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Generalized Results for Non Expansive Mapping in Metric Space

S.K.Tiwari, Divya Agrawal

Abstract:

In this paper we obtain some results for generalized based non-expansive mapping in metric space fr which we extend some well-known fixed point results.

1. Introduction:

The metric space is a fundamental tool of topology, functional analysis and non-linear analysis. The structure has attached a considerable attention from mathematicians because of the development of the fixed point theory.In this paper, we give sufficient conditions for establishing the existence of fixed points for single-valued and multi-valued nonexpansive mappings. We point out that the class of nonexpansive mappings considered herein contains the class of Banach contraction mappings. Also, some auxiliary facts on the convergence of Picard sequences and distance between fixed points of single-valued mappings are proved.

2. Preliminaries:

Definition 1: Let X be a non-empty set and let $d : X \times X \in [0 , \infty]$ be a distance

Function satisfying the conditions,

(d1) $d (x , x) = 0$;

(d2) $d (x , y) = d (y , x) = 0$ implies that $x = y$;

(d3) $d (x , y) = d (y , x)$;

(d4) $d (x , y) = d (x , z) + d (z , y)$ for all $x, y, z \in [X$.

If d satisfies the conditions from (d1) to (d4) then it is called metric on X, if d satisfy

Conditions (d2)to(d4) then it is called dislocated metric (d-metric) on X, and if d satisfy

Conditions (d2) and (d4) only then it is called dislocated quasi-metric(dq-metric) on X.

Clearly every metric space is a dislocated metric space but the converse is not necessarily true.

Definition 2: Let X be a non empty set and let $d: X \times X \to [0,\infty)$ be a function

Satisfying the Following conditions:

(i) $d (x, y) = d (y, x)$

(ii) $d (x, y)= d (y, x) = 0$ implies $x = y$.

(iii) $d (x, y) \leq d (x, z) + d (z, y)$ for all $x, y, z \in X$.

Then d is called dislocated metric (or simply d-metric) on X.

Definition 3: Let X be a nonempty set and let $d:X\times X\in [0, \infty)$be a function satisfying

Conditions

(i) d (x,y) = d (y,x) = 0 implies x=y

(ii) d (x , y) = d (x, z) + d (z, y) for all x,y,z ∈[X

Then d is called a dislocated quasi-metric on X . If d satisfies d(x,x) =0, then it is called a quasi-metric on X. If d satisfies d(x, y) = d(y ,x) then it is called dislocated metric .

Definition 4 : A sequence { xn } in dq-metric space (dislocated quasi-metric space) (X, d) is called Cauchy sequence if for given X_n> 0 n0 ∈N such that m,n ∈[n0 , implies

d (xm , xn) < ∈Xor d (xn , xm) < ∈ X i.e. min {d(xm ,xn) ,d (xn ,xm)} < ∈X.

Definition 5: A dq-metric space (X , d) is called complete if every Cauchy Sequence in it is a dq-convergent.

Definition 6: Let (X ,d) be a dq-metric space . A map T : X ×X is called Contraction if there exists 0 ≤X< 1 such that

d (Tx , Ty) ,d (x , y) for all x, y ∈X .

Definition 7: Let (X ,d1) and (Y , d2) be dq-metric spaces and let f : X ×Y be a Function. Then f is continuous to x0 ∈X, if for each sequence { xn} which is d1- q Convergent to x0 , the sequence {f (xn)} is d2-q convergent to f (x0) in Y.

Definition 8: Let (X; d) be a metric space and let CB(X) be the collection of all non-empty closed bounded subsets ofX.

For A; B∈ CB(X), define

H(A; B) = max[_δ (A; B); δ(B;A)]

where

δ (A; B) = supfδ(a; B) : a ∈A]

δ (B;A) = sup[d(b;A) : b ∈ B]

withd(a;C) = inf[d(a; x) : x ∈ C]

The function H : CB(X)_CB(X) → [0;+∞[is called the Pompeiu-Hausdor_ metric induced by the metric d.

Lemma 9 : Let (X ,d) be a dq-metric space and let f : X ×X is a contraction function then {f n (x0)} is a Cauchy sequence for each x0∈X .

Theorem 1 : Let (X, d) be a complete dq-metric space and let T : X ×X be a continuous mapping satisfying the following condition

d (Tx , Ty) ,ad (x , y) + b [d(x ,Tx) + d (y , Ty)]

Where a ,b are non negative, which may depends on both x and y, such that Sup{ a+2b: x,y∈X} < 1. Then T has unique fixed point.

Proof: Let { xn }be a sequence in X , defined as follows

Let x0 ∈X, T(x0) = x1, T(x1) =x2, -------------T(xn) =xn+1

Now consider

d (xn ,xn+1) = d (Txn-1 ,Txn)

ad (xn-1 , xn) + b [d(xn-1 ,Txn-1) + d (xn , Txn)]

= ad (xn-1 ,xn) + b [d(xn-1 , xn) + d (xn,xn+1)]

= ad (xn-1 , xn) + bd (xn-1 ,xn) + bd (xn,xn+1)

(a+b)d (∈) + bd (xn ,xn+1)

d (xn , xn+1) (x_n+− xn-1)-d (xn-1 ,xn)

d (xn , xn+1) d (xn,xn+1)+d (xn-1 ,xn) where xn-1 ,xn∈ X

Similarly we have d (xn-1 ,xn) ∈ d (xn-2 ,xn-1)

In this way , we get

d (xn , xn+1) ,xn,d (x0,x1).

Since 0 ≤ X< 1 so far n x→∞ , we have d (xn ,xn+1) →∞ . Similarly we show that d (xn+1 ,xn) . Hence {xn} is a Cauchy sequence in the complete dislocated quasi-metric space X.

So there is a point t0 ∈X such that xn ∈[t0 . Since T is continuous we have

T(t0) = lim x→∞ (xn) = limx→∞ xn+1 = t0.

Thus T(t0) =t0. Thus T has a fixed point.

Uniqueness: Let x be a fixed point of T. Then by given condition we have

d (x,x)= d (Tx , Tx)

ad (x , x) + b [d(x , Tx) + d (x,Tx)]

ad (x , x) + 2bd (x , x)

d (x , x) ∈ (a+2b) d (x , x)

which is true only if d(x , x) = 0,since 0 ≤(a+2b) < 1 and d(x , x) = 0.

Thus d (x , x) = 0, if x is fixed point of T .

Now let x,y be fixed point of T. That is Tx = x , Ty = y, then by given condition,

We have

d(x , y) = d (T x ,T y)

ad (x , y) + b [d(x , Tx) + d (y , Ty)]

ad (x , y) + b [d(x , x) + d (y , y)]

d (x , y) ∈ad (x ,y)

Similarly we have

d (y , x) ∈ad (y , x)

Hence | d (x , y) – d (y , x)| x a | d (x , y) – d (y , x)|

Which implies that d (x, y) = d (y , x), 0 ≤a < 1.

Again

d (x , y) ≤ad (x , y)

which gives d (x , y) = 0 since 0 ≤a < 1.

Further d (x , y) = d (y , x) = 0 gives x = y

Hence fixed point is unique.

Theorem 2. Let (α, d) be a complete metric space and f:X→X be non expansive mapping s.t.

$$d(f_x, f_y, f_z) \quad \leq \quad \left(\frac{d(x,f_y)+d(y,f_x)+d(y,f_z)+d(z,f_y)+d(x,f_z)+d(z,f_x)}{d(x,f_x)+d(y,f_y)+d(z,f_z)+1} + k\right) d(x,y,z)$$

for all (x,y,z)∈ β where k ∈[0,1[. Where β is binary relation on X. then prove that β is banach f – invariant if $(□_{□-1}, □_□)$∈ □ ,($□_□, □_{□-1}$)∈ □ for all n∈ □ and putting $□_□$ → □ □ □ and $□_{□+1}$ → □ □□ as n□ □.

Proof. Let $□_0$ ∈ □ s.t.

($□_0$□ $□_0$) ∈ □ and picard sequence of initial pair $□_0$ .if $□_{□-1}$ = $□_□$ for n∈ □. Let $□_{□-1}$ ≠ $□_□$ for all n∈ □ .

From ($□_1$, $□_2$) = ($□_0$, □ $□_0$) ∈ □

If a sequence ($□_1$, $□_2$) = (□$□_0$, $□^2$ $□_0$) ∈ □

=> ($□_{□-1}$, $□_□$) = ($□^{□-1}□_0$, $□^□□_□$) ∈ □

Therefore if x =$□_{□-1}$, y = $□_□$, z = $□_{□+1}$, we get

$$d(□_□, □_{□+1}) = d(□_{□-1}, □_□) \leq \left(\frac{□(□_{□-1}, □_{□+1})}{d(□_{□-1}, □_□)+d(□_□, □_{□+1})+1} + □\right) □(□_{□-1}, □_□)$$

$$\leq \left(\frac{□(□_{□-1}, □_□)+d(□_□, □_{□+1})}{d(□_{□-1}, □_□)+d(□_□, □_{□+1})+1} + □\right) □(□_{□-1}, □_□) \quad \text{for all } n\in □. \quad - \quad (□)$$

$$d(□_{□+1}, □_{□+2}) = d(□_□, □_{□+1}) \leq \left(\frac{□(□_□, □_{□+2})}{d(□_□, □_{□+1})+d(□_{□+1}, □_{□+2})+1} + □\right) □(□_□, □_{□+1})$$

$$\leq \left(\frac{□(□_□, □_{□-1})+d(□_{□+1}, □_{□+2})}{d(□_□, □_{□-1})+d(□_{□+1}, □_{□+2})+1} + □\right) □(□_□, □_{□+1})$$

By the above definition we get

$$d(□_{□+1}, □_{□+2}) \leq \left(\frac{□(□_{□-1}, □_□)+d(□_□, □_{□+1})+d(□_{□+1}, □_{□+2})}{d(□_{□-1}, □_□)+d(□_□, □_{□+1})+d(□_{□+1}, □_{□+2})+1} + □\right) □(□_{□-1}, □_□) = t\, d(□_{□-1}, □_□)$$

where $t = \frac{\square(\square_{0}, \square_{1}) + d(\square_{1}, \square_{2})}{d(\square_{0}, \square_{1}) + d(\square_{1}, \square_{2}) + 1} + \square \leq 1.$

Similarly,

$d(\square_{\square+1}, \square_{\square+2}) \leq \square\, d(\square_{\square}, \square_{\square+1})$,Where $t = \left(\frac{\square(\square_{1}, \square_{2}) + d(\square_{2}, \square_{3})}{d(\square_{1}, \square_{2}) + d(\square_{1}, \square_{3}) + 1} + \square\right) < 1.$

Since $\{\square_{\square}\}$ is a cauchy sequence. Since X is a complete metric space . Therefore $\{\square_{\square}\}$ is convergence to some $\square \in \square$. Then we reduce that $(\square_{\square}, \square)$ and $(\square_{\square+1}, \square) \in \square$.

Then putting $x = \square_{\square}$ and $y = \square$ in above equation, we get

$$d(\square_{\square+1}, \square_{\square}) = d(\square\, \square_{\square}, \square_{\square}) \leq \left(\frac{\square(\square_{\square}, \square_{\square}) + d(\square, \square\, \square_{\square})}{d(\square_{\square}, f\square_{\square}) + d(\square, \square_{\square}) + 1} + \square\right) \square(\square_{\square+1}, \square) = \left(\frac{\square(\square_{\square}, \square_{\square}) + d(\square, \square_{\square+1})}{d(\square_{\square}, \square_{\square+1}) + d(\square, \square_{\square}) + 1} + \square\right) \square(\square_{\square}, \square) \qquad (\square)$$

Similarly , putting $\square_{\square+1} = x$ and $y = \square$, we get

$$d(\square_{\square+2}, \square_{\square}) \leq \left(\frac{\square(\square_{\square+1}, \square_{\square}) + d(\square, \square\, \square_{\square+2})}{d(\square_{\square+1}, f\square_{\square+2}) + d(\square, \square_{\square}) + 1} + \square\right) \square(\square_{\square+2}, \square) = \left(\frac{\square(\square_{\square+2}, \square_{\square}) + d(\square, \square_{\square+2})}{d(\square_{\square+1}, \square_{\square+2}) + d(\square, \square_{\square}) + 1} + \square\right) \square(\square_{\square+1}, \square) \qquad - \quad (\square\,\square)$$

on taking n = +n on both sides of equation $(\square)\square\ \square\ \square(\square\ \square)$, we get

$d(\square, \square_{\square}) \leq 0$

$\Rightarrow d(\square, \square_{\square}) = 0 \Rightarrow z = \square_{\square}$ and hence $\square$ is a fixed point in f.

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