2}}{[\text{n}(\text{n}-2)]} = \text{K}(\varphi)$

Therefore,

$$\hat{F}(t) = \frac{(n-1)}{n}\frac{Q(t)-Q(a)}{Q(y)-Q(a)},$$

and

$$Var\left(\hat{F}(t)\right) = [Q(t) - Q(a)]^2\, Var\left[\left(\frac{\text{n}-1}{\text{n}}\right)\left(\text{Q(y)} - \text{Q(a)}\right)^{-1}\right]$$
$$= [Q(t) - Q(a)]^2 \frac{\varphi^{-2}}{[n(n-2)]}, \text{putting k} = -1 \text{ in (4.13)}$$
$$= \frac{\left(Q(t) - Q(a)\right)^2}{n(n-2)\left(Q(\theta) - Q(a)\right)^2}$$
$$= K(F(t)) \text{ , follows from(2.5).}$$

Then the results follow by solving $\hat{\hat{F}}$(t) = 1- $\hat{F}$(t) and using the fact that,

$$Var(1-\hat{F}(t)) = Var\left(\hat{F}(t)\right).$$

Example 4.1: We consider the following density given by Tate (1959), $f_2(x;\ \theta) = \frac{a\theta}{(\theta-a)x^2}$; $0 < a < x < \theta$; Here, $q(x) = \frac{1}{x^2}$, and $Q(x) = -\frac{1}{x}$.The maximum observation y is complete sufficient statistic and $Q(y) - Q(a) = \left[\frac{-1}{y} - \frac{-1}{a}\right] = \left[\frac{y-a}{ay}\right]$.Also, $Q(\theta) - Q(a) = \frac{\theta-a}{a\theta}$.

According to Theorem 4.1, $T_r(Y)= \binom{n+r}{n}\left[\frac{y-a}{ay}\right]^r$ is UMVUKBE of $\varphi = \left[\frac{\theta-a}{a\theta}\right]^r$ with

$Var\left(T_r(Y)\right) = \frac{r^2}{n(n+2r)}\left[\frac{\theta-a}{a\theta}\right]^{2r}$ = K(φ) = ψ.This Kiefer bound ψ has UMVUKBE

$T_{K,r}(Y) = \left(\frac{r}{n}\right)^2\left[\frac{Y-a}{ay}\right]^{2r}$ with its variance $\left(\frac{4r^6}{n^3\ (n+4r)(n+2r)^2}\right)\left[\frac{\theta-a}{a\theta}\right]^{4r}$ attaining its Kiefer bound K(ψ),r> $\frac{-n}{4}$.

Again F(t)= P[X $\leq$ t] $= \frac{(t-a)}{at}\frac{a\theta}{(\theta-a)} = \frac{\theta(t-a)}{t(\theta-a)}$ for t real.From Corollary 4.1, $\hat{F}(t) = \frac{(n-1)}{n}\frac{y(t-a)}{t(y-a)}$ is UMVUKBE of $F(t) = \frac{\theta(t-a)}{t(\theta-a)}$ with $Var\ (\hat{F}(t)) = \frac{\theta^2(t-a)^2}{n(n-2)t^2(\theta-a)^2} = K(\ F(t)) = K(\bar{F}(t))$,because Var($\hat{\bar{F}}$(t))= Var(1-$\hat{\bar{F}}$(t)). $\hat{\bar{F}}(t) = \frac{(n+t-a)y-nt}{nt(y-a)}$ is also UMVUKBE.

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