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On The Maximum Turns of 3D Lattice Paths

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Abstract.

We derive a precise formula of the maximum turns of unrestricted paths in a 3D lattice $\boldsymbol{L}(n_1, n_2, n_3)$ under the step set {<1, 0, 0>, <0, 1, 0>, <0, 0, 1>}. The contribution of this study is threefold. First, the proposed formula should play a critical role in the work that derive a precise formula of unrestricted lattice paths in 3D lattices $\boldsymbol{L}(n_1, n_2, n_3)$ with a given number of turns. Second, the result is identically a permutation statistics called maximum turns of three-item multiset $\{\mathrm{e}^{n_1}, \mathrm{n}^{n_2}, \mathrm{u}^{n_3}\}$ permutation. Third, the proposed formula can be applied to an open shop scheduling problem that deals with setup time among three types of machines.

Key words: lattice path, multiset permutation, open shop scheduling, turns

I. Introduction

This study focuses on a lattice path enumeration problem. Numerous early works have been devoted to this problem [1, 4, 5, 7]. To describe a lattice path enumeration problem, we have to take account of four factors, namely the lattice, the set of steps a path may take, the restrictions be imposed on a path, and the path characteristics. For further details, see [2]. In this study, we focus on the maximum turns of *unrestricted* paths in a 3D lattice $\boldsymbol{L}(n_1, n_2, n_3)$ under the step set {<1, 0, 0>, <0, 1, 0>, <0, 0, 1>}. By a *turn*, or precisely a *turn-point*, it means a point of a path where the direction of the path changes. Viewed in this light, maximum *turns* of unrestricted paths in a 3D lattice $\boldsymbol{L}(n_1, n_2, n_3)$ can be regarded as a permutation statistics called maximum *turns* of three-item *multiset* $\{\mathrm{e}^{n_1}, \mathrm{n}^{n_2}, \mathrm{u}^{n_3}\}$ permutation, here notation "e" denotes eastward, notation "n" denotes northward, and notation "u" denotes upward.[1] In multiset permutation, we call a *turn* occurs if two elements located in two consecutive positions are belonging to two different items.

The motivation of this study is inspired by trying to solve an *open shop* scheduling problem that concerns setup time between different types of machines [6]. Here, setup time means delay time that occurs when we arrange the route of a job from a machine of one type to a machine of another type. Note that the route of a job can be viewed as a permutation of a multiset and that a permutation of a multiset can be viewed as a path of a lattice. Therefore, the core of such a scheduling problem is intrinsically a lattice path enumeration problem. To

[1] A *multiset* is a set that each item in the set has a multiplicity which specifies how many times the item repeats.

address a lattice path enumeration problem with a given number of turns, the first step is to find its minimum and maximum turns. Although the maximum turns of *unrestricted* path in a 2D lattice is known, no studies, to my knowledge, have been reported on a 3D lattice. Thus, the attempt is to go up a notch, from a 2D lattice to a 3D lattice.

The remainder of this paper is structured as follows. In Section II, we derive a precise formula of maximum turns of *unrestricted* paths in a 3D lattice $\boldsymbol{L}(n_1, n_2, n_3)$. In Section III, the correctness of the proposed formula is confirmed by an algorithm that deals with the generation of *multiset* permutation [3], and results on some cases of $\boldsymbol{L}(n_1, n_2, n_3)$ are shown for demonstration. Finally, conclusions are summarized.

II. Results

The main result is Theorem 1 that a precise formula of maximum turns of *unrestricted* paths is proposed for a 3D lattice $\boldsymbol{L}(n_1, n_2, n_3)$ under the step set {<1, 0, 0>, <0, 1, 0>, <0, 0, 1>}. The proposed formula is derived from an analogy between lattice and building. The analogy is made by perceiving a 3D lattice $\boldsymbol{L}(n_1, n_2, n_3)$ as a 2D lattice $\boldsymbol{L}(n_1, n_2)$ built on the ground floor, and extented upward floor by floor to a total of n_3 floors. Such perception inspired our derivation of the proposed formula. Before we come to the derivation, we should emphasize that lattice points of a path can be classified into *turn-points* and *non-turn-points*. The classification is the key to understand the derivation of the proposed formula. Let t be the number of turns of a path. Note that throughout the paper all variables are positive integers and that, without loss of generality, we assume that $n_1 \leq \mathrm{n}_2$. For completeness, a result of the maximum turns of a 2D lattice is introduced without proof as follows.

***Lemma* 1**. For a path of a 2D lattice $\boldsymbol{L}(n_1, n_2)$, the minimum turns is one and the maximum turns is $2n_1 - 1$, if $n_1 = n_2$, or is $2n_1$, if $n_1 < n_2$.

In order to account for the Theorem 1, we will introduce the following three Lemmas first.

***Lemma* 2**. In a 2D lattice $\boldsymbol{L}(n_1, n_2)$, a path with t turns has $n_1 + n_2 - 1 - t$ *non-turn-points* where the path can extent upward to increase two turns, and has $t + 2$ *turn-points* where the path can extent upward to increase one turn.

Proof. It is clear that there are $n_1 + n_2 - 1$ points to be passed through for a path from the starting point (0, 0) to the end point (n_1, n_2) in a 2D lattice $\boldsymbol{L}(n_1, n_2)$. Therefore, a path with t turns has $n_1 + n_2 - 1 - t$ *non-turn-points* where the path can extent upward to increase two turns. It will involve two turns because for a path going upward at a *non-turn-point*, it begins (or creates) with a new turn on the floor the path belongs to, and end with another new turn on the upper floor. On the other hand, a path with t turns has $t + 2$ *turn-points* where the path can extent upward to increase one turn. It will involve only one turn because for a path going upward at a *turn-point*, it creates a new turn but at the same time it eliminates an old turn on the floor the path belongs to, and ends with another new turn on the upper floor. However, it will involve $t + 2$ *turn-points* because, in addition to the original t *turn-points*, the starting point (0, 0) and the target point (n_1, n_2) can also be a *turn-point* for going upward. ■

Just as Lemma 1 is the base of this study, Lemma 2 plays a pivotal role in our derivation of the proposed formula. By using Lemmas 1 and 2, we now can easily know about the minimum and maximum turns of a path in a 3D lattice $\boldsymbol{L}(n_1, n_2, 1)$.

***Lemma* 3**. For a path of a 3D lattice $\boldsymbol{L}(n_1, n_2, 1)$, the minimum turns is two and the maximum turns is

$$\begin{cases} 2n_1, if\ n_1 = n_2. \\ 2n_1 + 1, if\ n_1 = n_2 - 1. \\ 2n_1 + 2, if\ n_1 < n_2 - 1. \end{cases}$$

Proof. It is obviously that the minimum turns of a path in a 2D lattice $\boldsymbol{L}(n_1, n_2)$ is one. By Lemma 2, a path with one turn has three *turn-points* where the path can extent upward to increase one turn. Therefore, in a 3D lattice $\boldsymbol{L}(n_1, n_2, 1)$, the minimum turns of a path is two. As to the maximum turn, the following three cases help illustrate the analysis.

Case 1: $n_1 = n_2.$ By *Lemma* 1, the maximum turns of a path in a 2D lattice $\boldsymbol{L}(n_1, n_2)$ is $2\mathrm{n}_1 - 1$. Since $n_1 = n_2$, we have $2n_1 - 1 = n_1 + n_2 - 1$. By *Lemma* 2, a path with the maximum of $2n_1 - 1$ turns has zero *non-turn-point* where the path can extent upward to increase two turns. However, it still has $2n_1 + 1$ *turn-points* where the path can extent upward to increase one turn. Thus, the maximum turns of a path is $2n_1$.

Case 2: $n_1 = n_2 - 1$. By *Lemma* 1, the maximum turns in a 2D lattice $\boldsymbol{L}(n_1, n_2)$ is $2n_1$. Since $n_1 = n_2 - 1$, we have $2n_1 = n_1 + n_2 - 1$. By *Lemma* 2, a path with the maximum of $2n_1$ turns has zero *non-turn-point* where the path can extent upward to increase two turns. However, it still has $2n_1 + 2$ *turn-points* where the path can extent upward to increase one turn. Thus, the maximum turns of a path is $2n_1 + 1$.

Case 3: $n_1 < n_2 - 1$. By *Lemma* 1, the maximum turns in a 2D lattice $\boldsymbol{L}(n_1, n_2)$ is $2n_1$. Since $n_1 < n_2 - 1$, we have $2n_1 < n_1 + n_2 - 1$. By *Lemma* 2, a path with the maximum of $2n_1$ turns has at least one *non-turn-point* where the path can extent upward to increase two turns. However, since $n_3 = 1$, there is only one floor needed to extent upward. Thus, the maximum turns of a path is $2n_1 + 2$. ■

Now, we attempt to extend *Lemma* 3 further into a general 3D lattice $\boldsymbol{L}(n_1, n_2, n_3)$ as follows. Obviously, the minimum turns of a path in a 3D lattice $\boldsymbol{L}(n_1, n_2, n_3)$ is two.

***Lemma* 4**. Let v denotes the maximum turns of a path in a 3D lattice $\boldsymbol{L}(n_1, n_2, n_3)$. We have v as follows.

$$\begin{cases} 2n_1 - 1 + min(n_3, 2n_1 + 1), if\ n_1 = n_2. \\ 2(n_2 - 1) + min(n_1 - n_2 + n_3 + 1, 2n_1 + 2), if\ n_1 < n_2\ and\ n_3 > n_2 - n_1 - 1. \\ 2(n_1 + n_3), if\ n_1 < n_2\ and\ n_3 \le n_2 - n_1 - 1. \end{cases}$$

Proof. Let s denote the maximum turns of a path in a 2D lattice $\boldsymbol{L}(n_1, n_2)$. By Lemma 2, we known that a path with s turns has $n_1 + n_2 - 1 - s$ *non-turn-points* where the path can extent upward to increase two turns each time, and has $s + 2$ *turn-points* where the path can extent upward to increase one turn each time. If $n_3 > n_1 + n_2 - 1 - s$, it means that, after using all available *non-turn-points* to extent upward to increase two turns each time, there are still floors left to extent upward. Fortunately, there are $s + 2$ *turn-points* where can be used to extent upward to increase one turn each time. Therefore, if $n_3 > n_1 + n_2 - 1 - s$, then

$$v = s + 2(n_1 + n_2 - 1 - s) + min\,(n_3 - (n_1 + n_2 - 1 - s), s + 2). \quad (1)$$

On the other hand, if $n_3 \leq n_1 + n_2 - 1 - s$, it means that there are sufficient *non-turn-points* to extent upward each floor to reach finally the top floor, and increase two turns each time. Therefore,

$$v = s + 2n_3. \quad (2)$$

Now, when $n_1 = n_2$, by Lemma 1 we know that $s = 2n_1 - 1$. Therefore,

$$n_1 + n_2 - 1 - s = n_1 + n_2 - 1 - (2n_1 - 1) = 0.$$

Since $n_3 > 0$, the inequality $n_3 > n_1 + n_2 - 1 - s$ is always true. Thus, by (1),

$$v = 2n_1 - 1 + min(n_3, 2n_1 + 1).$$

On the other hand, when $n_1 < n_2$, by Lemma 1 we know that $s = 2n_1$. Therefore, if

$$n_3 > n_2 - n_1 - 1,$$

then

$$n_3 > n_1 + n_2 - 1 - 2n_1 = n_1 + n_2 - 1 - s.$$

Hence, by (1),

$$v = 2(n_2 - 1) + min(n_1 - n_2 + n_3 + 1, 2n_1 + 2).$$

Otherwise, by (2), $v = s + 2n_3 = 2(n_1 + n_3)$. ■

Although we have known how to compute the maximum turns of unrestricted lattice paths, there is *min* term within the computation. Next, we will further analyze Lemma 4 by eliminating the *min* terms within it, and obtain a concise formula as follows.

***Theorem* 1**. Let *v* denotes the maximum turns of a path in a 3D lattice $\boldsymbol{L}(n_1, n_2, n_3)$. Assume that $n_1 \leq n_2$. We have

$$v = \begin{cases} 2(n_1 + n_3), & if\ n_3 \leq n_2 - n_1 - 1. & (3) \\ n_1 + n_2 + n_3 - 1, & if\ n_2 - n_1 - 1 < n_3 < n_1 + n_2 + 1. & (4) \\ 2(n_1 + n_2), & if\ n_3 \geq n_1 + n_2 + 1. & (5) \end{cases}$$

Proof.

Case 1: $n_1 < n_2$.

By Lemma 4, when $n_3 > n_2 - n_1 - 1$, we need to compare the two expressions of $n_1 - n_2 + n_3 + 1$ and $2n_1 + 2$ in the *min* term. However, if

$$n_3 < n_1 + n_2 + 1,$$

then

$$n_1 - n_2 + n_3 + 1 < 2n_1 + 2,$$

otherwise

$$n_1 - n_2 + n_3 + 1 \geq 2n_1 + 2.$$

Therefore, by Lemma 4, if $n_2 - n_1 - 1 < n_3 < n_1 + n_2 + 1,$ then

$$v = 2(n_2 - 1) + n_1 - n_2 + n_3 + 1 = n_1 + n_2 + n_3 - 1,$$

otherwise,

$$v = 2(n_2 - 1) + 2n_1 + 2 = 2(n_1 + n_2).$$

On the other hand, by Lemma 4, if $n_3 \leq n_2 - n_1 - 1,$ then

$$v = 2(n_1 + n_3).$$

Case 2: $n_1 = n_2$.

By Lemma 4, we need to compare the two expressions of n_3 and $2n_1 + 1$ in the *min* term. It is obvious that if $n_3 \geq 2n_1 + 1$ then $v = 4n_1$, otherwise $v = 2n_1 + n_3 - 1$. First, we shall concentrate on the situation of $n_3 \geq 2n_1 + 1$.

However, the inequality $n_3 \geq 2n_1 + 1$ imply the inequality $n_3 \geq n_1 + n_2 + 1$,

and the equation $v = 4n_1$ imply the equation $v = 2(n_1 + n_2)$.

Thus, this part is in accordance with equation (5).

Note that, in case of $n_1 = n_2$, the inequality $n_3 \leq n_2 - n_1 - 1$ is impossible, thus, equation (3) does not exist here. Next, we shall concentrate on the situation of

$$n_2 - n_1 - 1 < n_3 < n_1 + n_2 + 1.$$

Similarly, the equation $v = 2n_1 + n_3 - 1$ means the equation

$$v = n_1 + n_2 + n_3 - 1.$$

Thus, this part is in accordance with equation (4).

Viewed in this light, the case of $n_1 = n_2$ can be regarded as a special case of $n_1 < \mathrm{n}_2$. And recall that, without loss of generality, we assume that $n_1 \leq \mathrm{n}_2$. Consequently, we have this concise formula. ∎

It is clear that *Lemma* 1 is a special case of *Theorem* 1, for $n_3 = 0$.

III. Examples

In this section, the correctness of the proposed formula is confirmed by an algorithm that deals with the generation of *multiset* permutation [3]. It must be noted that permutation generation of a three-item *multiset* $\{\mathrm{e}^{n_1}, \mathrm{n}^{n_2}, \mathrm{u}^{n_3}\}$ can be viewed as path generation of a 3D lattice $\boldsymbol{L}(n_1, n_2, n_3)$. We present maximum turns on some cases of 3D lattices $\boldsymbol{L}(n_1, n_2, 1)$ *and* $\boldsymbol{L}(n_1, n_2, 2)$ in Table 1.

Table 1. Maximum turns on some cases of $\boldsymbol{L}(n_1, n_2, n_3)$

$\boldsymbol{n_1}$	**3**	**3**	**4**	**4**	**3**	**3**	**2**	**2**	**2**	**2**
$\boldsymbol{n_2}$	**4**	**4**	**4**	**4**	**5**	**5**	**5**	**5**	**6**	**6**
$\boldsymbol{n_3}$	**1**	**2**	**1**	**2**	**1**	**2**	**1**	**2**	**1**	**2**
Total Paths	280	1260	630	3150	504	2520	168	756	252	1260
Max. turns	7	8	8	9	8	9	6	8	6	8

The figures in the bottom raw (Max. turns) of Table 1 provide evidence for the correctness of the proposed formula.

IV. Conclusions

We derive a precise formula of the maximum turns of unrestricted lattice paths in a 3D integer rectangular lattice $\boldsymbol{L}(n_1, n_2, n_3)$ under the step set {<1, 0, 0>, <0, 1, 0>, <0, 0, 1>}. The contribution of this study is threefold. First, the proposed formula should play a critical role in the work that derive a precise formula of unrestricted lattice paths in 3D lattices $\boldsymbol{L}(n_1, n_2, n_3)$ with a given number of turns. Second, the result is identically a permutation statistics called maximum *turns* of three-item *multiset* $\{0^{n_1}, 1^{n_2}, 2^{n_3}\}$ permutation. Third, the proposed formula can be applied to an open shop scheduling problem that deals with setup time among three types of machines. In brief, there are three points to be emphasized. First, we classify the lattice points of a path into *turn-points* and *non-turn-points*. The classification is the key to understand the derivation of the proposed formula. Second, in a 2D lattice $\boldsymbol{L}(n_1, n_2)$, each path with t turns has $n_1 + n_2 - 1 - t$ *non-turn-points* where the path can extent upward to increase two turns, and has $t + 2$ *turn-points* where the path can extent upward to increase one turn. Third, a path of 3D lattice $\boldsymbol{L}(n_1, n_2, n_3)$ can be viewed as a permutation of three-item *multiset* $\{\mathrm{e}^{n_1}, \mathrm{n}^{n_2}, \mathrm{u}^{n_3}\}$. By using a *multiset* permutation algorithm, we present results on some cases of $\boldsymbol{L}(n_1, n_2, n_3)$. The consistent results support the correctness of the proposed formula.

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