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Generalization of Fourier and Mellin Transforms

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Abstract:
Integral transform, such as Fourier and Mellin are helpful in providing techniques for solving problems in differential equations. It can be used as a mathematical tool to solve these problems. In this paper, we find the correlation of Fourier transform and Mellin transforms of some function $f: \mathrm{R} \to \mathrm{C}$ and characterizing condition in terms of Mellin transform. This will not only be helpful in solving various differential and integral equations, but also have versatile applications in many fields of applied science.

Key words: Fourier transform, Mellin transform, Integral transform, Inverse Mellin transform.

Introduction:

The Fourier and Mellin transforms have been used collectively in many fields separately. On combining these two transforms i.e. Fourier-Finite Mellin transforms is obtained. Fourier-finite Mellin transforms are used for solving differential and integral equations.The algebraic properties of Fourier and Mellin transforms are (briefly) workedout in a series of exercises in [8]. For the more algebraically inclined, one can develop an abstract theory ofconvolution and Fourier analysis on groups. See [7], or [14] for a full treatment. [1], [19], [11], and [15] all cover the Mellin transform, the last two in theprobability context. [3]contains an extensive table of Mellin transforms (aswell as Fourier, Laplace, and other transforms).A more abstract view is provided by [20],which includes a treatment of integral transforms of (Schwartz) distributions. [5] contains a very completetreatment of properties of the Mellin transform, with proofs. But here in this paper we find the correlation of Fourier and Infinite Mellin transforms and how Mellin transform can be obtained from a Fourier transform for some function $f: R \to C$, which is helpful for solving differential equations with manifold applications.
Important and interesting as those cited references are, we know that the Fourier transform has an inherent complex structure and is (more) naturally defined on R (rather thanR+).Here we consider [Q] and use the below criteria

(a) The integral transform is unitary.

(b) The integral kernel is Fourier.

(a) and (b) are understood by well-known Fourier-transform properties:

$$\hat{f}(x) = F(f)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixy} f(y)dy$$

$$f(x) = \bar{F}(\hat{f})(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixy} \hat{f}(y)dy$$

so that we have $\bar{F}F = id$, where id denotes the identity operator, and the kernel is$k(xy) = e^{ixy}/\sqrt{2\pi}$ Thus, the Fourier transform itself is unitary, with a kernel $k(xy)$, satisfying both (a) and (b) above.

A characterizing condition on$k(xy)$ in terms of the Mellin transform has been obtained in [6], [15]

$$\tilde{k}(s)\tilde{k}(1-s) = 1$$

Where : $$\tilde{k}(s) = \int_{R_+} x^{s-1}\, k(x)dx\,,\ \ Re(s) > 0$$

is the Mellin transform of k.

1. Interrelation of Mellin and Fourier transform

The Mellin transform of f with argument $s \in C$ is defined as

$$F(s) = M[f]s = \int_0^{\infty} f(u)u^{s-1}ds$$

where$a \leq Re(s) \leq b$ (if 1 converges for $s = c \in R$, then it can be easily shown that it converges for$s = c + it, t \in R$)

For $(s) = M[f]s$, the inverse Mellin transform is

$$F(x) = M^{-1}\{M[f]\}x = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\, x^{-s}ds$$

For the existence of inverse the condition is that $F(s)x^{-s}$ is analytic in a strip $(a,b) \times (-i\infty, i\infty)$ such that$c \in (a,b)$ [5].

In many engineering texts, statistics, the Fourier transform is defined as

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x}dx$$

Using the transform T and substituting $\xi = -(\frac{\eta-c}{2\pi i})$ for all real $c \geq 0$

$$\hat{T}\hat{f}(\xi) = \int_{-\infty}^{\infty} f(e^x)e^{2\pi i(\frac{\eta-c}{2\pi i})x}dx$$

On substituting $y = e^x$ we get

$$\hat{T}\hat{f}\left(-\frac{\eta-c}{2\pi i}\right) = \int_{-\infty}^{\infty} f(e^x)e^{(\eta-c)\log y}\frac{1}{y}dy$$

$$= \int_{-\infty}^{\infty} f(y)y^{-c}y^{\eta}\frac{1}{y}dy$$

$$= \int_{-\infty}^{\infty} f^*(y)y^{\eta-1}dy\, for f^*(y) = f(y)y^{-c}$$

(An aside on the substitution$\xi = -(\frac{\eta-c}{2\pi i})$. The factor of 2π is a consequenceof the way we define the Fourier transform. In statistics, and in many engineering texts, the Fourier transform is defined as $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{i\xi x}dx$ (essentiallymeasuring frequency in radians per time unit instead of cycles per time unit),which simplifies the derivation of the Mellin transform from the Fourier transform. For a summary of the different ways the Fourier transform and its inverse are represented, see [13].)

The same technique can be used to derive Mellin inverse formula from Fourier inverse transform as

$$f(y) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i log\,(y)\xi}d\xi$$

On substituting $\xi = -(\frac{\eta-c}{2\pi i})$

$$f(y\,) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{f}\left(-\frac{\eta-c}{2\pi i}\right)e^{-(\eta-c)\log y}d\eta$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{f}\left(-\frac{\eta}{2\pi i}\right)y^c y^{-\eta}d\eta$$

$$f(y)y^{-c} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{f}\left(-\frac{\eta}{2\pi i}\right)y^{-\eta}d\eta$$

$$= f^*y$$

In some cases the transformation T provides an easier way to invert Mellin transforms, by using Fourier inversion techniques.

Special case:

For a suitable function$f: R \to C$,

The Fourier transform of f is the integral

$$Ff: R \to C, \quad (Ff)y = \int_R f(x)e^{-2\pi ixy}dx$$

Also the inverse Fourier transform of $g: R \to C$ is the integral

$$F^{-1}g: R \to C, \quad (F^{-1}g)x = \int_R g(y)e^{2\pi ixy}dy$$

If the function f and g are well enough behaved then $g = F^{-1}f$ iff $f = F^{-1}g$ (by using Fourier inversion formula)

The exponential map is a topological isomorphism

$$exp: (R, +) \to (R^+, \cdot)$$

Essentially Mellin transform, Inverse Mellin transform, Inversion formula are the Fourier counterparts of the exponential map passed through the isomorphism.

For a suitable function on the positive real axis

$$f : R^{+} \to C$$

a corresponding function on the positive real axis can be

$$\tilde{f} : R \to C, \qquad \tilde{f} = f \ o \ exp$$

Then the Fourier transform of $\tilde{f}$ is $F\tilde{f} : R \to C$, where

$$(F\tilde{f})y = \int_{R} \tilde{f}\,(x) e^{-2\pi i x y} dx$$

$$= \int_{R} f(e^{x})(e^{x})^{-2\pi i y} \frac{de^{x}}{e^{x}} \ , \quad (on\ substituting\ x = e^{x})$$

$$= \int_{R^{+}} f(t)\,(t)^{-2\pi i y} \frac{dt}{t}$$

$$= \int_{R^{+}} f(t)\,(t)^{s} \frac{dt}{t} \ , \qquad\qquad s = -2\pi i y$$

If $f(t)$ decreases at least as a polynomial in t as $t \to 0^{+}$ and f decreases rapidly as $t \to \infty$, then in fact the integral converges on a complex right half plane of s- values $(Re(s) > \sigma_0)$ where $\sigma_0 < 0$. More generally, if $f(t)$ behaves asymptotically as $t^{-\sigma_0}$ where $\sigma_0 \in R$ as $t \to \sigma^{+}$, and $f(t)$ decreases rapidly as $t \to \infty$, then the integral converges on a complex right half plane $(Re(s) > \sigma_0)$

The Mellin transform of such functions is defined as

$$Mf : \{Re(s) > \sigma_0\} \to C \ , \qquad (Mf)(s) = \int_{R^{+}} f(t)\,(t)^{s} \frac{dt}{t}$$

The condition that $(F\tilde{f})(y)$ is small for large $|y|$ says that $(Mf)(s)$ is small for s far from the real axis.

Example: If the function is taken as negative exponential then Mellin transform gives the gamma function

$$\mathrm{r}(s) = \int_{R>0} e^{-t}\,(t)^{s} \frac{dt}{t} \qquad Re(s) > 0$$

Let, $g = Mf$, so that $g(s) = Mf(s) = (F\tilde{f})(y)$, when $s = -2\pi i y$

The next question is how to recover f from g. Since g is simply the Fourier transform of f up to a coordinate change, f must be essentially the inverse Fourier transform of g In other words $\tilde{f}$ is exactly the inverse Fourier transform of $F\tilde{f}$

$$\tilde{f}(x) = \int_{R} (F\tilde{f}) y e^{2\pi i x y} dy$$

can be written as

$$f(e^x) = \int_R g(s)\,(e^x)^{-s} dy \quad , \quad \text{where } s = -2\pi i y$$

$$= \frac{1}{2\pi i}\int_{Re(s)=0} g(s)\,(e^x)^{-s} ds \qquad \text{integrating upwards}$$

i,e,

$$f(t) = \frac{1}{2\pi i}\int_{Re(s)=0} g(s)\,(t)^{-s} ds$$

By contour integration the vertical line of the integration can be shifted horizontally within the right half plane of convergence with no effect on the integral. Thus inverse Mellin transform of g is inevitably

$$M^{-1}g: R^+ \to C \;, \qquad (M^{-1}g)(t) = \frac{1}{2\pi i}\int_{Re(s)=\sigma} g(s) t^{-s}\, ds \quad \text{for any suitable } \sigma$$

Thus, the Mellin inversion formula says that if the function f and g are well enough behaved then $g = Mf$ iff $f = M^{-1}g$

Now, easily the integral given below can also be evaluated

$$f(t) = \int_{s=\sigma-i\infty}^{\sigma+i\infty} \Gamma(s) t^{-s} ds \quad , \qquad \sigma > 0$$

2. Generalized Characterizing condition in terms of Mellin transform

Theorem : Let $k(xy)$ be a Fourier kernel (on R) such that (a) and (b) are satisfied. This holds if and only if

$$\begin{pmatrix} \kappa_+(1-s) & \kappa_-(1-s) \\ \kappa_-(1-s) & \kappa_+(1-s) \end{pmatrix}\begin{pmatrix} \overline{\kappa}_+(s) & \overline{\kappa}_-(s) \\ \overline{\kappa}_-(s) & \overline{\kappa}_+(s) \end{pmatrix} = I_2 \; for\ all\ s \in C,\ 0 < Re\ s < 1 \qquad (1.1)$$

where $\kappa_+$and κ_- are respectively the Mellin transform of $k_+$and k_- of the Fourier kernel k, $\overline{\kappa}_\pm(s) = \overline{\kappa_\pm(\bar{s})}$, I_2 is 2×2 identity matrix and s can be analytically continued on C.

Proof: For any C^∞ function f with compact support on R, we have

$$f_+(x) + f_-(x)$$

let $k(xy)$ be the Fourier kernel satisfying (a) and (b), and let

$$g(x) = \int_{R^+} k(xy) f(y) dy$$

then

$$g_+(x) = \int_{R^+} k_+(xy) f_+(y) dy + \int_{R^+} k_-(xy) f_-(y) dy \qquad (1.2)$$

and

$$g_-(x) = \int_{R^+} k_-(xy)f_+(y)dy + \int_{R^+} k_+(xy)f_-(y)dy \qquad (1.3)$$

taking Mellin transform of (1.2) and (1.3) and using

$$Mh(s) = Mp(s)Mf(1-s), where \quad Mh(s) = \int_{R_+} x^{s-1}h(x)ds \ , h(x) = \int_{R_+} p(xy)f(y)dy$$

we get

$$\gamma_+(s) = ҡ_+(s)\phi_+(1-s) + ҡ_-(s)\phi_-(1-s)$$

$$\gamma_-(s) = ҡ_-(s)\phi_+(1-s) + ҡ_+(s)\phi_-(1-s)$$

where $\gamma_\pm$, $ҡ_\pm$ and $\phi_\pm$ are respectively the Mellin transform of $g_\pm$, $k_\pm$, $f_\pm$, we thus have

$$\begin{pmatrix} \gamma_+(s) \\ \gamma_-(s) \end{pmatrix} = \begin{pmatrix} ҡ_+(s) & ҡ_-(s) \\ ҡ_-(s) & ҡ_+(s) \end{pmatrix} \begin{pmatrix} \phi_+(1-s) \\ \phi_-(1-s) \end{pmatrix} \qquad (1.4)$$

similarly for reciprocal relation

$$f(x) = \int_R \bar{k}(xy)g(y)dy$$

we obtain

$$\begin{pmatrix} \phi_+(s) \\ \phi_-(s) \end{pmatrix} = \begin{pmatrix} \bar{ҡ}_+(s) & \bar{ҡ}_-(s) \\ \bar{ҡ}_-(s) & \bar{ҡ}_+(s) \end{pmatrix} \begin{pmatrix} \gamma_+(1-s) \\ \gamma_-(1-s) \end{pmatrix} \qquad (1.5)$$

from (1.4) and (1.5) we conclude (1.1).

Further reading:

Mathematical science is not the only one application area for the Mellin transform and Fourier transform,but many other applications also make the Fourier transform and its variantsuniversal elsewhere in almost all branches of science and engineering, i.e, it has versatile applications in many fields. The Mellin transform is used in computer science for analysis of algorithms (see, for example, [17, ch. 9-10]); it hasapplications to analytic number theory [10]; and Mellin himself developed it inconnection with his researches in the theory of functions, number theory, andpartial differential equations [18]. The result (1.1) can also be extended in 3×3 identity matrix that will further improve the characterizing condition of Mellin transform.

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