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Some Generalizations of Differential Polynomials of Meromorphic Functions

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Abstract : In this paper, we prove two theorems on differential polynomials of meromorphic functions sharing a non zero polynomial which generalize a result of Xiao-Min Li and Ling Gao[4]. Also we prove two theorems of differential polynomials of meromorphic functions sharing (1,2) and (1, 1) which ultimately generalize two results of Jin-Dong Li[7].

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1. Introduction:

Let f be a non constant meromorphic fuction. This always means that f is defined in the whole complex plane. For standard notations of the Nevanlinna theory such as $T(r,f), N(r,f), S(r,f)$ etc., we refer to [1, 2].

Let f and g be two such meromorphic fuctions. Let c be a finite complex number. We say that f and g share the same value c CM(counting multiplicity) if $f-c$ and $g-c$ have the same zeros with the same multiplicities. Similarly we say that f and g share c IM(ignoring multiliplicities)if $f-c$ and $g-c$ have the same zeros ignoring multiplicities. Also if $\frac{1}{f}$ and $\frac{1}{g}$ share 0 CM(resp. IM), we say that f and g share ∞ CM(resp. IM).

Definition 1.1 [3] : Let k be any positive integer and f be a meromorphic function. Then for any $a \in C \cup \{\infty\}$, the notation $N_{k)}(r,a;f)$ means the counting function of those a-points of f (counting multiplicities) whose multiplicity are not greater than k and $\overline{N}_{k)}(r,a;f)$ means the counting function of those a-points of f whose multiplicities are not greater than k, where each a point is counted only once. Similarly, we denote $N_{(p}(r,a;f)$ to mean the counting function of those a-points of f (counting multiplicities) whose multiplicities are not smaller than p and also denote $N_{(p}(r,a;f)$ to mean the counting function of those a-points of f whose multiplicities are not smaller than p, where each a - point is counted only once.

Definition 1.2[4] : Let a be any value in the extended complex plane, and let k be any arbitrary non negative integer. We define

$$\delta_k(a;f) = 1 - \limsup_{r\to\infty} \frac{N_k(r,a;f)}{T(r,a;f)}.$$

Definition 1.3[7] Let k be a non negative integer or infinity. For any complex number $a \in C \cup \{\infty\}$, we denote $E_k(a,f)$ as the set of all a-points of f, where an a-point of multiplicity p is counted p times if $p \le k$ and k + 1 times if p > k. If $E_k(a, f) = E_k(a, g)$, one says that f,g share the value a with weight k. We say f,g share (a, k) this means that f,g share the value a with weight k. Then it is clear that if f,g share (a,k) then f,g share (a,q) for all integers q with $0 \le q \le k$.

In 2010, Xiao-Min Li and Ling Gao [4] proved the following theorem.

Theorem 1 .4[4]: Let f and g be two transcendental meromorphic functions, let $P \ne 0$ be a polynomial and $n \ge 15$ be an integer. If $(f^n(f-1))' - P$ and $(g^n(g-1))' - P$ share 0 CM and $\Theta(\infty, f) > \frac{2}{n}$ then f = g.

In 2011, Jin-Dong Li[7], proved the following theorems.

Theorem 1.5[7]: Let $f(z)$ and $g(z)$ be two non constant meromorphic functions and let n, k be two positive integers with $n > 3k+11$. *If* $\Theta(\infty, f) > \frac{2}{n}$, $[f^n(z)(f(z)-1)]^{(k)}$ *and* $[g^n(z)(g(z)-1)]^{(k)}$ share (1, 2) then $f(z) = g(z)$ *or* $[f^n(z)(f(z)-1)]^{(k)}.[g^n(z)(g(z)-1]^{(k)} \equiv 1.$

Theorem 1.6[7]: Let $f(z)$ and $g(z)$ be two non constant meromorphic functions and let n, k be two positive integers with $n > 5k+14$. *If* $\Theta(\infty, f) > \frac{2}{n}$, $[f^n(z)(f(z)-1)]^{(k)}$ and $[g^n(z)(g(z)-1)]^{(k)}$ share (1, 1) then $f(z)=g(z)$ or $[f^n(z)(f(z)-1]^{(k)}.\ [g^n(z)(g(z)-1]^{(k)} \equiv 1$

In this paper, we have studied the behavior of certain weighted sharing of non linear differential polynomials generated by a transcendental meromorphic functions as well as the behavior of certain non linear differential polynomials generated by a transcendental meromorphic functions sharing one point CM. In fact, we offer certain new theorems, the above mentioned theorems follows as a consequence from our new theorems.

2. Main Theorems

In this section, we will prove the following four theorems :

Theorem 2.1: Let f and g be two transcendental meromorphic functions, let n > 17 be a positive integer and n is not divisible by 2 and let $P \neq 0$ be a polynomial. If $[f^n(f^2-1)]' - P$ and $[g^n(g^2-1)]' - P$ share 0 CM then, $f = g$.

Theorem 2.2: Let f and g be two transcendental meromorphic functions, let n > 3m + 11 be a positive integer where m > 2 is also a positive integer and n is not divisible by m and let $P \neq 0$ be a polynomial. If $[f^n(f^m-1)]' - P$ and $[g^n(g^m-1)]' - P$ share 0 CM, then $f = g$.

Theorem 2.3: Let f and g be two non constant meromorphic function and let n, m and k be three positive integers with n > 3m + 3k + 8. If $(f^n(f^m-1))^{(k)}$ and $(g^n(g^m-1))^{(k)}$ share (1,2) then $f = g$ or $[f^n(f^m-1)]^{(k)}.[g^n(g^m-1)]^{(k)} \equiv 1$.

Theorem 2.4: Let f and g be two non constant meromorphic function and let n, m and k be three positive integers with n > 4m + 5k + 10. If $(f^n(f^m-1))^{(k)}$ and $(g^n(g^m-1))^{(k)}$ share (1,1) then $f = g$ or $[f^n(f^m-1)]^{(k)}.[g^n(g^m-1)]^{(k)} \equiv 1$.

Before proving the theorems, we state some existing results in the form of lemmas, which will be used in the sequel.

Lemma 2.5[4] : Let f and g be two transcendental meromorphic functions such that $f^{(k)} - P$ and $g^{(k)} - P$ share 0 CM, where k is a positive integer, $P \neq 0$ is a polynomial. If

$$\Delta_1 = (k+2)\Theta(\infty, f) + 2\Theta(\infty, g) + \theta(0, f) + \theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) > k+7$$

and

$$\Delta_2 = (k+2)\Theta(\infty, g) + 2\Theta(\infty, f) + \theta(0, g) + \theta(0, f) + \delta_{k+1}(0, g) + \delta_{k+1}(0, f) > k+7$$

Then either $f^{(k)} g^{(k)} = P^2$ or f = g.

Lemma 2.6[6] : Let f be a transcendental meromorphic function and $P(f) = a_n f^n + ... + a_2 f^2 + a_1 f^1 + a_0$ Then $T(r, P(f) = nT(r, f) + 0(1)$.

Lemma 2.7[7]: Let f and g be two non constant meromorphic functions and let $k(\geq 1), l(\geq 1)$ be two positive integers. Suppose that $f^{(k)}$ and $g^{(k)}$ share $(1, l)$.

(i) If l = 2 and

$\Delta_1=(k+2)\Theta(\infty,g)+2\Theta(\infty,f)+\theta(0,f)+\theta(0,g)+\delta_{k+1}(0,f)+\delta_{k+1}(0,g)>k+7$ then either $f^{(k)}g^{(k)}\equiv 1 or\ f\equiv g$.

(ii) If $l=1$ and

$\Delta_2=(k+3)\Theta(\infty,f)+(k+2)\Theta(\infty,g)+\Theta(0,f)+\Theta(0,g)+2\delta_{k+1}(0,f)+\delta_{k+1}(0,g)>2k+9$

then either $f^{(k)}g^{(k)}\equiv 1 or\ f\equiv g$.

Proof of Theorem 2.1: Let $F=f^n(f^2-1)$ and $G=g^n(g^2-1)$

and let

$\Delta_1=3\Theta(\infty,F)+2\Theta(\infty,G)+\Theta(0,F)+\Theta(0,G)+\delta_2(0,F)+\delta_2(0,G)$

and

$\Delta_2=3\Theta(\infty,F)+2\Theta(\infty,F)+\Theta(0,G)+\Theta(0,F)+\delta_2(0,G)+\delta_2(0,F)$

Now,

$$\begin{aligned}\Theta(0,F) &= 1-\limsup_{r\to\infty}\frac{\overline{N}(r,\frac{1}{F})}{T(r,F)}\\ &= 1-\limsup_{r\to\infty}\frac{\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{f^2-1})}{(n+2)T(r,f)}\\ &= 1-\limsup_{r\to\infty}\frac{3T(r,f)}{(n+2)T(r,f)}\\ &= \frac{n-1}{n+2}\end{aligned}$$

and

$$\begin{aligned}\Theta(\infty,F) &= 1-\limsup_{r\to\infty}\frac{\overline{N}(r,F)}{T(r,F)}\\ &= 1-\limsup_{r\to\infty}\frac{\overline{N}(r,f)}{(n+2)T(r,f)}\\ &\ge 1-\limsup_{r\to\infty}\frac{T(r,f)}{(n+2)T(r,f)}\\ &= \frac{n-1}{n+2}\end{aligned}$$

Similarly, $\Theta(0,G)\ge\frac{n+1}{n+2}$ and $\Theta(\infty,G)\ge\frac{n+1}{n+2}$

Now,

$$\delta_2(0,F)=1-\limsup_{r\to\infty}\frac{N_2(r,\frac{1}{F})}{T(r,F)}$$

$$= 1 - \limsup_{r \to \infty} \frac{N_2\left(r, \frac{1}{f^n(f^2-1)}\right)}{T(r,F)}$$

$$\geq 1 - \limsup_{r \to \infty} \frac{2\overline{N}(r, \frac{1}{f}) + N(r, \left(r, \frac{1}{f^2-1}\right)}{(n+2)T(r,f)}$$

$$\geq 1 - \frac{4T(r,f)}{(n+2)T(r,f)}$$

$$= \frac{n-2}{n+2}$$

Similarly, $\delta_2(0,G) \geq \frac{n-3}{n+2}$.

Therefore,

$$\Delta_1 \geq 5\frac{n+1}{n+2} + 2\frac{n-1}{n-2} + 2\frac{n-2}{n+2}$$

$$= \frac{9n-1}{n+2}$$

$$= 9 - \frac{19}{n+2}$$

$$> 8 \; if \; n > 17$$

Similarly, $\Delta_2 > 8 \, if n > 17$

Now since f and g are transcendental meromorphic functions, so F and G are transcendental meromorphic functions.

From the conditions that $F' - P$ and $G' - P$ share 0 CM together with $\Delta_1 > 8$ and $\Delta_2 > 8$ and the lemma(2.5), we get either $F'G' = P^2 \; or \; F = G.$ We discuss the following two cases :

Case I : Suppose $F'G' = P^2$.

i.e., $[f^n(f^2-1)]'[g^n(g^2-1)]' = P^2$

i.e., $f^{n-1}(f^2 - \frac{n}{n+2})f' g^{n-1}(g^2 - \frac{n}{n+2})g' = \frac{p^2}{(n+2)^3}$

Let $z_0 \notin \{z : P(z) = 0\}$ be a point such that $f^2(z_0) = \frac{n}{n+2}$, with multiplicity p. Then z_0 is a pole of g with multiplicity q(say).

Therefore, $p + p - 1 = nq - q + 2q + q + 1$

$i.e., 2p - 1 = (n+2)q + 1 \geq (n+3)$

$i.e., p \geq \frac{n+4}{2}$

Again let $z_1 \notin \{z: P(z)=0\}$ be a zero of f with multiplicity r then z_1 be a pole of g with multiplicity s(say).

Therefore, $nr - r + r - 1 = ns - s + 2s + s + 1$

$i.e., 2s = n(r-s) - 2 \geq (n-2)$

$So, s \geq \frac{n-2}{2}.$

So, $nr = (n+2)s + 2 \geq \frac{(n+2)(n-2)}{2} + 2 = \frac{n^2 - 4 + 4}{2}$

Therefore, $r \geq \frac{n}{2}$

Now, any pole of g must be either a zero of f or points for which $f^2 - \frac{n}{n+2} = 0$ or a zero of f' (consider those zeros for which f is not zero or $\pm\sqrt{\frac{n}{n+2}}$).

So,

$$\overline{N}(r,g) \leq \overline{N}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{f-\sqrt{\frac{n}{n+2}}}) + \overline{N}(r,\frac{1}{f+\sqrt{\frac{n}{n+2}}}) + \overline{N}_0(r,\frac{1}{f'})$$

$$\leq \frac{2}{n} N(r,\frac{1}{f}) + \frac{2}{n+4} N(r,\frac{1}{f-\sqrt{\frac{n}{n+2}}} + \frac{2}{n+4} N(r,\frac{1}{f+\sqrt{\frac{n}{n+2}}}) + \overline{N}_0(r,\frac{1}{f'})$$

$$\leq (\frac{2}{n} + \frac{4}{n+4})\, T(r,f) + \overline{N}_0(r,\frac{1}{f'})$$

where $\overline{N}_0(r,\frac{1}{f'})$ refers to those zeros of f' which occur at points other than roots of the equation $f\left(f^2 - \frac{n}{n+2}\right) = 0$

Now, from the second fundamental theorem, we have

$$2T(r,g) \leq \overline{N}(r,\frac{1}{g}) + \overline{N}(r,g) + \overline{N}(r,\frac{1}{g-\sqrt{\frac{n}{n+2}}}) + \overline{N}(r,\frac{1}{g+\sqrt{\frac{n}{n+2}}} - \overline{N}_0(r,\frac{1}{g'}) + S(r,g)$$

$$i.e.\ 2T(r,g) \leq \frac{2}{n} N(r,\frac{1}{g}) + \left(\frac{2}{n} + \frac{4}{n+4}\right) T(r,f) + \left(\frac{2}{n+4}\right) N(r,\frac{1}{g-\sqrt{\frac{n}{n+2}}}) + \left(\frac{2}{n+4}\right)$$

$$N(r,\frac{1}{g+\sqrt{\frac{n}{n+2}}}) + \overline{N_0}(r,\frac{1}{f'}) - \overline{N_0}(r,\frac{1}{g'}) + S(r,g)$$

So, $(2 - \frac{2}{n} - \frac{4}{n+4}) T(r,g) \leq (\frac{2}{n} + \frac{4}{n+4}) T(r,f) + \overline{N}_0(r,\frac{1}{f'}) - \overline{N}_0(r,\frac{1}{g'}) + S(r,g)$

Similarly we get

$$\left(2-\frac{2}{n}-\frac{4}{n+4}\right)T(r,f)\leq(\frac{2}{n}+\frac{4}{n+4})T(r,g)+\overline{N}_0(r,\frac{1}{g^{/}})-\overline{N}_0(r,\frac{1}{f^{/}})+S(r,f)$$

Adding these two, we get

$$(2-\frac{4}{n}-\frac{8}{n+4})\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g)$$

Which is a contradiction for given n.

Case II : Suppose F = G.

i.e., $f^n(f^2-1)=g^n(g^2-1)$ ………………(1)

Let $h=\frac{g}{f}$

If possible, suppose that h is a non constant. From equation (1) we have,

$$f^2=\frac{h^n-1}{h^{n+2}-1}$$

Now we assume that $h^n\neq1$ for otherwise we have trivial solution. So we must assume that n is not divisible by 2. By simple calculation it can be shown that the number of common zeros of h^n-1 *and* $h^{n+2}-1$ is at most 2 and hence $h^{n+2}-1$ has at least n zeros which are not the zeros of h^n-1 . We denote these n zeros by a_p, p = 1,2,……n.. Now f^2 can not have any simple pole and hence we conclude that h - a_p = 0 has no simple root for p = 1,2,……n. Hence $\Theta(a_p;h)\geq\frac{1}{2}$ for p = 1, 2,…..,n which is not possible for given n. This means that our assumption that h is non constant, is wrong. Therefore h is constant. Now if $h\neq1$, this means f will become a constant, which is clearly not the case. So $h=1$ and hence f = g.

This completes the proof.

Proof of Theorem 2.2: Let $F=f^n(f^m-1)$ *and* $G=g^n(g^m-1)$ and let

$\Delta_1=3\Theta(\infty,F)+2\Theta(\infty,G)+\Theta(0,F)+\Theta(0,G)+\delta_2(0,F)+\delta_2(0,G)$ ………………(2)

and

$\Delta_2=3\Theta(\infty,G)+2\Theta(\infty,F)+\Theta(0,G)+\Theta(0,F)+\delta_2(0,F)+\delta_2(0,G)+\delta_2(0,F)$ ……………(3)

Now,

$$\Theta(0,F) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,\frac{1}{F})}{T(r,F)}$$

$$= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,\frac{1}{f}) \ + \ \overline{N}(r,\frac{1}{f^m - 1})}{(n+m)T(r,f)}$$

$$\geq 1 - \limsup_{r\to\infty} \frac{(m+1)T(r,f)}{(n+m)T(r,f)}$$

$$= \frac{n-1}{n+m}$$

and,

$$\Theta(\infty,F) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,F)}{T(r,F)}$$

$$= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,F)}{(n+m)T(r,f)}$$

$$\geq 1 - \limsup_{r\to\infty} \frac{T(r,f)}{(n+m)T(r,f)}$$

$$= 1 - \frac{1}{n+m}$$

$$= \frac{n+m-1}{n+m}$$

and

$$\delta_2(0,F) = 1 - \limsup_{r\to\infty} \frac{N_2(r,\frac{1}{F})}{T(r,F)}$$

$$= 1 - \limsup_{r\to\infty} \frac{N_2(r,\frac{1}{f^n(f^m - 1})}{T(r,F)}$$

$$\geq 1 - \limsup_{r\to\infty} \frac{2\overline{N}(r + \frac{1}{f}) + N(r,\frac{1}{f^m - 1})}{(n+m)T(r,f)}$$

$$\geq 1 - \frac{(m+2)t(r,f)}{(n+m)T(r,f)}$$

$$= \frac{n-2}{n+m}$$

Applying similar logic, we have

$\Theta(0,G)=\dfrac{n-1}{n+m}$

$\Theta(\infty,G)=\dfrac{n+m-1}{n+m}$

$and\,\Theta(0,G)=\dfrac{n-2}{n+m}$

Now putting these values in (2), we get

$$\Delta_1 \geq 5\frac{n+m-1}{n+m}+2\frac{n-1}{n+m}+2\frac{n-2}{n+m}$$

$$=\frac{9n+5m-11}{n+m}$$

$$>8 \; if \;\; n>3m \; +11 \qquad \text{..................(4)}$$

Similarly we get from (3)

$$\Delta_2 >8 \; if \;\; n>3m+11 \qquad \text{..................(5)}$$

Now since f and g are transcendental meromorphic functions, so F and G are transcendental meromorphic functions.

From the condition that $F^{/}-P$ and $F^{/}-G$ share 0 CM, together with the inequality (4) and (5) and the lemma (2.5), we get either $F^{/}G^{/}=P^2$ or F = G. We discuss the following two cases :

Case I: Suppose that $F^{/}G^{/}=P^2$

i.e., $[f^n(f^m-1)]^{/}[g^n(g^m-1]^{/}=P^2$ ....(6)

Now, $[f^n(f^m-1)]^{/}=(n+m)f^{n-1}(f^m-\dfrac{n}{n+m})f^{/}$

and $[g^n(g^m-1)]^{/}=(n+m)g^{n-1}(g^m-\dfrac{n}{n+m})g^{/}$

Putting these two values in equation (6), we get

$$f^{n-1}(f^m-\frac{n}{n+m})f^{/}.g^{n-1}(g^m-\frac{n}{n+m})g^{/}=\frac{P^2}{(n+m)^2}$$

Let z_1, z_2, z_3,..., $z_m \notin \{z:P(z)=0\}$ be points such that $f^m(z_i)=\dfrac{n}{n+m}$ for i = 1,2, ...m and also let the multiplicity of z_1 is p. Then z_1 is a pole of g of multiplicity q (say).

Therefore,

$$p+p-1 \quad = np-q+mq+q+1$$

$$2p \quad =(n+m)q+2\geq (n+m)+2$$

$$p \quad \geq \frac{n+m+2}{2}$$

Hence, $\Theta(z_1,f)\geq 1-\dfrac{2}{n+m+2}$

Similarly,

$\Theta(z_2, f) \geq 1 - \frac{2}{n+m+2}$

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And $\Theta(z_m, f) \geq 1 - \frac{2}{n+m+2}$

Adding we get, $\Theta(z_1, f) + \ldots\ldots + \Theta(z_m, f) \geq m - \frac{2m}{n+m+5} > 2$

if n > 3m +11 and m > 2 where n and m are both integers, which is impossible.

So, $F'G' \neq P^2$

Case II: Suppose F = G.

i.e., $[f^n(f^m - 1) = g^n(g^m - 1)$ ………………(7)

Let $h = \frac{g}{f}$ If possible suppose that h is non constant. Then it follows from equation (7) that,

$$f^m = \frac{h^n - 1}{h^{n+m} - 1}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Now we assume that $h^n \neq 1$ for otherwise we have trivial solution. So we must assume that n is not divisible by m. By simple calculation it can be shown that the number of common zeros of $h^n - 1$ *and* $h^{n+m} - 1$ is at most m and hence $h^{n+m} - 1$ has at least n zeros which are not the zeros of $h^n - 1$. We denote these n zeros by a_p, p = 1,2,……n..

Now, f^m can not have any simple pole and hence we conclude that h - a_p = 0 has no simple root for p = 1,2,……n. where $a_p = \exp\left(\frac{2\pi ip}{n+m}\right)$. Hence $\Theta(a_p; h) \geq \frac{1}{2}$ for p = 1, 2,…..,n which is impossible for given n. Therefore h is a constant. if $h \neq 1$, it follows that f is a constant, which is a absurd. So h = 1 and hence f = g.

This proves the theorem.

Remark 2.8: The Theorem (1.4) follows from the Theorem (2.2) as a particular case if we take m = 1.

Proof of Theorem 2.3: Let F(z) = f^n (f^m – 1) and G(z) = g^n (g^m – 1).

Also let

$$\Delta_1 = 2\Theta(\infty, F) + (k+2)\Theta(\infty, G) + \Theta(0, F) + \Theta(0, G) + \delta_{k+1}(0, F) + \delta_{k+1}(0, G)$$

Now,

$$\Theta(0,F)=1-\limsup_{r\to\infty}\frac{\overline{N}\left(r,\frac{1}{F}\right)}{T(r,F)}$$

$$=1-\limsup_{r\to\infty}\frac{\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f^m-1}\right)}{T(r,f)}$$

$$\geq 1-\limsup_{r\to\infty}\frac{(1+m)T(r,f)}{(n+m)T(r,f)}$$

$$=\frac{n-1}{n+m}$$

and,

$$\Theta(\infty,F)=1-\limsup_{r\to\infty}\frac{\overline{N}(r,F)}{T(r,F)}$$

$$=1-\limsup_{r\to\infty}\frac{\overline{N}(r,f)}{T(r,F)}$$

$$\geq 1-\limsup_{r\to\infty}\frac{T(r,f)}{(n+m)T(r,f)}$$

$$=\frac{n+m-1}{n+m}$$

Similarly, $\Theta(0,G)\geq\frac{n-1}{n+m}$ and $\Theta(\infty,G)\geq\frac{n+m-1}{n+m}$

And

$$\delta_{k+1}(0,F)=1-\limsup_{r\to\infty}\frac{N_{k+1}\left(r,\frac{1}{F}\right)}{T(r,F)}$$

$$\geq 1-\limsup_{r\to\infty}\frac{(k+1)\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^m-1}\right)}{(n+m)T(r,f)}$$

$$\geq 1-\limsup_{r\to\infty}\frac{(k+1+m)T(r,f)}{(n+m)T(r,f)}$$

$$=\frac{n-k-1}{n+m}$$

Similarly, $\delta_{k+1}(0,G)\geq\frac{n-k-1}{n+m}$

So,

$$\Delta_1\geq 2\frac{n+m-1}{n+m}+(k+2)\frac{n+m-1}{n+m}+2\frac{n-1}{n+m}+2\frac{n-k-1}{n+m}$$

$$=\frac{(k+4)(n+m-1)+2(n-1)+2(n-k-1)}{(n+m)}$$

$$= (k+8) - \frac{4m+3k+8}{n+m}$$
$$> (k+7) \ if \ \ n+m > 4m+3k+8 \ i.e. \ n > 3m+3k+8$$

Now, $F^{(k)}$ and $G^{(k)}$ share (1,2) together with the condition that $\Delta_1 > k + 7$ and the lemma (2.7) that either $F^{(k)}.G^{(k)} = 1$ or F = G.

Now we discuss the following two cases:

Case I: $F^{(k)}.G^{(k)} = 1$ that is $[f^n(f^m — 1)]^{(k)} .[g^n (g^m – 1)]^{(k)} = 1$

Case II: F = G that is $f^n(f^m – 1) = g^n (g^m – 1)$

Let $h = \frac{g}{f}$. If possible suppose that h is not a constant. We have

$$f^m = \frac{h^n - 1}{h^{n+m} - 1}$$

We assume that $h^n \neq 1$ for otherwise we have trivial solution. So we must assume that n is not divisible by 2. By simple calculation it can be shown that the number of common zeros of $h^n - 1$ and $h^{n+m} - 1$ is at most m and hence $h^{n+m} - 1$ has at least n zeros which are not the zeros of $h^n - 1$. We denote these n zeros by a_p, p = 1, 2, ……………., n. Since f^m(m > 1) has no simple pole, it follows that h — a_p = 0 has no simple root for p = 1, 2, ..., n. Hence $\Theta(a_p; h) \geq \frac{1}{2}$ for p = 1, 2, ………, n. Which is impossible. Therefore h is a constant. If $h \neq 1$, it follows that f is a constant, which is not the case. So h = 1 and therefore f = g.

This proves the theorem.

Remark 2.9: The Theorem (1.5) follows from the Theorem (2.3) as a particular case if we take m = 1.

Proof of Theorem 2.4: Let F = f^n (f^m — 1) and G = g^n(g^m — 1)

Also let,

$$\Delta_2 = (k+3)\Theta(\infty, F) + (k+2)\Theta(\infty, G) + \Theta(0, F) + \Theta(0, G) + 2\delta_{k+1}(0, F) + \delta_{k+1}(0, G)$$

As in the previous theorem, we have

$$\Theta(\infty, F) \geq \frac{n+m+1}{n+m}, \Theta(\infty, G) \geq \frac{n+m-1}{n+m}$$
$$\Theta(0, F) \geq \frac{n-1}{n+m}, \Theta(0, G) \geq \frac{n-1}{n+m}$$
$$\delta_{k+1}(0, F) \geq \frac{n-k-1}{n+m}, \delta_{k+1}(0, F) \geq \frac{n-k-1}{n+m},$$

So,

$$\Delta_2 \geq (k+3)\frac{n+m-1}{n+m} + (k+2)\frac{n+m-1}{n+m} + 2\frac{n-1}{n+m} + 3\frac{n-k-1}{n+m}$$
$$= \frac{(2k+5)(n+m-1) + 2(n-1) + 3(n-k-1)}{n+m}$$

$$= \frac{(2k+10)(n+m)-5m-5k-10}{n+m}$$

$$= 2k+10-\frac{5m+5k+10}{n+m}$$

$$> 2k+9 \text{ if } n+m > 5m+5k+10 \text{ i.e., } n > 4m+5k+10$$

Now, $F^{(k)}$ and $G^{(k)}$ share (1, 1) together with condition that $\Delta_2 > 2k + 9$ and the lemma (2.7) that either $F^{(k)}G^{(k)} = 1$ or F = G.

The remaining proof is similar to the proof of the Theorem (2.3).

Remark 2.10: The Theorem (1.6) follows from the Theorem (2.4) as a particular case if we take m = 1.

CONFLICT OF INTEREST : None

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