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"Reflexivity and completeness of normed almost linear space"

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Abstract: *In this paper we established some results on reflexivity and completeness of normed almost linear space. The reflexivity of a normed almost linear space X with respect to the sub spaces* W_X *and* V_X *are described through some results.*

1.Introduction

The notion of almost linear space and normed almost linear space was introduced by G. Godini [1 - 3]. All spaces involved in this work are over the real field ℝ. Sang Han Lee [4] introduced the algebraic dual space and algebraic double dual space of the almost linear space X and define algebraic reflexivity of the almost linear space X. Sung Mo Im and Sang Han Lee [7] characterizes the reflexivity of normed almost linear space without basis. Sung Mo Im and Sang Han Lee [9] proved that the dual space X^* of a normed almost linear space is complete. Basing all the above results in this paper we established some results relating to reflexivity of normed almost linear space X with respect to the sub spaces W_X and V_X of normed almost linear space X.

2. Preliminaries

Definition 2.1: Let $X^* = \{ f \in X^{\#} : |||f||| < \infty\}$, then the space X^* together with $|||\,.\,|||$ defined by $|||f||| = \sup\{ |f(x)| : |||x||| \le 1\}$ is called the dual space of the normed almost linear space X.

Definition 2.2: The dual space of the dual space X^* is called bi-dual space or second dual space of X and is denoted by X^{**}.

Definition 2.3: For every normed almost linear space X, there is a natural map F: $X \to X^{**}$ such that $F(x)(f) = f(x)$, for every $x \in X$ and for every $f \in X^*$ where $f : X \to \mathbb{R}$, $F(x) \in X^{**}$ where $F(x) : X^* \to \mathbb{R}$.

Definition 2.4: The normed almost linear space X is called reflexive when the natural map F: $X \to X^{**}$ is an isomorphism

Proposition 2.5: Let $(X, |||\,.\,|||)$ be a normed almost linear space. Then for each $x \in X$ there exists $f_x \in X^*$ such that $|||f_x||| = 1$ and $f_x(x) = |||x|||$.■

Proposition 2.6: Let $(X, |||\,.\,|||)$ be a normed almost linear space. Then for each $f \in (W_X)^*$ there exists $f_1 \in W_{X^*}$ such that $f_1\backslash W_X = f$ and $|||f_1||| = |||f|||$ and $f_1(v + w) = f(w)$ for each $v \in V_X$ and $w \in W_X$.■

Proposition 2.7: Let $(X, |||\,.\,|||)$ be a normed almost linear space and split as $X = WX+VX$. Then for each $f\in(VX)*$ there exists $f1\in VX*$ such that $f1|VX=f$ and $|||f_1||| = |||f|||$.■

Proposition2.8: If a normed almost linear space X is reflexive, then $X = W_X + V_X$. ■

Proposition 2.9: If a normed almost linear space X splits as $X = W_X + V_X$ and f is an almost linear functional on X then $f \in W_{X^\#}$ if and only if $f/V_X = 0$. ■

Proposition 2.10: If a normed almost linear space X splits as $X = W_X + V_X$, then (i). $V_{X^{**}}$ is isomorphic with $(V_X)^{**}$ and (ii). $W_{X^{**}}$ is isomorphic with $(W_X)^{**}$. ■

Proposition 2.11: If ω_Y is one-to-one then I is one-to-one and onto $L(X_1,(Y_1,C_1))$. And $L(X,(Y,C))$ is a normed almost linear space iff $L(X_1,(Y_1,C_1))$ is a normed almost linear space.

For proof of propositions 2.5 to 2.11 refer [3 – 9]

3. Main results

Theorem 3.1: For any x in a normed almost linear space X, we have

$$|||x||| = sup\{\frac{|f(x)|}{||f||} : \ f \in X^*, f \neq 0\}.$$

Proof: For any $x \in X$, by Proposition 2.5, there exists $f_x \in X^*$ such that $|||f_x|||$=1 and $f_x(x) = |||x|||$.

So we have $|||x||| = \frac{|f_x(x)|}{|||f_x|||} \leq \ sup\{\frac{|f(x)|}{|||f|||} : f \in X^*, f \neq 0\}$.

From $|f(x)| \leq |||f|||\ |||x|||$, we have $sup\left\{\frac{|f(x)|}{||f||} : f \in X^*, f \neq 0\right\} \leq |||\ x\ |||$ for each $f \in X^*$. Hence

$|||x||| = sup\{\frac{|f(x)|}{|||f|||} : f \in X^*, f \neq 0\}$.■

For a normed almost linear space X and $f \in X^*$, an equivalent formula for f is

$|||\ f\ ||| = \sup_{|||x|||=1}|f(x)| = \sup\frac{|f(x)|}{||x||}\ (x \neq 0)$ **3.1**

An isomorphism T of a normed almost linear space X onto a normed almost linear space Y is a bijective linear operator $T: X \rightarrow Y$ which preserves the norm, that is, for all $x \in X$, $|||T(x)||| = |||x|||$.Then X is called isomorphic with Y.

For $x \in X$ let Q_x be the functional on X^* defined, as in the case of a normed linear space, by

$Q_x(f) = f(x)(f \ \in X^*)$. **3.2**

Then Q_x is an almost linear functional on X^* and $|||Q_x||| \leq |||x|||$.

Hence Q_x is an element of X^{**}, by definition of X^{**}. **3.3**

This defines a mapping $C : X \rightarrow X^{**}$ by $C(x) = Q_x$. **3.4**

C is called the canonical mapping of X into X^{**}.

If the canonical mapping C of a normed almost linear space X onto X^{**} defined by (3.1) is an isomorphism, then X is said to reflexive. ■

Theorem 3.2: For a normed almost linear space X, the canonical mapping C defined by (3.4) is a linear operator and preserves the norm.

Proof: By (3.1) and Theorem (3.1), we have

$$||Q_x|| = \sup_{f \neq 0} \frac{|Q_x(f)|}{||f||} = \sup_{f \neq 0} \frac{|f(x)|}{||f||} = ||x|| \text{ for each } x \in X.$$

Hence C preserves the norm.

Let $x, y \in X$ and $\alpha \in R$.

For each $f \ \in X^*$, we have $Q_{x+y}(f) = f(x+y) = f(x) + f(y) = Q_x(f) + Q_y(f)$,

$$Q_{\alpha x}(f) = f(\alpha x) = (\alpha o f)(x) = (\alpha o Q_x)(f).$$

Thus C $(x+y)$ = C(x) + C(y) and C (αx) = $\alpha\ o$ C(x).

Therefore C is a linear operator. ■

Theorem 3.3: If a normed almost linear space X splits as $X = W_X + V_X$, then V_{X^*} is isomorphic with $(V_X)^*$and W_{X^*} is isomorphic with $(W_X)^*$.

Proof: Since $x^*\backslash V_X \in (V_X)^*$ for each $x^* \in V_{X^*}$, we can define an operator $T: V_{X^*} \to (V_X)^*$ by $T(x^*) = x^*\backslash V_X$ for each $x^* \in V_{X^*}$. For $x^*, y^* \in V_{X^*}$ and $\alpha, \beta \in R$, we have $T(\alpha o x^* + \beta o y*v = \alpha o x* + \beta o y*v = x*\alpha v + y*\beta v = T(x*)\alpha v + T(y*)\beta v = \alpha o T(x*))v + (\beta o T(y*)v = \alpha o T(x* + \beta o T(y^*)](v)$ for each $v \in V_X$.

Hence T is a linear operator.

If $x^* \neq y^* \in V_{X^*}$, then $x^*(v) \neq y^*(v)$ for some $v \in V_X$.

So $T(x^*) \neq T(y^*)$. Hence T is injective.

For each $v^* \in (V_X)^*$, there exists $x^* \in V_{X^*}$ such that $x^*\backslash V_X = v^*$ by Proposition 2.5.

Hence T is surjective.

For any $v^* \in V_{X^*}$, $|||v^*||| \geq ||| v^*\backslash V_X|||$ = $|||T(v^*)|||$.

Also, if $x = v + w \in X, v \in V_X, w \in W_X$with $|||\ x\ ||| \leq 1$, then $|||\ v\ ||| \leq 1$ and $v^*(x) = v^*(v)$. So we have $|||v^*||| = \sup\{|v^*(x)| : x \in X, |||x||| \leq 1\} \leq \sup\{|v^*(v)| : v \in V_X, |||v||| \leq 1\}$

$$= \sup\{|T(v^*)(v)| : v \in V_X, |||v||| \leq 1\} = |||T(v^*)|||$$

Hence T preserves the norm. Therefore V_{X^*} is isomorphic with $(V_X)^*$.

Similarly apply Proposition 2.9and Proposition 2.6, we can show that an operator $T': W_{X^*} \to (W_X)^*$, $T'(x^*) = x^*\backslash W_X (x^* \in W_{X^*})$, is an isomorphism. ■

Theorem 3.4: If a normed almost linear space X is reflexive, then V_X and W_X are reflexive.

Proof: By Proposition 2.8, $X = W_X + V_X$ since X is reflexive.

Let $C: X \to X^{**}$ be the canonical isomorphism, and let $C': V_X \to (V_X)^{**}$ be the canonical mapping. We will show that C' is bijective. Let $v^{**} \in (V_X)^{**}$.

By theorem3.3, $T: V_{X^*} \to (V_X)^*$, $T(v^*) = v^*\backslash V_X (v^* \in V_{X^*})$, is an isomorphism.

Since $x^*\backslash V_X \in (V_X)^*$ for each $x^* \in X^*$, we can define a functional $\bar{v}^{**}: X^* \to R$ by $\bar{v}^{**}: (x^*) = v^{**}(x^*\backslash V_X)$ for each $x^* \in X^*$. Then $\bar{v}^{**} \in V_{X^{**}}$.

Since C is an isomorphism of X onto X^{**}, there exists $v \in V_X$ such that $C(v) = \bar{v}^{**}$.

For this $v \in V_X$, $C'(v) = v^{**}$.

Indeed, for each $v^* \in (V_X)^*$, there exists $\bar{v}^* \in V_{X^*}$ such that $\bar{v}^*\backslash V_X = v^*$ by Proposition 2.7. So, we have $v^{**}(v^*) = v^{**}(\bar{v}^*\backslash V_X) = \bar{v}^{**}(\bar{v}^*) = C(v)(\bar{v}^*) = \bar{v}^*(v) = v^*(v) = C'(v)(v^*)$.

Hence C' is surjective.

If $v_1 \neq v_2$ in V_X, then $C(v_1) \neq C(v_2)$ in X^{**} since C is an isomorphism.

Choose $f \in X^*$, such that $C(v_1)(f) \neq C(v_2)(f)$, i.e., $f(v_1) \neq f(v_2)$.

For this $f \in X^*$, $f\backslash V_X \in (V_X)^*$ and $f\backslash V_X\,(v_1) \neq f\backslash V_X\,(v_2)$.

So, we have $C'(v_1) \neq C'(v_2)$.

Hence C' is injective.

Therefore C' is an isomorphism.

Similarly, we can show that W_X is reflexive. ■

Theorem 3.5: Let X be a split normed almost linear space as $X = W_X + V_X$. If V_X and W_X are reflexive, then X is reflexive.

Proof: Note that $X^* = W_{X^*} + V_{X^*}$ and $X^{**} = W_{X^{**}} + V_{X^{**}}$.

Let $C': V_X \to (V_X)^{**}$ and $C'': W_X \to (W_X)^{**}$ be the canonical isomorphism, and let $C: X \to X^{**}$ be the canonical map.

We will show that C is bijective.

Let $v^{**} \in V_{X^{**}}$. By Proposition 2.11, we have $v^{**}(x^*) = v^{**}(v^*)$ for each $x^* = v^* + w^* \in X^*$, $v^* \in V_{X^*}$, $w^* \in W_{X^*}$. And $v^{**}\backslash V_{X^*} \in (V_{X^*})^*$.

Recall that $T: V_{X^*} \to (V_X)^*$, $T(v^*) = v^*\backslash V_X\ (v^* \in V_{X^*})$, is an isomorphism.

Define a functional $\bar{v}^{**}: (V_X)^* \to R$ by $\bar{v}^{**}(v^*\backslash V_X) = v^{**}(v^*)$ for each $v^*\backslash V_X \in (V_X)^*$.Then $\bar{v}^{**} \in (V_X)^{**}$.

Since C' is an isomorphism of V_X onto $(V_X)^{**}$, there exists $v \in V_X$ such that $C'(v) = \bar{v}^{**}$.

For this $v \in V_X$, $C(v) = v^{**}$.

Indeed, $v^{**}(x^*) = v^{**}(v^*) = \bar{v}^{**}(v^*\backslash V_X) = C'(v)(v^*\backslash V_X) = v^*\backslash V_X(v) = v^*(v) = x^*(v) = C(v)(x^*)$ for each $x^* = v^* + w^* \in X^*$ with $v^* \in V_{X^*}$, $w^* \in W_{X^*}$.

Similarly, for each $w^{**} \in W_{X^{**}}$, there exists $w \in W_X$ such that $C(w) = w^{**}$.

Hence, for each $x^{**} = v^{**} + w^{**} \in X^{**}$ with $v^{**} \in V_{X^{**}}$, $w^{**} \in W_{X^{**}}$, there exists

$x = v + w \in X$ with $v \in V_X$, $w \in W_X$ such that $C(x) = C(v) + C(w) = v^{**} + w^{**} = x^{**}$. Hence

C is surjective.

If $w_1 \neq w_2$ in W_X, then $C''(w_1) \neq C''(w_2)$ in $(W_X)^{**}$ since C'' is an isomorphism.

Choose $f \in (W_X)^*$ such that $C''(w_1)(f) \neq C''(w_2)(f)$, i.e., $f(w_1) \neq f(w_2)$.

By proposition 2.6, there exists $f_1 \in X^*$ such that $f_1 \setminus W_X = f$ and $\|f_1\| = \|f\|$.

For this f_1, we have $C(w_1)(f_1) \neq C(w_2)(f_1)$ since $f_1(w_1) \neq f_1(w_2)$.

Hence $C(w_1) \neq C(w_2)$. Similarly $C(v_1) \neq C(v_2)$ for $v_1 \neq v_2$ in V_X.

Therefore C is injective since C is a linear operator and hence X is reflexive ■

Lemma 3.6: A normed almost linear space $(X, |||\,.\,|||)$ is complete iff $(E_X, \|\,.\,\|)$ is a Banach space and X_1 is norm-closed in E_X.

Proof: Suppose X complete.

Then X_1 is complete in the $\|\,.\,\|$ of E_X and so closed in E_X.

We show now that E_X is a Banach space.

Let $\{z_n\}_{n=1}^{\infty} \subset E_X$ be a Cauchy sequence.

We can suppose that for each $n \in N$ we have $\| z_n - z_{n+p} \| < \frac{1}{2^{n+1}}$ for each $p \geq 1$

Let $z_1 = x_1 - y_1$, $x_1, y_1 \in X_1$.

Since $\| z_2 - z_1 \| < 1/2^2$, there *exist* $x_2, y_2 \in X_1$ such that $z_2 - z_1 = x_2 - y_2$ and

$\| x_2\| + \| y_2 \| < 1/2^2$.

Then $z_2 = (x_1 + x_2) - (y_1 + y_2)$ where $\| x_2\| < 1/2^2$, $\| y_2 \| < 1/2^2$.

By induction on n we find two sequences $\{x_i\}_{i=1}$, $\{y_i\}_{i=1}^{\infty} \subset X_1$ such that for each $n \in N$ we have

$z_n = (\sum_{i=1}^{n} x_i) - (\sum_{i=1}^{n} y_i)$ and for n≥2 we have $\| x_n\| < 1/2^n$, $\| y_n\| < 1/2^n$. For each n∈ N, let $\overline{x_n} = \sum_{i=1}^{n} x_i \in X_1$ and $\overline{y_n} = \sum_{i=1}^{n} y_i \in X_1$. Clearly $\{\overline{x_n}\}_{n=1}^{\infty}$ and $\{\overline{y_n}\}_{n=1}^{\infty}$ are Cauchy sequences and since X_1 is complete, there exist $\bar{x}, \bar{y} \in X_1$ such that $lim_{n\to\infty}\| \overline{x_n} - \bar{x} \| = 0$ and $lim_{n\to\infty}\| \overline{y_n} - \bar{y} \| = 0$. Then for $z = \bar{x} - \bar{y} \in E_X$ we have $lim_{n\to\infty}\| z_n - z \| = 0$, i.e., E_X is a Banach space. The converse part is obvious. ■

Theorem 3.7: Let X be a complete normed almost linear space, Y a normed almost linear space such that ω_Y is one-to-one and c⊂Y a closed convex cone such that L(X,(Y,C)) is a normed almost linear space. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence in L(X,(Y,C)) such that $lim_{n\to\infty}\, \rho_Y(T_n(x), T(x)) = 0$ for each x∈ X. Then the sequence$\{|||T_n|||\}_{n=1}^{\infty}$ is bounded and T ∈ L(X,(Y,C)).

Proof: Since ω_Y is one-to-one and C closed, it is easy to show that T ∈ L(X,(Y,C)).

Now for each $x \in X$, $|||x|||\leq 1$ we have

$|||T(x)||| = \|\omega_Y T(x)\| \leq \|\omega_Y(T(x)) - \omega_Y(T_n(x))||| + ||| \omega_Y(T_n(x)) ||| = \rho_Y(T_n(x),$ $T(x)) +$ $|||T_n(x)||| \leq \rho_Y(T_n(x), T(x)) + |||T_n|||$ for each nεN, and so if we show that $\{|||T_n|||\}_{n=1}^{\infty}$ is bounded, then T ∈ L(X,(Y,C)). Since ω_Y is one-to-one, by hypotheses and Proposition 2.11, L(X_1,(Y_1,C_1)) is a normed almost linear space.

Now $\omega_{L(X,(Y,C))}$ (T_n) ∈ K, n ∈ N.

Then $\omega_{L(X,(Y,C))}$ (T_n) / X_1 = $\widetilde{T_n}$ ∈ L(X_1,(Y_1,C_1)) and $\omega_Y T_n = \widetilde{T}_n \omega_x$, n ∈ N.

Hence by hypothesis we have that $lim_{n\to\infty}\, \rho_Y(T_n(x) = 0$ for each $x \in X$.

$T(x)) = lim_{n\to\infty}\| \omega_Y(T_n\,(x)) - \|\omega_Y(T(x))\|$

$= lim_{n\to\infty}\| \widetilde{T_n}\,(\omega_x(x)) - \omega_Y(T(x)) \|$ and so for each $\bar{x} \in X_1$ the sequence $\{T_n(\bar{x})\}_{n=1}^{\infty}$ converges to an element of Y_1.

Let z ∈ E_X, z=$\overline{x_1}$ - $\overline{x_2}$, $\overline{x_\iota} \in X_1$, i=1,2.

Then $\{\omega_{L(X,(Y,C))}\,(T_n)\,(z)\}_{n=1}^{\infty}$ converges to an element of E_Y.

By Lemma (3.6), E_X is a Banach space.

Hence by Banach- Steinhaus theorem the sequence $\{\|\omega_{L(X,(Y,C))}(T_n)\|\}_{n=1}^{\infty}$ is bounded. Since

$\|\omega_{L(X,(Y,C))}(T_n)\| = |||\, T_n|||$ for each n∈N, the sequence $\{||||T_n|||\}_{n=1}^{\infty}$ is bounded. ■

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