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Two Steps Iterative Methods for Solving Nonlinear Equations

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Abstract
In this paper, we suggest and analyze some two steps iterative methods for solving nonlinear equations by a series expansion of the nonlinear function. We have shown that these methods are cubic convergence methods. Several examples are given to illustrate the efficiency of these methods and their comparison with other methods. These new methods can be viewed as significant modification and improvement of the Newton method and it's variant.
Keywords: Nonlinear equation, Order of convergence, Taylor series expansion, Asymptotic convergence.

1. Introduction

We consider a single variable nonlinear equation

$$f(x) = 0 \qquad (1)$$

Finding zeros of a single variable nonlinear equation (1) efficiently, is an interesting very old problem in numerical analysis and has many application in applied sciences. In recent years, researchers have developed many iterative methods for solving equation (1). These methods can be classified as one-step, two-step, and three-steps, see [1-4]. These methods have been proposed using Taylors series, decomposition techniques and quadrature rules, etc. Soheili [1], Noor [2], Bahgat [3] and Mir et al [4] have proposed many two-step methods.

In this paper we proposed some two-steps iterative methods based on series expansion of nonlinear equation. We have proved that the methods are of order three convergences. The methods and their algorithms are described in section 2. The convergence analysis of the methods is discussed in section 3. In section 4, we tested the performance of the algorithms on some selected functions. Comparison of the methods performance was also made with performance of some existing methods. It was noted that in some cases, the proposed methods are comparable with known existing methods and in many cases gives better results.

2. Development of method

Consider the nonlinear equation of type (1). If α is a simple root and x_0 is an initial guess close to α. Using the Taylor's series expansion of the function $f(x)$, we have

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0) \qquad (2)$$

from which we have

$$x = x_0 - \frac{2f(x_0)}{2f'(x_0) + (x - x_0)f''(x_0)} \quad (3)$$

This formulation allows us to suggest the following iterative methods for solving the nonlinear equations.

Algorithm 2.1

For a given initial choice x_n, find the approximate solution x_{n+1} by the iterative schemes:

$$x_{n+1} = x_n - \frac{2f(x_n)}{2f'(x_n) + (x_{n+1} - x_n)f''(x_n)} \quad (4)$$

Equation (4) is an implicit method, since x_{n+1} is on both side of the equation, which itself a difficult problem. Note that if $f''(x_0) = 0$, the algorithm 2.1 will reduce to the famous Newton's Method given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (5)$$

From Algorithm 2.1, we suggest the following iterative methods.

Algorithm 2.2

For a given initial choice x_n, find the approximate solution x_{n+1} by the iterative schemes:

$$y_n = x_n - \frac{f(x)}{f'(x_n)} \quad (6a)$$

$$x_{n+1} = x_n - \frac{2f(x_n)}{2f'(x_n) + (y_n - x_n)f''(x_n)}, n = 0,1,2, \ldots \quad (6b)$$

If we use Taylor's series to approximate $f''(x_n)$ in (6b) with $\frac{f'(y_n)-f'(x_n)}{y_n-x_n}$, where y_n is as defined in (5). This enables us to suggest an iterative scheme which does not involve the second derivatives.

Algorithm 2.3

For a given initial choice x_n, find the approximate solution x_{n+1} by the iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \quad (7a)$$

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n) + f'(y_n)}, n = 0,1,2, \ldots \quad (7b)$$

If we approximates y_n with Halley's method, we suggest the following iterative scheme.

Algorithm 2.4

For a given initial choice x_n, find the approximate solution x_{n+1} by the iterative schemes:

$$y_n = x_n - \frac{2f(x)f'(x_n)}{2\left(f'(x_n)\right)^2 - f(x_n)f''(x_n)} \quad (8a)$$

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n) + f'(y_n)}, n = 0,1,2, \ldots \quad (8b)$$

3. Convergence Analysis

Theorem 2.1

Let $\alpha \in I$ be a simple zero of sufficiently differentiable function $f: I \subset R \to R$ for an open interval I. If x_0 is sufficiently close to α then the iterative method defined by Algorithm 2.2 has third-order convergence.

Proof

Let α be a simple zero of f, and $e_n = x_n - \alpha$. Using Taylor expansion around $x = \alpha$ and taking into account $f(\alpha) = 0$, we get

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + \cdots] \qquad (9)$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + \cdots] \qquad (10)$$

$$f''(x_n) = f'(\alpha)[2c_2 + 6c_3e_n + 12c_4e_n^2 + 20c_5e_n^3 + 30c_6e_n^4 + \cdots] \qquad (11)$$

Where

$$c_k = \frac{1}{k!}\frac{f^k(\alpha)}{f'(\alpha)}, \quad k = 1,2,3, \ldots \text{ and } e_n = x_n - r$$

Combining (9) and (10) in (6a) we have;

$$y_n = \alpha + c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 - (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 + \ \ldots \qquad (12)$$

where;

$$y_n - x_n = -e_n + c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 - (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 + \ \ldots \qquad (13)$$

$$(y_n - x_n)f''(x_n) = f'(\alpha)[-2c_2e_n + (2c_2^2 - 6c_3)e_n^2 + (10c_2c_3 - 12c_4 - 4c_2^3)e_n^3$$
$$+(12c_3^2 - 12c_2c_3 + 6c_2c_4 + 8c_2^4 - 14c_2^2c_3)e_n^4 + \cdots \qquad (14)$$

$$2f'(x_n) = f'(\alpha)[2 + 4c_2e_n + 6c_3e_n^2 + 8c_4e_n^3 + 10c_5e_n^4 + \cdots] \qquad (15)$$

$$2f(x_n) = f'(\alpha)[2e_n + 2c_2e_n^2 + 2c_3e_n^3 + 2c_4e_n^4 + \cdots \qquad (16)$$

$$2f'(x_n) + (y_n - x_n)f''(x_n) = f'(\alpha)[2 + 2c_2e_n + 2c_2^2e_n^2 + (10c_2c_3 - 4c_4 - 4c_2^4)e_n^3$$
$$+(12c_3^2 - 12c_2c_3 + 6c_2c_4 + 8c_2^4 - 14c_2^2c_3 + 10c_5)e_n^4 + \cdots \qquad (17)$$

Thus,

$$\frac{2f(x_n)}{2f'(x_n) + (y_n - x_n)f''(x_n)} = e_n + (c_3 - c_2^2)e_n^3 + O(e_n^4) \qquad (18)$$

Which shows that Algorithm 2.1 has third –order convergence.

Theorem 2.2

Let $\alpha \in I$ be a simple zero of sufficiently differentiable function $f: I \subset R \to R$ for an open interval I. If x_0 is sufficiently close to α then the iterative method defined by Algorithm 2.3 has third-order convergence.

Proof

Let α be a simple zero of f, and $e_n = x_n - \alpha$. Using Taylor expansion around $x = \alpha$ and taking into account $f(\alpha) = 0$, we get

$$f'(y_n) = f'(\alpha)[1 + 2c_2^2e_n^2 + 4(c_2c_3 - c_2^3)e_n^3 + (-11c_2^2c_3 + 6c_2c_4 + 8c_2^4)e_n^4 + \cdots] \qquad (19)$$

Using (10) and (19),

$$f'(x_n) + f'(y_n) = 2 + 2c_2e_n + (2c_2^2 + 3c_3)e^2 + 4(c_2c_3 - c_2^3 + c_4)e_n^3$$
$$+(-11c_2^2c_3 + 6c_2c_4 + 8c_2^4 + 5c_5) + \cdots \qquad (20)$$

Thus;

$$2\frac{f(x_n)}{f'(x_n) + f'(y_n)} = e_n + \left(\frac{1}{3}c_3 + c_2^2\right)e^3 + o(e^4) \qquad (21)$$

Equation (21) establishes that algorithm 2.3 has order of convergence equal to three. This completes the proof of the theorem.

Theorem 2.3

Let $\alpha \in I$ be a simple zero of sufficiently differentiable function $f: I \subset R \rightarrow R$ for an open interval I. If x_0 is sufficiently close to α then the iterative method defined by Algorithm 2.4 has third-order convergence.

Proof

Let α be a simple zero of f, and $e_n = x_n - \alpha$. Using Taylor expansion around $x = \alpha$ and taking into account $f(\alpha) = 0$, we get

$$y_n = \alpha + (c_2^2 - c_3)e_n^3 + (-3c_2^3 + 6c_2c_3 - 3c_4) + O(e_n^4) \quad (22)$$

$$f(y_n) = f'(\alpha)(y_n - \alpha) + \frac{1}{2!}f''(\alpha)(y_n - \alpha)^2 + \frac{1}{3!}f'''(\alpha)(y_n - \alpha)^3 + \frac{1}{4!}f^4(\alpha)(y_n - \alpha)^4 + O((y_n - \alpha)^5) \quad (23)$$

$$= f'(\alpha)[(c_2^2 - c_3)e_n^3 + (-3c_2^3 + 6c_2c_3 - 3c_4)e_n^4 + O(e_n^5)] \quad (24)$$

$$f'(y_n) = f'(\alpha)[1 + c_2(y_n - \alpha) + c_3(y_n - \alpha)^2 + c_4(y_n - \alpha)^3 + O(y_n - \alpha)^4]$$
$$= f'(\alpha)[1 + (2c_2^3 - 2c_2c_3)e_n^3 + O(e_n^4)] \quad (25)$$

Thus;

$$2\frac{f(x_n)}{f'(x_n) + f'(y_n)} = e_n - \frac{1}{2}c_3e_n^3 + O(e_n^4) \quad (26)$$

This means that the method defined by algorithm 2.4 is cubically convergent. Since asymptotic convergence of Newton method, Soheili method (SM) [1], and Bahgat method [3] are c_2, c_3 and $\frac{c_2}{3}$ respectively, it shows that the convergence rate of Algorithm 2.2, Algorithm 2.3 and Algorithm 2.4 with asymptotic convergence of $(c_3 - c_2^2)$, $\left(\frac{1}{3}c_3 + c_2^2\right)$ and $\frac{c_3}{2}$ respectively are better.

4 Numerical Examples

In this section, we present some examples to illustrate the efficiency of our proposed methods which are given by the Algorithm 2.2, 2.3 and 2.4. We compare the performance of these Algorithms with that of Newton Method (NM), the method of Soheili (SM) and Bahgat (BM). Displayed in Table 1 and 2 are the number of iterations (NT) and number of function evaluation (NFE) required to achieve the desired approximate root x_n. The following stopping criteria were used.

$$i. \quad |x_{n+1} - x_n| < \varepsilon \qquad ii. \quad f(x_{n+1}) < \varepsilon \quad (27)$$

where $\varepsilon = 10^{-15}$.

We used the following functions, some of which are same as in [1-3]

$$\begin{cases} f_1(x) = e^{X^2+7X-30} - 1 \\ f_2(x) = x^3 - 10 \\ f_3(x) = x^2 - e^x - 3x + 2 \\ f_4(x) = sin^2(x) - x^2 + 1 \\ f_5(x) = x^{10} - 1 \\ f_6(x) = 11x^{11} - 1 \\ f_7(x) = \sin\left(\frac{1}{x}\right) - x \end{cases} \quad (28)$$

Table 1, presents number of iteration (NI) comparison of Algorithms 2.2, 2.3, and 2.4 with NM, SM and BM, in given precision. The numerical computations in Table 1 are performed using MathLab R2007b.

Table 1. Comparison between other methods

Functions	x_0	NI					
		NM	**SM**	**BM**	**Alg 2.2**	**Alg 2.3**	**Alg 2.4**
f_1	4	20	13	8	10	14	11
f_2	1.5	7	5	4	3	4	3
f_3	2	6	4	4	4	4	4
f_4	-1	7	5	4	4	4	3
f_5	1.5	10	7	5	5	6	5
f_6	0.7	8	6	5	4	5	3
f_7	2	6	4	4	4	4	3

Table 2. Comparison of number of function evaluation of different iterative methods

Functions	x_0	**NM**	**SM**	**Alg 2.2**	**Alg 2.3**	**Alg 2.4**
f_1	4	40	52	40	42	44
f_2	1.5	14	20	12	12	12
f_3	2	12	16	16	12	16
f_4	-1	14	25	16	12	12
f_5	1.5	20	28	20	18	15
f_6	0.7	16	24	16	15	9
f_7	2	12	16	16	12	9

In Table 3, the CPU time (per second) of the proposed algorithms, (NM), (SM) and (BM) are compared.

Table 3.The CPU time (per second) of algorithm

Functions	**NM**	**SM**	**BM**	**Alg 2.2**	**Alg 2.3**	**Alg 2.4**
f_1	0.171875	0.109375	0.062500	0.08413	0.11779	0.09255
f_2	0.078125	0.031250	0.02500	0.01875	0.02500	0.01875
f_3	0.046875	0.031250	0.02500	0.02310	0.02310	0.02310
f_4	0.046875	0.046875	0.02500	0.03750	0.03750	0.028125
f_5	0.078125	0.046875	0.039062	0.03348	0.04018	0.03348
f_6	0.062500	0.031250	0.039062	0.02083	0.02604	0.015625
f_7	0.046875	0.031250	0.02500	0.02412	0.02412	0.02344

The computational efficiency of an iterative method of order p, requiring N function evaluations per iteration, is most frequently calculated by Ostrowski-Traub's efficiency index [5,6]

$$E = p^{\frac{1}{N}} \tag{29}$$

It can be seen that the methods (Al 2.2), (Al 2.3) requires three function evaluations per iteration with efficiency indexes $3^{\frac{1}{3}} \approx 1.4422$. This efficiency index is superior to the efficiency indexes of Steffensen's and Newton's method which are equal to 1.414.

5 Conclusions

The problem of locating zeros of nonlinear equations occurs frequently in scientific work. In this paper, we have introduced three iterative methods for solving nonlinear equations. We have proven that these methods have third-order convergence. The efficiency indexes for the methods is 1.4422 greater than the efficiency indexes of both Steffensen's and Newton's method.

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