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Local Fractional Series Expansion Method for Solving Laplace And Schrodinger Equations on Cantor Sets within Local Fractional Operators

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Abstract

In this paper, we proposed a local fractional series expansion method (LFSEM) to solve the Laplace and Schrodinger equations on Cantor sets. Some examples are given to illustrate the efficiency and accuracy of the proposed method to obtain analytical solutions to differential equations within the local fractional derivatives.

Keywords

fractional Laplace equation; fractional Schrodinger equation; Local fractional series expansion method; Cantor set.

1. Introduction

Many problems of physics and engineering are expressed by ordinary and partial differential equations, which are termed boundary value problems. We can mention, for example, the wave, the Klein-Gordon, the Schrodinger's, the Advection, the Burgers, the Boussinesq, and the Fisher equations and others [1].

Several analytical and numerical techniques were successfully applied to deal with differential equations, fractional differential equations, and local fractional differential equations [1-10] . The techniques include the heat-balance integral [11], the fractional Fourier [12], the fractional Laplace transform [12], the harmonic wavelet [13, 14], the local fractional Fourier and Laplace transform [15], local fractional variational iteration [16, 17,18], the local fractional decomposition [19], and the generalized local fractional Fourier transform [21] methods.

The main idea of this paper is to present the local fractional series expansion method for effective solutions of the Laplace and Schrodinger equations on Cantor sets involving local fractional derivatives. The paper has been organized as follows. Section 2, the basic mathematical tools are reviewed. In Section 3 gives a local fractional series expansion method. Some illustrative examples are shown in Section 4. The conclusions are presented in Section 5.

2. A Brief Review of the Local Fractional Calculus

Definition 1. [19-24]. Suppose that there is the relation

$$|f(x)-f(x_0)|<\varepsilon^{\alpha} \quad ,0<\alpha\leq 1 \tag{2.1}$$

with $|x-x_0|<\delta$, for $\varepsilon,\delta>0$ and $\varepsilon,\delta\in R$, then the function $f(x)$ is called local fractional continuous at $x=x_0$ and it is denoted by $\lim_{x\to x_0} f(x)=f(x_0)$.

Definition 2.[19-24]. Suppose that the function $f(x)$ satisfies condition (2.1), for $x\in(a,b)$; it is so called local fractional continuous on the interval (a,b), denoted by $f(x)\in C_\alpha(a,b)$.

Definition 3.[19-24]. In fractal space, let $f(x)\in C_\alpha(a,b)$, local fractional derivative of $f(x)$ of order α at $x=x_0$ is given by

$$D_x^{\alpha} f(x_0)=\frac{d^{\alpha}}{dx^{\alpha}} f(x)\Big|_{x=x_0} = f^{(\alpha)}(x_0)=\lim_{x\to x_0}\frac{\Delta^{\alpha}(f(x)-f(x_0))}{(x-x_0)^{\alpha}} \tag{2.2}$$

where $\Delta^{\alpha}(f(x)-f(x_0))\cong\Gamma(\alpha+1)(f(x)-f(x_0))$.

The formulas of local fractional derivatives of special functions [19] used in the paper are as follows:

$$D_x^{(\alpha)} ag(x)=aD_x^{(\alpha)} g(x), \tag{2.3}$$

$$\frac{d^{\alpha}}{dx^{\alpha}}\left(\frac{x^{n\alpha}}{\Gamma(1+n\alpha)}\right)=\frac{x^{(n-1)^{\alpha}}}{\Gamma\left(1+(n-1)\alpha\right)}, \quad n\in N \tag{2.4}$$

Definition 4.[19-24]. A partition of the interval $[a,b]$ is denoted as (t_j, t_{j+1}), $j = 0,\ldots,N-1$, $t_0 = a$ and $t_N = b$ with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_0, \Delta t_1, ...\}$.Local fractional integral of $f(x)$ in the interval $[a,b]$ is given by

$$ {}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha . \tag{2.5}$$

The formulas of local fractional integrals of special functions [19] used in the paper are as follows:

$$ {}_0I_x^{(\alpha)} a g(x) = a\, {}_0I_x^{(\alpha)} g(x), \tag{2.6}$$

$$ {}_0I_t^{(\alpha)} \left(\frac{x^{n\alpha}}{\Gamma(1+n\alpha)} \right) = \frac{x^{(n+1)^\alpha}}{\Gamma\big(1+(n+1)\alpha\big)} ,\ n \in N \tag{2.7}$$

3. Analysis of the method

Let us consider a given local fractional differential equation

$$ \frac{\partial^{n\alpha} u(x,t)}{\partial t^{n\alpha}} = L_\alpha u(x,t) , \tag{3.1}$$

where L_α is a linear local fractional derivative operator with respect to x, $n \in \{1,2\}$.

In accordance with the results in [24], there are multiterm separated functions of independent variables t and x reads as

$$ u(x,t) = \sum_{i=0}^{\infty} T_i(t) X_i(x) , \tag{3.2}$$

where $T_i(t)$ and $X_i(x)$ are local fractional continuous functions.

In view of (3.2), we have

$$ T_i(t) = \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} , \tag{3.3}$$

so that

$$ u(x,t) = \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} X_i(x) . \tag{3.4}$$

Making use of (3.4), we get

$$ \frac{\partial^{n\alpha} u(x,t)}{\partial t^{n\alpha}} = \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} X_{i+n}(x) , \tag{3.5}$$

$$ L_\alpha u(x,t) = L_\alpha \left(\sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} X_i(x) \right) = \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} \big(L_\alpha X_i\big)(x) . \tag{3.6}$$

In view of (3.5) and (3.6), we have

$$\sum_{i=0}^{\infty}\frac{1}{\Gamma(1+i\alpha)}t^{i\alpha}X_{i+n}(x)=\sum_{i=0}^{\infty}\frac{t^{i\alpha}}{\Gamma(1+i\alpha)}(L_{\alpha}X_i)(x)\,. \tag{3.7}$$

Hence, from (3.7), the recursion reads as follows:

$$X_{i+n}(x)=(L_{\alpha}X_i)(x)\,, \tag{3.8}$$

with $n=1$; we arrive at the following relation:

$$X_{i+1}(x)=(L_{\alpha}X_i)(x)\,, \tag{3.9}$$

with $n=2$; we may rewrite (3.8) as:

$$X_{i+2}(x)=(L_{\alpha}X_i)(x) \tag{3.10}$$

By the recursion formulas, we can obtain the solution of (3.1) as

$$u(x,t)=\sum_{i=0}^{\infty}\frac{t^{i\alpha}}{\Gamma(1+i\alpha)}X_i(x) \tag{3.11}$$

4. Applications to Laplace and Schrodinger Equations on Cantor Sets

In this section, four examples for Laplace and Schrodinger equations on Cantor sets will demonstrate the efficiency of local fractional series expansion method.

Example 1. Let us consider the Laplace equation on Cantor set

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}+\frac{\partial^{2}u(x,t)}{\partial x^{2\alpha}}=0 \tag{4.1}$$

and subject to the fractal value conditions

$$u(x,0)=-E_{\alpha}(x^{\alpha})\,,\ \frac{\partial^{\alpha}u(x,0)}{\partial t^{\alpha}}=0 \tag{4.2}$$

Following (3.10), we have recursive formula

$$\begin{aligned}
&X_{i+2}(x)=-\frac{\partial^{2\alpha}X_i(x)}{\partial x^{2\alpha}},\\
&X_0(x)=u(x,0)=-E_{\alpha}(x^{\alpha}),\\
&X_1(x)=\frac{\partial^{\alpha}u(x,0)}{\partial t^{\alpha}}=0,
\end{aligned} \tag{4.3}$$

Hence, using the relations (4.3), the recursive calculations yield

$$\begin{aligned}
&X_0(x)=u(x,0)=-E_{\alpha}(x^{\alpha}),\\
&X_1(x)=\frac{\partial^{\alpha}u(x,0)}{\partial t^{\alpha}}=0,\\
&X_2(x)=-\frac{\partial^{2\alpha}X_0(x)}{\partial x^{2\alpha}}=E_{\alpha}(x^{\alpha}),
\end{aligned} \tag{4.4}$$

$$X_3(x) = -\frac{\partial^{2\alpha} X_1(x)}{\partial x^{2\alpha}} = 0,$$

$$X_4(x) = -\frac{\partial^{2\alpha} X_2(x)}{\partial x^{2\alpha}} = -E_\alpha(x^\alpha), \tag{4.5}$$

$$X_5(x) = -\frac{\partial^{2\alpha} X_3(x)}{\partial x^{2\alpha}} = 0,$$

$$X_6(x) = -\frac{\partial^{2\alpha} X_4(x)}{\partial x^{2\alpha}} = E_\alpha(x^\alpha),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

and so on.

Therefore, through (4.5) we get the solution

$$u(x,t) = -E_\alpha(x^\alpha)\sum_{i=0}^{\infty}(-1)^i \frac{t^{2i}}{\Gamma(1+2i\alpha)}$$

$$= -E_\alpha(x^\alpha)\cos_\alpha(t^\alpha) \tag{4.6}$$

Example 2. Let us consider the Laplace equation on Cantor set

$$\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} + \frac{\partial^{2} u(x,t)}{\partial x^{2\alpha}} = 0 \tag{4.7}$$

and subject to the fractal value conditions

$$u(x,0) = 0\ , \ \frac{\partial^{\alpha} u(x,0)}{\partial t^{\alpha}} = -E_\alpha(x^\alpha) \tag{4.8}$$

By using (3.10), we have

$$X_{i+2}(x) = -\frac{\partial^{2\alpha} X_i(x)}{\partial x^{2\alpha}},$$
$$X_0(x) = u(x,0) = 0, \tag{4.9}$$
$$X_1(x) = \frac{\partial^{\alpha} u(x,0)}{\partial t^{\alpha}} = -E_\alpha(x^\alpha).$$

Then, through the iterative relations (4.9), we have

$$X_0(x) = 0,$$

$$X_1(x) = \frac{\partial^{\alpha} u(x,0)}{\partial t^{\alpha}} = -E_\alpha(x^\alpha). \tag{4.10}$$

$$X_2(x) = -\frac{\partial^{2\alpha} X_0(x)}{\partial x^{2\alpha}} = 0,$$

$$X_3(x) = -\frac{\partial^{2\alpha} X_1(x)}{\partial x^{2\alpha}} = E_\alpha(x^\alpha),$$

$$X_4(x) = -\frac{\partial^{2\alpha} X_2(x)}{\partial x^{2\alpha}} = 0, \tag{4.11}$$

$$X_5(x) = -\frac{\partial^{2\alpha} X_3(x)}{\partial x^{2\alpha}} = -E_\alpha(x^\alpha),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

and so on.

Finally, we obtain

$$u(x,t) = -E_\alpha(x^\alpha)\sum_{i=0}^{\infty}(-1)^i \frac{t^{(2i+1)\alpha}}{\Gamma(1+(2i+1)\alpha)}$$

$$= -E_\alpha(x^\alpha)\sin_\alpha(t^\alpha) \tag{4.12}$$

Example 3. Let us consider the Schrodinger equation on Cantor set

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + i\frac{\partial^2 u(x,t)}{\partial x^{2\alpha}} = 0, \tag{4.13}$$

with the initial conditions

$$u(x,0) = \sin_\alpha(x^\alpha) \tag{4.14}$$

Following (3.10), we have recursive formula

$$X_{i+1}(x) = -i\frac{\partial^{2\alpha} X_i(x)}{\partial x^{2\alpha}},$$
$$X_0(x) = u(x,0) = \sin_\alpha(x^\alpha) \tag{4.15}$$

Hence, using the relations (4.15), the recursive calculations yield

$$X_0(x) = u(x,0) = \sin_\alpha(x^\alpha),$$

$$X_1(x) = -i\frac{\partial^{2\alpha} X_0(x)}{\partial x^{2\alpha}} = i\sin_\alpha(x^\alpha),$$

$$X_2(x) = -i\frac{\partial^{2\alpha} X_1(x)}{\partial x^{2\alpha}} = i^2\sin_\alpha(x^\alpha),$$

$$X_3(x) = -i\frac{\partial^{2\alpha} X_2(x)}{\partial x^{2\alpha}} = i^3\sin_\alpha(x^\alpha), \tag{4.16}$$

$$X_4(x) = -i\frac{\partial^{2\alpha} X_3(x)}{\partial x^{2\alpha}} = i^4\sin_\alpha(x^\alpha),$$

$$\cdots\cdots\cdots\cdots\cdots$$

and so on.

Therefore, through (4.16) we get the solution

$$u(x,t) = \sin_\alpha(x^\alpha)E_\alpha\left(i^\alpha t^\alpha\right) \tag{4.17}$$

Example 4. Let us consider the Schrodinger equation on Cantor set

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} + i\frac{\partial^{2} u(x,t)}{\partial x^{2\alpha}} = 0, \tag{4.18}$$

with the initial conditions

$$u(x,0) = E_{\alpha}(x^{\alpha}) \tag{4.19}$$

Following (3.10), we have recursive formula

$$\begin{aligned} X_{i+1}(x) &= -i\frac{\partial^{2\alpha} X_i(x)}{\partial x^{2\alpha}}, \\ X_0(x) &= u(x,0) = E_{\alpha}(x^{\alpha}) \end{aligned} \tag{4.20}$$

Hence, using the relations (4.20), the recursive calculations yield

$$\begin{aligned} X_0(x) &= u(x,0) = E_{\alpha}(x^{\alpha}), \\ X_1(x) &= -i\frac{\partial^{2\alpha} X_0(x)}{\partial x^{2\alpha}} = -i\,E_{\alpha}(x^{\alpha}), \\ X_2(x) &= -i\frac{\partial^{2\alpha} X_1(x)}{\partial x^{2\alpha}} = i^{2} E_{\alpha}(x^{\alpha}), \\ X_3(x) &= -i\frac{\partial^{2\alpha} X_2(x)}{\partial x^{2\alpha}} = -i^{3} E_{\alpha}(x^{\alpha}), \\ X_4(x) &= -i\frac{\partial^{2\alpha} X_3(x)}{\partial x^{2\alpha}} = i^{4} E_{\alpha}(x^{\alpha}), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{4.21}$$

and so on.

Therefore, through (4.21) we get the solution

$$u(x,t) = E_{\alpha}(x^{\alpha}) E_{\alpha}\left(-i^{\alpha} t^{\alpha}\right) = E_{\alpha}\left((x-it)^{\alpha}\right) \tag{4.22}$$

5. Conclusion

In this work, the nondifferentiable solution for the Laplace and Schrodinger equations involving local fractional derivative operators is investigated by using the local fractional series expansion method (LFSEM). In this context, the suggested method is a potential tool for development of approximate solutions of local fractional differential equations with fractal initial value conditions, which, of course, draws new problems beyond the scope of the present work.

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