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Wrapped Log Kumaraswamy Distribution and its Applications

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ARTICLE INFO	ABSTRACT
Published Online: 10 October 2018 Corresponding Author: **K.K. Jose**	A new circular distribution called Wrapped Log Kumaraswamy Distribution (WLKD) is introduced in this paper. We obtain explicit form for the probability density function and derive expressions for distribution function, characteristic function and trigonometric moments. Method of maximum likelihood estimation is used for estimation of parameters. The proposed model is also applied to a real data set on repair times and it is established that the WLKD is better than log Kumaraswamy distribution for modeling the data.

KEYWORDS: Log Kumaraswamy-Geometric distribution, Wrapped Log Kumaraswamy distribution, Trigonometric moments.

1. Introduction

Kumaraswamy (1980) introduced a two-parameter distribution over the support [0, 1], called Kumaraswamy distribution (KD), for double bounded random processes for hydrological applications. The Kumaraswamy distribution is very similar to the Beta distribution, but has the important advantage of an invertible closed form cumulative distribution function. Kumaraswamy (1976, 1978) has showed that the well known probability distribution functions such as the normal, log-normal, beta and empirical distributions such as Johnson's and polynomial – transformed - normal, etc., do not fit well in the case of hydrological data, such as daily rainfall, daily stream flow, etc. and developed a new probability density function known as the sine power probability density function. The probability density function (pdf) is

$$g(x;a,b) = abx^{(a-1)}(1-x^a)^{(b-1)}, x \in [0,1]$$

where a and b are non- negative shape parameters.

The cumulative distribution function (cdf) is

$$G(x;a,b) = 1-(1-x^a)^b$$

The raw moment is $m_n = bB(1+\frac{n}{a},b)$, where B (.) is the beta function.The mean is obtained as $\frac{b\Gamma(1+\frac{1}{a})\Gamma(b)}{\Gamma(1+\frac{1}{a}+b)}$. The median of the distribution is $(1-2^{\frac{-1}{b}})^{\frac{1}{a}}$ and the mode is $(\frac{a-1}{ab-1})^{\frac{1}{a}}$, $a \rangle 1$, $b \rangle 1$.

This distribution has many applications in many natural phenomena whose outcomes have lower and upper bounds, such as the heights of individuals, scores obtained on a test, atmospheric temperatures, hydrological data, etc. Also, this distribution could be appropriate in situations where scientists use probability distributions which have infinite lower and (or) upper bounds to fit data, where as in reality the bounds are finite. Gupta and Kirmani (1988) discussed the connection between non-homogeneous Poisson process (NHPP) and record values. Hence the results in this paper may be used for some applied situations such as preventive maintenance. Kumaraswamy distribution possesses many of the properties of the beta distribution but has some advantages in terms of tractability. This distribution appears to have received considerable interest in hydrology and related areas, see Sundar and Subbiah (1989), Fletcher and Ponnambalam (1996), Seifi et al. (2000), and Ponnambalam et al. (2001). Kumaraswamy distributions are special cases of the three parameter distribution with density

$$g(x) = \frac{a}{B(\gamma,b)} x^{\gamma(a-1)}(1-x^a)^{(b-1)}, x \in [0,1], a,b > 0. \quad (1)$$

Kumaraswamy distributions give rise to many special cases. Kumaraswamy (a,1) distribution is the power function distribution, where as Kumaraswamy (1,a) distribution is the distribution of one minus the power function random variable. Kumaraswamy (1,1) distribution is the uniform distribution. The Kumaraswamy (2,b) distribution has the generating variate $\sqrt{x_1^2+x_2^2}$ when (x_1,x_2) follows a bivariate Pearson Type II distribution (Fang, (1990)). The Kumaraswamy distribution is relatively much appreciated in comparison to the beta distribution, and has a simple form which can be unimodal, increasing, decreasing or constant, depending on the parameter values. Cordeiro and de Castro (2011) defined the Kumaraswamy G distribution specified by

(2)

the cdf and the pdf

$$G(x;a,b)=1-(1-G(x)^{a})^{b} \quad (4)$$

and

$$g(x;a,b)=abg(x)G(x)^{(a-1)}(1-G(x)^{a})^{(b-1)} \quad (5)$$

where $x>0$, $g(x)=dG(x)=dx, a>0$ and $b>0$ are shape parameters.

The Kumaraswamy G families of distributions are more flexible than the baseline distribution in the sense that the families allow for greater flexibility of tail properties. Their second benefit is their ability to fit skew data that cannot be properly fitted by existing distributions. Tractability and effectiveness for modeling censored data require, among other things, closed form expressions for the cdf. So, the Kumaraswamy G distributions can be tractable and effective models for censored data.

In the last several decades various forms of Kumaraswamy G family of distributions have appeared in the literature. For more details see de Pascoa et al. (2011), Kazemi et al. (2011), El-Sherpieny et al. (2011,2014), de Santana et al. (2012), Saulo et al. (2012), Shahbaz et al. (2012), Correa et al. (2012), Bourguignion et al. (2013), Nadarajah et al. (2012a), Paranaiba et al.(2013), Elbatal (2013a, 2013b, 2013c), Muthulakshmi and Selvi (2013), Shams (2013), Zubair (2013), Lemonte et al. (2013), Cordeiro et al. (2014), Gomes et al. (2014), Huang and Oluyede (2014) and Eldin et al.(2014).

2. Log Kumaraswamy distribution (LKD)

Lemonte et al. (2013) introduced the log-exponentiated Kumaraswamy distribution which can be useful to model lifetime data. Log Kumaraswamy distribution as a special case of log-exponentiated Kumaraswamy distribution studied by Lemonte et al. (2013).For more details see Lemonte et al. (2013) and Akinsete et al.(2014). Let X follows Kumaraswamy distribution with parameters a and b. By using transformation technique, $Y=-\log(1-X)$ follows log Kumaraswamy distribution with pdf

$$g(y;a,b)= abe^{-y}(1-e^{-y})^{(a-1)}[1-(1-e^{-y})^{a}]^{(b-1)}, y>0 \quad (6)$$

The cumulative distribution function (cdf) is

$$G(y)=1-[1-(1-e^{-y})^{a}]^{b} \quad (7)$$

The random variable (rv) Y is said to follow the log Kumaraswamy distribution (LKD). It may be noted that the LKD model is related to the T-X family of distributions introduced by Alzaatreh et al. (2013) through the log Kumaraswamy geometric distribution (LKGD). For details, see Akinsete et al. (2014). The Kumaraswamy Geometric distribution is a special case of the T-Geometric family studied in Alzaatreh et al. (2012), by taking the transformed rv T to have the Kumaraswamy distribution and the transformer rv X to have the Geometric distribution.

3. Elementary concepts of Circular Distributions

The major statistical techniques used to analyze the human performance are linear, in which the assumptions are often easy to specify and provide good mathematical solutions for modeling a wide range of data. Most of the problems dealing with different fields like biological scenario do not lend themselves to strict linear representation (see Batschelet (1981) and Zar (1999)). It was found that frequently those data which cannot be modeled in a linear manner are data produced from circular scales. These data points are distributed on a circle instead of the points on the real number line. Circular scales produce cyclic or periodic data that complicate usual analytical procedures. The difficulties found in evaluating circular data are largely a manifestation of the special interval level status the circular scale represents. Circular scales do not have a true zero point. That is, they are circular means that any designation of high or low or more or less is purely arbitrary.

Even though circular data has a long history, there is no major statistical language which provides direct support for circular statistics. The basic statistical assumption in circular statistics is that the data are randomly sampled from a population of directions. Observations arise either from direct measurement of angles or they may arise from the measurement of times reduced modulo some period and converted into angles according to the periodicity of time, such as days or years. They are commonly summarized as locations on a unit circle or as angles over a 360^{0} or 2π radians range, with the endpoints of each range corresponding to the same location on the circle. The important characteristic that differentiates circular data from linear data is its wrap-around nature with no maximum or minimum. That is, the "beginning" coincides with the "end", that is, $0=2\pi$ and in general the measurement is periodic with θ being the same as $\theta+2\pi$ for any integer p.

Differences between the theories of statistics on the line and on the circle can be attributed to the fact that the circle is a closed curve while the line is not. Thus, distribution functions, characteristic functions and moments on the circle have to be defined by taking into account the natural periodicity of the circle. The circular distribution is a probability distribution whose total probability is concentrated on the circumference of unit circle. Since each point on the circumference represents a direction, it is a way of assigning probabilities to different directions or defining a directional distribution. Circular distributions are of two types: discrete and continuous. In continuous case, the probability density function $g(\theta)$ exists and has the following properties

$$(i) g(\theta)\geq 0;\forall\theta, (ii)\int_{0}^{2\pi} g(\theta)d\theta=1 and (iii) g(\theta) = g(\theta+2\pi k); k=0,\pm1,\pm2,\ldots$$

where $g(\theta)$ is periodic with period 2π. Therefore, if X is a rv defined on real line, then the corresponding circular (wrapped) rv X_w is defined as $X_w=x(mod 2\pi)$ and is clearly a many valued function given by

$$X_w(\theta)=\{g(\theta+2\pi k)|k\in Z\}$$

Thus, given a circular rv X_w defined in $[0, 2\pi)$, through the transformation $(\theta + 2\pi k)$, with unobservable variable $k \in Z$, we extend the support of X_w to R so that we can apply an in line density function g(x) to the argument $(\theta + 2\pi k)$. The wrapped circular pdf $g(\theta)$ corresponding to the density function g(x) of a linear rv X is defined as $g(\theta) = \sum_{k=-\infty}^{\infty} g(\theta + 2\pi k); \theta \in [0, 2\pi)$
(for more details, see Jammalamadakka and SenGupta, 2001).

Recently, statisticians took much interest in the study of circular distribution because of their applicability in various areas. Jammalamadakka and Kozubowski (2004) discussed circular distributions obtained by wrapping the classical exponential and Laplace distributions on the real line around the circle. Rao et al (2007) derived new circular models by wrapping the well known life testing models like log normal, logistic, Weibull and extreme-value distributions. Roy and Adnan (2012) developed a new class of circular distributions namely wrapped weighted exponential distribution.In another work, Roy and Adnan (2012) explored wrapped generalized Gompertz distribution and discussed its application to Ornithology. Recently, Jacob and Jayakumar (2013) derived a new family of circular distribution by wrapping geometric distribution and studied its properties. Rao et al (2013) discussed the characteristics of wrapped Gamma distribution. Adnan and Roy (2014) derived wrapped variance Gamma distribution and showed its applicability to wind direction. Joshi and Jose (2017) introduced a wrapped Lindley distribution and applied it for a biological data. Recently, Jammalamadakka and Kozubowski (2017) introduced a general approach to obtain wrapped circular distributions through mixtures.

4. Wrapped Log Kumaraswamy distribution (WLKD)

Let Y follows the Log Kumaraswamy distribution with pdf (6). Then the wrapped Log Kumaraswamy rv is defined as $\theta = X(mod 2\pi)$, such that $\theta \in [0, 2\pi)$. The pdf of the corresponding wrapped distribution is given by

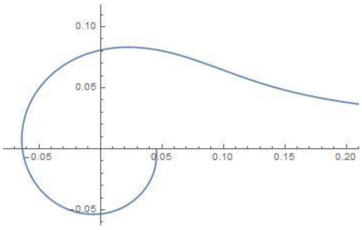
$$g(\theta) = \sum_{m=-\infty}^{\infty} g(\theta + 2\pi m)$$

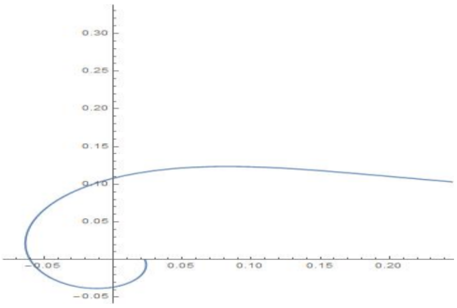
$$= ab \frac{e^{-\theta}}{1-e^{-2\pi}} \left[\frac{1-e^{-2\pi}-e^{-\theta}}{1-e^{-2\pi}}\right]^{a-1} \left[1 - \left(\frac{1-e^{-2\pi}-e^{-\theta}}{1-e^{-2\pi}}\right)^{a}\right]^{b-1}$$

where $\theta \in [0, 2\pi)$, $a > 0$ and $b > 0$.

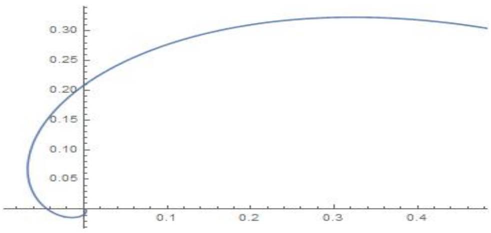
It can be easily verified that (8) is a form of density function of Kumaraswamy distribution multiplied by a constant $\frac{e^{-\theta}}{1-e^{-2\pi}}$.

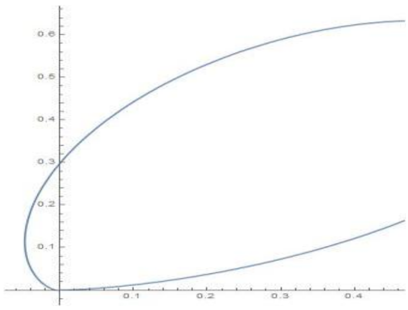
The density plot for different values of the parameters are given in the following figures.



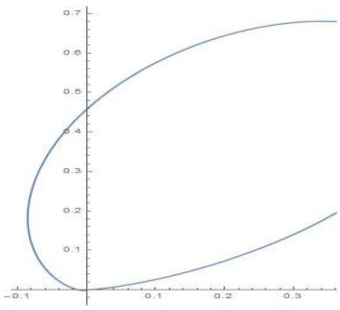


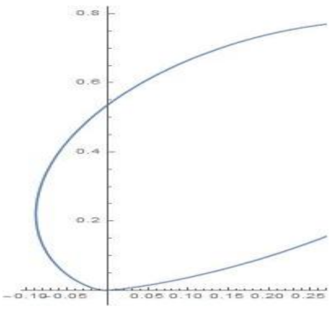
Density for a=0.2,b=0.1; Density for a=0.1,b=0.3





Density for a=1,b=1; Density for a=3,b=3





Density for a=4,b=3; Density for a=5,b=4

5. Cumulative Distribution Function of Wrapped Log Kumaraswamy distribution

Let g(θ) be the probability density function of a continuous rv Θ. Hence g(θ) is a non-negative periodic function with period 2π such that g($\theta + 2\pi$) = g(θ) and $\int_0^{2\pi} g(\theta)d\theta = 1$. The cumulative distribution function G(θ) can be defined over any interval (θ_1, θ_2) by G(θ_2) - G(θ_1) = $\int_{\theta_1}^{\theta_2} g(\theta)d\theta$ and is specified by its cumulative distribution function (cdf). Suppose that an initial direction and orientation of the unit circle have been chosen (generally 0^0 direction and anticlockwise orientation). Then G(θ) is defined as

$$G(\theta) = \int_0^\theta g(\theta)d\theta$$

Obviously it follows that, G(2π) = 1.

The cdf of (8) is given by

$$G(\theta) = \int_0^\theta g(\theta)d\theta$$

$$= \int_0^\theta ab\frac{e^{-\theta}}{1-e^{-2\pi}}\left[\frac{1-e^{-2\pi}-e^{-\theta}}{1-e^{-2\pi}}\right]^{a-1}\left[1-\left(\frac{1-e^{-2\pi}-e^{-\theta}}{1-e^{-2\pi}}\right)^a\right]^{b-1} d\theta.$$

On simplification, we get the cdf as

$$G(\theta) = \left[1-\left(1-\frac{1}{1-e^{-2\pi}}\right)^a\right]^b - \left[1-\left(1-\frac{e^{-\theta}}{1-e^{-2\pi}}\right)^a\right]^b.$$

The survival function and hazard rate function can be obtained from the corresponding density function and distribution function.

6. Characteristic Function and Trigonometric Moments

The characteristic function of a random angle θ is the doubly-infinite sequence of complex numbers $\{\phi_p : p = 0, \pm1, \ldots\}$ given by

$$\phi_p = E(e^{ip\theta})$$
$$= \int_0^{2\pi} e^{ip\theta} dG(\theta)$$

$$= \int_0^{2\pi} e^{ip\theta} ab\frac{e^{-\theta}}{1-e^{-2\pi}}\left[\frac{1-e^{-2\pi}-e^{-\theta}}{1-e^{-2\pi}}\right]^{a-1}\left[1-\left(\frac{1-e^{-2\pi}-e^{-\theta}}{1-e^{-2\pi}}\right)^a\right]^{b-1} d\theta$$

On simplification, we get

$$\phi_p = ab\sum_{j,k=0}^{\infty}\frac{(-1)^{k+j}\binom{b-1}{k}\binom{a(k+1)-1}{j}(1-e^{-2\pi(j-ip+1)})}{(1-e^{-2\pi})^{j+1}}$$

By the definition of trigonometric moments, we have

$\phi_p = \alpha_p + i\beta_p;\ p = \pm1, \pm2, \ldots$ where

$$\alpha_p = E(\cos p\theta)$$
$$= \int_0^{2\pi} (\cos p\theta)g(\theta)d\theta$$

$$= \int_0^{2\pi} (\cos p\theta)ab\frac{e^{-\theta}}{1-e^{-2\pi}}\left[\frac{1-e^{-2\pi}-e^{-\theta}}{1-e^{-2\pi}}\right]^{a-1}\left[1-\left(\frac{1-e^{-2\pi}-e^{-\theta}}{1-e^{-2\pi}}\right)^a\right]^{b-1} d\theta$$

On simplification, we get

$$\alpha_p = ab\sum_{j,k=0}^{\infty}\frac{(-1)^{j+k}\binom{b-1}{k}\binom{a(k+1)-1}{j}[(j+1)(1-e^{-2\pi(j+1)}\cos 2\pi p)+pe^{-2\pi(j+1)}\sin 2\pi p]}{((j+1)^2+p^2)(1-e^{-2\pi})^{j+1}} \quad (11)$$

$$\beta_p = E(\sin p\theta)$$
$$= \int_0^{2\pi} (\sin p\theta)g(\theta)d\theta$$

$$= \int_0^{2\pi} (\sin p\theta)ab\frac{e^{-\theta}}{1-e^{-2\pi}}\left[\frac{1-e^{-2\pi}-e^{-\theta}}{1-e^{-2\pi}}\right]^{a-1}\left[1-\left(\frac{1-e^{-2\pi}-e^{-\theta}}{1-e^{-2\pi}}\right)^a\right]^{b-1} d\theta$$

On simplification, we get (9)

$$\beta_p = ab\sum_{j,k=0}^{\infty}\frac{(-1)^{j+k}\binom{b-1}{k}\binom{a(k+1)-1}{j}[p(1-e^{-2\pi(j+1)}\cos 2\pi p)-(j+1)e^{-2\pi(j+1)}\sin 2\pi p)]}{((j+1)^2+p^2)(1-e^{-2\pi})^{j+1}} \quad (12)$$

According to Jammalamadaka and SenGupta (2001), an alternative expression for the PDF of the wrapped distribution using the trigonometric moments is given by

$$g(\theta) = \frac{1}{2\pi}(1 + 2\sum_{p=1}^{\infty} \alpha_p cos(p\theta) + \beta p sin(p\theta); \theta \in [0, 2\pi)$$

Substituting the values of α_p and β_p given in (11) and (12) in (13), we get an alternative expression for the PDF of the wrapped log Kumaraswamy distribution.

7. Maximum Likelihood Estimation

In this section, the maximum likelihood estimators of the unknown parameters (a, b) of the WLKD distribution are derived. Let $\theta_1, \theta_2, \theta_3, \ldots \theta_n$ be a random sample of size n from WLKD distribution, then the the likelihood function is

$$L = (ab)^n\frac{e^{-\sum_{i=1}^n \theta i}}{(1-e^{2\pi})^n}\prod_{i=1}^n\left[\frac{1-e^{-2\pi}-e^{-\theta i}}{1-e^{-2\pi}}\right]^{a-1}\prod_{i=1}^n\left[1-\left(\frac{1-e^{-2\pi}-e^{-\theta i}}{1-e^{-2\pi}}\right)^a\right]^{b-1} (10)$$

The log likelihood function is given by

$$logL = n\log(ab) - \sum_{i=1}^{n} \theta i + (a-1)\sum_{i=1}^{n} \log(1 - e^{-2\pi} - e^{-\theta i}) - na\log(1-e^{-2\pi}) + (b-1)\sum_{i=1}^{n} \log\left[1-\left(\frac{1-e^{-2\pi}-e^{-\theta i}}{1-e^{-2\pi}}\right)^{a}\right]$$

The partial derivatives of the log likelihood with respect to a and b are obtained as

$$\frac{\partial \log L}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} \log(1-e^{-2\pi}-e^{-\theta i}) - n\log(1-e^{-2\pi}) - (b-1)\sum_{i=1}^{n} \frac{\left[\frac{1-e^{-2\pi}-e^{-\theta i}}{1-e^{-2\pi}}\right]^{a} \log\left[\frac{1-e^{-2\pi}-e^{-\theta i}}{1-e^{-2\pi}}\right]}{\left[1-\left(\frac{1-e^{-2\pi}-e^{-\theta i}}{1-e^{-2\pi}}\right)^{a}\right]} \quad (15)$$

$$\frac{\partial \log L}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} \log\left[1-\left(\frac{1-e^{-2\pi}-e^{-\theta i}}{1-e^{-2\pi}}\right)^{a}\right] \quad (16)$$

Inorder to estimate the parameters,we have to solve the normal equations

$$\frac{\partial \log L}{\partial a} = 0$$

Since (17) cannot be solved analytically, numerical iteration technique is used to get a solution for a. One may use the nlm package in R software to solve this equation to obtain the mle of a. Also we consider

$$\frac{\partial \log L}{\partial b} = 0 \quad (14)$$

From (18) the MLE of b can be obtained as

$$\hat{b} = \frac{-n}{\sum_{i=1}^{n} \log\left[1-\left(\frac{1-e^{-2\pi}-e^{-\theta i}}{1-e^{-2\pi}}\right)^{a}\right]}$$

8. Data Analysis

The following application shows the effectiveness of the wrapped Log Kumaraswamy distribution (WLKD) than the LKD model. We analyze a real data set, on the active repair times (hours) for an airborne communication transceiver, taken from Jorgensen (1982). The data are given in Table 1, and summary statistics are given in Table 2.

Table 1: Active repair times (hours)

0.50	0.60	0.60	0.70	0.70	0.70	0.80	0.80
1.00	1.00	1.00	1.00	1.10	1.30	1.50	1.50
1.50	1.50	2.00	2.00	2.20	2.50	2.70	3.00
3.00	3.30	4.00	4.00	4.50	4.70	5.00	5.40
5.40	7.00	7.50	8.80	9.00	10.20	22.0	24.50

Here we compute the estimates of the unknown parameters with respect to the two distributions, namely WLKD(a, b) and LKD(a, b) and obtain the values for log likelihood, Akaike information criterion (AIC) and Bayesian information criterion (BIC). Based on these values given in the Table 2, we can conclude that the wrapped form of the Log Kumaraswamy distribution is better than the linear form of Log Kumaraswamy distribution for modeling the data on repair times.

Table 2: Summary statistics

Distribution	Estimates		-logL	AIC	BIC
WLKD (a,b)	36.3313	1.6384	22.8224	49.6448	45.849
LKD (a,b)	1.5801	0.2763	94.7417	193.4834	192.6876

9. Conclusion

In this paper, a new circular distribution namely wrapped Log Kumaraswamy distribution is introduced and studied. The pdf and cdf of the distribution are derived and the shapes of the density function for different values of the parameters are obtained. Expressions for characteristic function and trigonometric moments are derived. Method of maximum likelihood estimation is used for estimating the parameters. For exploring the validity of the model, we apply it to a real data set. The performance of the proposed wrapped model is compared with that of linear model using log-likelihood, AIC and BIC. It is concluded that the wrapped log Kumaraswamy distribution is a better model for the given data set than the linear log Kumaraswamy distribution.

Conflicts of Interest: Nil

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