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Path Double Covering Number of a Graph

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ABSTRACT

A path double cover of a graph G is a collection $\wp$ of paths in G such that every edge of G belongs to exactly two paths in $\wp$. The minimum cardinality of a path double cover is called the path double covering number of G and is denoted by $\eta_{PD}(G)$.In this paper we determine the exact value of this parameter for several classes of graphs.

Key words: Graphoidal covers; path double covers; path double covering number of a graph; bicyclic graphs.

INTRODUCTION

A graph is a pair $G=(V,E)$, where V is the set of vertices and E is the set of edges. Here we consider only nontrivial,finite, connected undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For graph theoretic terminology we refer to Harary [7].The concept of graphoidal cover was introduced by B.D Acharya and E. Sampathkumar [1] and the concept of acyclic graphoidal cover was introduced by Arumugam and Suresh Suseela [4].The reader may refer [7] and [2] for the terms not defined here.

Let $p=(v_1,v_2,v_3,\ldots,v_r)$ be a path or a cycle in a graph $G=(V,E)$. Then vertices $(v_2,v_3,\ldots,v_{r-1})$ are called internal vertices of P and v_1 and v_r are called external vertices of P. Two paths P and Q of a graph G are said to be internally disjoint if no vertex of G is an internal vertex of both P and Q .If $P=(v_0,v_1,v_2,\ldots v_r)$ and $Q=(v_r=w_0,w_1,w_2,\ldots,w_r)$ are two paths in G then the walk obtained by concatenating P and Q at v_r is denoted by $P\circ Q$ and the path $(v_r,v_{r-1},\ldots,v_0)$ is denoted by P^{-1} .[3]. Bondy [5] introduced the concept of path double cover of a graph. This was further studied by Hao Li [8].

Definition 1.1[3]**:** A path double cover (PDC) of a graph G is a collection $\mathcal{P}$ of paths in G such that every edge of *G* belongs to exactly two paths in $\mathcal{P}$.

The collection $\mathcal{P}$ may not necessarily consist of distinct paths in *G* and hence it cannot be treated as a set in the standard sense. For any graph $G=(V,E)$, let $\mathcal{P}$ denote the collection of all paths of length one each path appearing twice in the collection. Clearly $\mathcal{P}$ is a path double of *G* and hence the set of all path double covers of G is non-empty.

Arumugam and Meena [3] introduced the concept of path double covering number of a graph *G*.

Definition 1.2[3]**:** The minimum cardinality of a path double cover of a graph G is called path double covering number of *G* and is denoted by $\eta_{PD}(G)$

In [9] it has been observed that for any graph *G* $\eta_{PD}(G)\leq 2q$ and equality holds if and only if *G* is isomorphic to qK_2 and the following results have been proved .

Theorem1.3[3]**:** Let $\mathcal{P}$ be any path double cover of a graph G. Then $|P|=2q-ip$ where $ip=\sum_{p\varepsilon P} i(p)$ where i(p) is the number of internal vertices of $\mathcal{P}$.

Theorem1.4[3]**:** $\eta_{PD}=2q-i$ where i= maxiP the maximum being taken over all path double covers $\mathcal{P}$ of *G*

Theorem 1.5 [3]**:** Let *G* be a graph with $\delta=1$,if there exists a path double cover $\mathcal{P}$ such that every non pendant vertex of *G* is an internal vertex of d(v)paths in $\mathcal{P}$ then $\mathcal{P}$ is minimum path double cover and $\eta_{PD}=|P|$

Theorem 1.6 [3]**:** For any tree *T*, $\eta_{PD}(T)=n$ where n is the number of pendent vertices of *T*.

Theorem 1.7 [3]**:** For any graph *G*, $\eta_{PD}(T)\geq\Delta$.Further for any tree *T*, $\eta_{PD}(T)=\Delta$ if and only if *T* is homeomorphic to a star.

Definition 1.8 [6]: A triangular cactus is a connected graph all of whose blocks are triangles. A triangular snake is a triangular cactus whose block-cutpoint-graph is a path (a triangular snake is obtained from a path $v_1,v_2,...,v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i=1,2,...,n-1$).

Definition 1.9 [6]: A double triangular snake consists of two triangular snakes that have a common path. That is a double triangular snake is obtained from a path $v_1, v_2, ..., v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, ..., n-1$ and to a new vertex u_i for $i = 1, 2, ..., n-1$.

Definition 1.20 [6]: The book B_m is the graph $S_m \times P_2$ where S_m is the star with m + 1 vertex

Definition 1.21 [6]: A gear graph denoted G_n is a graph obtained by inserting an extra vertex between each pair of adjacent vertices on the perimeter of a wheel graph W_n. Thus, G_n has 2n+1 vertices and 3n edges. Gear graphs are examples of square graphs, and play a key role in the forbidden graph characterization of square graphs. Gear graphs are also known as cogwheels and bipartite wheels.

Definition 1.22 [6]: A helm graph, denoted Hn is a graph obtained by attaching a single edge and node to each node of the outer circuit of a wheel graph Wn.

Definition 1.23 [6]: A graph G is called the flower graph with n petals if it has 3n+1 vertices which form an n- cycle.

Definition [6]: A shell Sn is the graph obtained by taking n-3 concurrent chords in a cycle Cn on n vertices. The vertex at which all the chords are concurrent is called the apex vertex. The shell is also called fan Fn-1.

i.e.. $S_n = F_n\text{-}1 = P_n\text{-}1 + K_1$.

Definition 1.24 [6]: The cartesian product of two paths is known as grid graph which is denoted by $P_m \times P_n$. In particular the graph $L_n = P_n \times P_2$ is known as ladder graph.

Definition 1.25 [6]: A web graph is the graph obtained by joining the pendant vertices of a helm to form a cycle and then adding a single pendant edge to each vertex of this outer cycle.

Theorem 2.1: Let G be a triangular snake graph, then $\eta_{PD}(G) = 4$

Proof: Let $V(G) = \{v_1, v_2, v_3, \ldots, v_n, w_1, w_2, ..., w_{n-1}\}$, n is odd.

The path double covering of G is as follows.

$P_1 = \{v_1, v_2, v_3, \ldots, v_n\}$

$P_2 = \{v_1, \mathrm{w}_1, v_2, \mathrm{w}_2, \ldots, v_{n-1}, \mathrm{w}_{n-1}, v_n\}$

$P = 2P_1 \cup 2P_2$ is a path double cover of G .

$\eta_{PD}(G) \leq 4 = \Delta$

Since $\eta_{PD}(G) \geq \Delta = 4$

$\eta_{PD}(G) = \Delta = 4$ is a minimum path double covering number of G.

Theorem 2.2: Let G be a triangular cactus graph, then $\eta_{PD}(G)=2n$, where n is the number of triangles in the graph.

Proof: Let $V(G)=\{v_0,v_{11},v_{12},v_{21},v_{22},\ldots,v_{n1},v_{n2}\}$

Case1: n is even and n > 4

The path double covering of G is as follows.

$P_i=\{v_{i1},v_{i2},v_0,v_{i+12},v_{i+11}\},\ i=1,3,5,...,n-1$

$P_j=\{v_{j+12},v_{j+11},v_0,v_{j1},v_{j2}\}\quad j=1,3,5,7,...,n-1$

$P_k=\{v_{k1},v_0,v_{k2}\},k=1\text{to } n$

$P=\{P_i\}\cup\{P_j\}\cup\{P_k\}$ is a path double cover of G .

$|P|=\frac{n}{2}+\frac{n}{2}+n=2n$

$\eta_{PD}(G)\le 2n=\Delta$

Since $\eta_{PD}(G)\ge\Delta=2n$

$\eta_{PD}(G)=\Delta=2n$ is a minimum path double covering number of G.

Case 2: n is odd and n >5

The path double covering of G is as follows.

$P_1=\{v_{11},v_{12},v_0,v_{n1},v_{n2}\}$

$P_2=\{v_{21},v_{22},v_0,v_{n2},v_{n1}\}$

$P_3=\{v_{12},v_{11},v_0,v_{21},v_{22}\}$

$P_i=\{v_{i1},v_{i2},v_0,v_{i+1,2},v_{i+1},1\},i=3,5,7,\ldots,n-2$

$P_j=\{v_{j+1,2},v_{j+1,1},v_0,v_{j1},v_{j2}\},\text{j}=3,5,7,\ldots,n-2$

$P_k=\{v_{k1},v_0,v_{k2}\},\text{k}=1,2,...,\ n$

$P=\{P_1,P_2,P_3,P_i,P_j,P_k\}$ is a path double cover of G .

$|P|=3+\frac{n-1}{2}-1+\frac{n-1}{2}-1+n=2n$

$\eta_{PD}(G)\le 2n=\Delta$

Since $\eta_{PD}(G)\ge\Delta=2n$

$\eta_{PD}(G)=\Delta=2n$ is a minimum path double covering number of G.

Theorem 2.3: Let G be a flower graph with n petals then $\eta_{PD}(G)=2n$

Proof: Let $V(G)=\{v_0,v_{11},v_{12},v_{21},v_{22},\ldots,v_{n1},v_{n2}\}$

Case1: n is even.

The path double covering of G is as follows.

$P_i=\{v_{i1},v_{i2},v_{i3},v_0,v_{i+13},v_{i+12},v_{i+11}\}i=1,3,5,\ldots,n-1$

$P_j=\{v_{i+13},v_{i+12},v_{i+11},v_0,v_{i1},v_{i2},v_{i3}\}i=1,3,5,\ldots,n-1$

$P_k=\{v_{i1},v_0,v_{i3}\},i=1\text{to } n$

$P=\{P_i,P_j,P_k\}$ is a path double cover of G .

$|P| = n-1+n-1+n = 2n$

$\eta_{PD}(G) = 2n = \Delta$ is a minimum path double covering of G.

Case 2: n is odd.

The path double covering of G is as follows.

$P_1 = \{v_{11}, v_{12}, v_{13}, v_0, v_{23}, v_{22}, v_{21}\}$

$P_2 = \{v_{n3}, v_{n2}, v_{n1}, v_0, v_{21}, v_{22}, v_{23}\}$

$P_3 = \{v_{13}, v_{12}, v_{11}, v_{n3}, v_{n2}, v_{n1}\}$

$P_i = \{v_{i1}, v_{i2}, v_{i3}, v_0, v_{i+3}, v_{i+2}, v_{i+1}\}, i = 3,5,\ldots,n-2$

$P_j = \{v_{i+13}\, v_{i+12}, v_{i+11}, v_0, v_{i1}, v_{i2}, v_{i3}\}, i = 3,5,\ldots,n-2$

$P_k = \{v_{i1}, v_0, v_{i3}\}, i = 1 \text{ to } n$

$P = \{P_1, P_2, P_3, P_i, P_j, P_k\}$ is a path double cover of G .

$|P| = 3+\frac{n-1}{2}-1+\frac{n-1}{2}-1+n = 2n$

$\eta_{PD}(G) = 2n = \Delta$ is a minimum path double covering of G.

Theorem 2.4: Let G be a $P_m(QS_n)$ graph. The path double covering of G is $\eta_{PD}(G) = 4n-2$

Proof: Let $V(G) = \{v_1, \ldots, v_m, l_{i1} \ldots l_{in}, r_{i1} \ldots r_{in}, w_{i1} \ldots, w_{in}\}$ $i = 1 \text{to} n$

Case1: n is even.

The path double covering of G is as follows.

$P_i = \{w_{in}, r_{in}, w_{in-1}, r_{in-1}, \ldots, w_{i1}, l_{i1}, v_i, v_{i+1}, l_{i+11}, w_{i+11}, l_{i+12}, w_{i+12}, \ldots, l_{i+1n}, w_{i+1n}\}$ $i = 1$ to $n-1$

$Q_i = \{w_{in}, r_{in}, w_{in-1}, r_{in-1}, \ldots, w_{i1}, r_{i1}, v_i, v_{i+1}, r_{i+1,1}, w_{i+1,1}, r_{i+1,2}, w_{i+1,2}, \ldots, r_{i+1,n}, w_{i+1,n}\}$ $i = 1$ to $n-1$

$R_i = \{v_i, l_{i1}, w_{i1}, l_{i2}, w_{i2}, \ldots, l_{in}, w_{in}\}$ $i = 1,2,\ldots,n$

$S_i = \{v_i, r_{i1}, w_{i1}, r_{i2}, w_{i2}, \ldots, r_{in}, w_{in}\}$ $i = 1,2,\ldots,n$

$\therefore \eta_{PD}(G) = n-1+n-1+2n = 4n-2$

Case 2: n is odd.

The path double covering of G is as follows.

$P_i = \{w_{in}, r_{in}, w_{in-1}, r_{in-1}, \ldots, w_{i1}, l_{i1}, v_i, v_{i+1}, l_{i+11}, w_{i+11}, l_{i+12}, w_{i+12}, \ldots, l_{i+1n}, w_{i+1n}\}$ $i = 1$ to $n-2$

$Q_i = \{w_{in}, r_{in}, w_{in-1}, r_{in-1}, \ldots, w_{i1}, r_{i1}, v_i, v_{i+1}, r_{i+1,1}, w_{i+1,1}, r_{i+1,2}, w_{i+1,2}, \ldots, r_{i+1,n}, w_{i+1,n}\}$ $i = 1$ to $n-2$

$R_i = \{v_i, l_{i1}, w_{i1}, l_{i2}, w_{i2}, \ldots, l_{in}, w_{in}\}$ $i = 1,2,\ldots,n$

$S_i = \{v_i, r_{i1}, w_{i1}, r_{i2}, w_{i2}, \ldots, r_{in}, w_{in}\}$ $i = 1,2,\ldots,n$

$P_{n-1} = \{w_{nn}, l_{nn}, \ldots, w_{n1}, l_{n1}, v_n, v_{n-1}\}$

$Q_{n-1} = \{w_{nn}, r_{nn}, \ldots, w_{n1}, r_{n1}, v_n, v_{n-1}\}$

$\therefore \eta_{PD}(G) = n-1+n-1+2n = 4n-2$

Theorem 2.5: Let G be a $C_m(QS_n)$ graph. The path double covering of G is $\eta_{PD}(G) = 4n$

Proof: Let $V(G) = \{v_1, \ldots, v_m, l_{i1} \ldots l_{in}, r_{i1} \ldots r_{in}, w_{i1} \ldots, w_{in}\}$ $i = 1 \text{to} n$

Case1: n is even.

The path double covering of G is as follows.

$P_i=\{w_{in},r_{in},w_{in-1},r_{in-1},\ldots,w_{i1},l_{i1},v_i,v_{i+1},l_{i+11},w_{i+11},l_{i+12},w_{i+12},\ldots,l_{i+1n},w_{i+1n}\}$ $i=1$ to $n-1$

$Q_i=\{w_{in},r_{in},w_{in-1},r_{in-1},\ldots,w_{i1},r_{i1},v_i,v_{i+1},r_{i+1,1},w_{i+1,1},r_{i+1,2},w_{i+1,2},\ldots,r_{i+1,n},w_{i+1,n}\}$ $i=1$ to $n-1$

$R_i=\{v_i,l_{i1},w_{i1},l_{i2},w_{i2},\ldots,l_{in},w_{in}\}$ $i=1,2,...,n$

$S_i=\{v_i,r_{i1},w_{i1},r_{i2},w_{i2},\ldots,r_{in},w_{in}\}$ $i=1,2,...,n$

$P_n=(v_1,v_n)$

$\psi=\{P_i,Q_i,R_i,S_i,2P_n\}$

$\therefore \eta_{PD}(G)=n-1+n-1+2n+2=4n$

Case 2: n is odd.

The path double covering of G is as follows.

$P_i=\{w_{in},r_{in},w_{in-1},r_{in-1},\ldots,w_{i1},l_{i1},v_i,v_{i+1},l_{i+11},w_{i+11},l_{i+12},w_{i+12},\ldots,l_{i+1n},w_{i+1n}\}$ $i=1$ to $n-2$

$Q_i=\{w_{in},r_{in},w_{in-1},r_{in-1},\ldots,w_{i1},r_{i1},v_i,v_{i+1},r_{i+1,1},w_{i+1,1},r_{i+1,2},w_{i+1,2},\ldots,r_{i+1,n},w_{i+1,n}\}$ $i=1$ to $n-2$

$R_i=\{v_i,l_{i1},w_{i1},l_{i2},w_{i2},\ldots,l_{in},w_{in}\}$ $i=1,2,...,n$

$S_i=\{v_i,r_{i1},w_{i1},r_{i2},w_{i2},\ldots,r_{in},w_{in}\}$ $i=1,2,...,n$

$P_{n-1}=\{w_{nn},l_{nn},\ldots,w_{n1},l_{n1},v_n,v_{n-1}\}$

$Q_{n-1}=\{w_{nn},r_{nn},\ldots,w_{n1},r_{n1},v_n,v_{n-1}\}$

$P_n=(v_1,v_n)$

$\psi=\{P_i,Q_i,R_i,S_i,2P_n\}$

$\therefore \eta_{PD}(G)=n-1+n-1+2n+2=4n$

Theorem 2.6: Let G be a Ladder graph. The path double covering of G is $\eta_{PD}(G)=3$

Proof: Let $V(G)=\{u_1,u_2,u_3,...,u_n,l_1,l_2,...,l_n\}$

The path double covering of G is as follows.

$$P_1=\begin{cases} l_1,u_1,u_2,l_2,l_3,u_3,u_4,\ldots,l_{i-1},l_i,u_i,u_{i+1},\ldots,u_n,l_n, \text{if n is even} \\ l_1,u_1,u_2,l_2,l_3,u_3,u_4,\ldots,l_{i-1},l_i,u_i,u_{i+1},\ldots,l_n,u_n, \text{if n is odd} \end{cases}$$

$$P_2=\begin{cases} l_1,l_2,u_2,u_3,l_3,l_4,u_4,\ldots,l_{n-1},l_n,u_n, \text{if n is even} \\ l_1,u_1,u_2,l_2,l_3,u_3,u_4,\ldots,u_{n-1},u_n,l_n, \text{ if n is odd} \end{cases}$$

$$P_3=\{u_n,u_{n-1},u_{n-2},\ldots,u_2,u_1,l_1,l_2,\ldots,l_{n-1},l_n\}$$

$\eta_{PD}(G)=\Delta=3$ is a minimum path double covering of G.

Theorem 2.7: Let G be a fan graph with n vertices. The path double covering of G is $\eta_{PD}(G)=n$

Proof: Let $V(G)=\{x_1,x_2,\ldots,x_n\}$

The path double covering of G is as follows.

$P_1=\{x_1x_2x_3\ldots x_n\}$

$P_2=\{x_nx_1x_2x_3\ldots x_{n-1}\}$

$P_3=\{x_{n-1}x_nx_1\}$

Let $G_1=G-\{P_1,P_2,P_3\}$ is a tree with n-3 pendant vertices.

$\eta_{PD}(G_1)=n-3$

$\eta_{PD}(G)=n-3+3=n$

$\eta_{PD}(G)=\Delta=n$ is a minimum path covering of G.

Theorem 2.8: Let G be a mobius graph. The path double covering of G is $\eta_{PD}(G)=4$

Proof: Let $V(G)=\{v_1,v_2,\ldots,v_n,w_1,w_2,\ldots,w_n\}$

The path double covering of G is as follows.

$P_1=\{v_1,v_2,\ldots,v_{n-1},v_n,w_n,w_{n-1},\ldots,w_2,w_1\}$

$P_2=\{v_n,w_1,v_1,v_2,w_2,w_3,v_3,v_4,\ldots,v_{n-1},w_{n-1},w_n\}$

$P_3=\{v_1,w_1,w_2,v_2,v_3,\ldots,w_{n-1},w_n\}$

$P_4=\{w_1,v_n,w_n,v_1\}$

$P=\{P_1,P_2,P_3,P_4\}$ is a minimum path covering of G

$\eta_{PD}(G)=4=\Delta$ is a minimum path covering of G.

Theorem 2.9: Let G be a shell graph with n vertices. The path double covering of G is $\eta_{PD}(G)=n$

Proof: Let $V(G)=\{v_1,v_2,\ldots,v_n\}$

The path double covering of G is as follows.

$P_1=\{v_1,v_2,\ldots,v_{n-1},v_n\}$

$P_2=\{v_1,v_n,v_{n-1},\ldots,v_3,v_2\}$

$P_3=\{v_2,v_1,v_3\}$

$P_4=\{v_n,v_1,v_{n-1}\}$

$P_5=\{v_3,v_1,v_{n-1}\}$

$G_1=G-\{P_1,P_2,P_3,P_4,P_5\}$ is a tree with n-5 pendant vertices.

$\eta_{PD}(G_1)=n-5$ (By corollary 1.1)

$\eta_{PD}(G)=n-5+5=n\leq\Delta$ is a minimum path covering of G.

Since $\eta_{PD}(G)\geq\Delta=n$

$\therefore \eta_{PD}(G)=\Delta$

Theorem 2.10: Let G be a gear graph with n vertices. The path double covering of G is $\eta_{PD}(G)=n+1$

Proof: Let $V(G)=\{v_0,v_1,v_2,\ldots,v_n,w_1,w_2,\ldots,w_n\}$

The path double covering of G is as follows.

$P_1=\{v_0,v_1,t_1,v_2,t_2,v_3,t_3,\ldots.,v_n,t_n\}$

$P_2=\{v_0,v_n,t_n,v_1,t_1,v_2,t_2,\ldots.,v_{n-1},t_{n-1}\}$

$P_3=\{t_{n-1},v_n,v_0,v_1,t_n\}$

$G_1=G-\{P_1,P_2,P_3\}$ is a tree with n-2 pendant vertices.

$\eta_{PD}(G_1) = n-2$ (By corollary 1.1)

$\eta_{PD}(G) = n-2+3 = n+1$ is a minimum path covering of G.

Theorem 2.11: Let G be a web graph with n vertices. Then $\eta_{PD}(G) = n+1$

Proof: Let $V(G) = \{v_0, v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n\}$

Her $w_1, w_2, \ldots, w_n$ are pendant vertices.

v_i is adjacent to v_0 and w_i

The path double covering of G is as follows.

$P_1 = \{w_1, v_1, v_0, v_2, v_3, ..., v_n, w_n\}$

$P_2 = \{v_0, v_n, v_1, v_2, \ldots, v_{n-1}, w_{n-1}\}$

$P_3 = \{v_0, v_{n-1}, v_n, v_1, v_2, w_2\}$

$P_4 = \{w_{n-1}, v_{n-1}, v_0, v_n, w_n\}$

$P_5 = \{w_2, v_2, v_0, v_1, w_1\}$

$G_1 = G - \{P_1, P_2, P_3, P_4, P_5\}$ is a tree with n-4 pendant vertices.

$\eta_{PD}(G_1) = n-4$ (By corollary 1.1)

$\eta_{PD}(G) = n-4+5 = n+1$ is a minimum path covering of G.

Theorem 2.12: Let G be a double triangular snake graph, then $\eta_{PD}(G) = 6 = \Delta$

Proof: Let $V(G) = \{v_1,, v_2, v_3, \ldots, v_n, w_1, w_2, \ldots, w_n, u_1, u_2, \ldots, u_n\}$, n is odd.

The path double covering of G is as follows.

$P_1 = \{v_1, w_1, v_2, w_2, v_3, w_3, \ldots, v_{n-1}, w_{n-1}, v_n\}$

$P_2 = \{v_1, u_1, v_2, u_2, v_3, u_3, \ldots, v_{n-1}, u_{n-1}, v_n\}$

$P_3 = \{v_1, w_1, v_2, v_3, w_3, v_4, v_5, w_5, \ldots, w_{n-2}, v_{n-1}, v_n\}$

$P_4 = \{v_1, u_1, v_2, v_3, u_3, v_4, v_5, u_5, \ldots, u_{n-2}, v_{n-1}, v_n\}$

$P_5 = \{v_1, v_2, u_2, u_3, v_4, u_4, u_5, v_6, ..., v_{n-1}, u_{n-1}, v_n\}$

$P_6 = \{v_1, v_2, w_2, v_3, v_4, w_4, v_5, v_6, \ldots, v_{n-1}, w_{n-1}, v_n\}$

$\eta_{PD}(G) = \Delta = 6$ is a minimum path double covering number of G.

Note: Observe that for the following graphs the path double covering number is $\eta_{PD}(G) = \Delta$

1. t-ply
2. Multipleshell
3. Book graph

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