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Common Fixed Points for φ-pairs in Complex Valued Metric Spaces

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Abstract

We define the concept of φ -mappings and prove a common fixed point theorem for a φ -pair in complex valued metric spaces.

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1. Introduction

The metric fixed point theory is very important and useful in mathematics because of its applications in various areas such as variational and linear inequalities, approximation theory, physics, and computer science. There were many authors introduced the generalizations of metric spaces. One such generalization is a G-metric space proposed by Mustafa and Sims [9]. Another such generalization is a cone metric space initiated by Huang and Zhang [4]. Very recently, Azam et. al.[1] introduced the concept of complex valued metric spaces which is more general than the usual notion of metric spaces and studied some fixed point results for mappings satisfying generalized contractive conditions. In this work, we define φ-mappings in complex valued metric spaces and prove a common fixed point theorem for a φ-pair in this setting. Also, our result is supported by an illustrative example.

2. Preliminaries

We recall some basic facts and definitions in complex valued metric spaces.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. We can define a partial ordering $\precsim$ on $\mathbb{C}$ by $z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$. Thus, $z_1 \precsim z_2$ if one of the following holds

(C1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$;

(C2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$;

(C3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$;

(C4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

In particular, we will write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (C2), (C3), and (C4) is satisfied and we will write $z_1 \prec z_2$ if only (C4) is satisfied.

Definition 2.1. *([1])* Let X be a nonempty set. Suppose that the mapping $d: X \times X \to \mathbb{C}$ satisfies the following conditions*:*

(i) $0 \leq d(x,y)$, for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$, for all $x, y \in X$;

(iii) $d(x,y) \leq d(x,z) + d(z,y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Example 2.2. *([12])* Let $X = \mathbb{C}$. Define the mapping $d: X \times X \to \mathbb{C}$ by

$$d(z_1, z_2) = e^{-ik} |z_1 - z_2|$$

where $k \in \mathbb{R}$. Then (X, d) is a complex valued metric space.

Definition 2.3. *([1])* Let (X, d) be a complex valued metric space, (x_n) be a sequence in X and $x \in X$.

(i) If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$, then (x_n) is said to be convergent , (x_n) converges to x and x is the limit point of (x_n).
We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ *as* $n \to \infty$.

(ii) If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then (x_n) is said to be Cauchy sequence.

(iii) If every Cauchy sequence in X is convergent, then (X, d) is said to be a complete complex valued metric space.

Lemma 2.4. *([1])* Let (X, d) be a complex valued metric space and let (x_n) be a sequence in X. Then (x_n) converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma 2.5. *([1])* Let (X, d) be a complex valued metric space and let (x_n) be a sequence in X. Then (x_n) is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.

Definition 2.6. *([7])* Let T and S be self mappings of a nonempty set X. The mappings T and S are weakly compatible if $T(S(x)) = S(T(x))$ whenever $T(x) = S(x)$.

Definition 2.7. A mapping $T : X \to X$ in a complex valued metric space (X, d) is said to be expansive if there is a real constant $c > 1$ satisfying

$$cd(x,y) \preccurlyeq d(T(x), T(y))$$

for all $x, y \in X$.

Main Results

In this section, we always suppose that $\mathbb{C}$ is the set of complex numbers, $\preccurlyeq$ is a partial ordering on $\mathbb{C}$, and $P = \{z \in \mathbb{C} : Re(z) \geq 0 \text{ and } Im(z) \geq 0\}$. Throughout the paper we denote by $\mathbb{N}$ the set of all positive integers.

Definition 3.1. Let (z_n) be a sequence in P. If for every $c \in P$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that $z_n \prec c$ for all $n > n_0$, then we say that $z_n \to 0$.

Definition 3.2. A nondecreasing mapping $\varphi: P \to P$ is called a φ -mapping if

(φ_1) $\varphi(0) = 0$ *and* $0 \precnsim \varphi(z) \precnsim z$ *for* $z \in P \setminus \{0\}$,

(φ_2) $\varphi(z) \prec z$ *for every* $0 \prec z$,

(φ_3) $\lim_{n\to\infty} \varphi^n(z) = 0$ *for every* $z \in P \setminus \{0\}$.

Definition 3.3. Let (X,d) be a complex valued metric space. The mappings $T,S: X \to X$ are called φ -pair if there exists a φ -mapping satisfying

$$d(T(x),T(y)) \preccurlyeq \varphi(d(S(x),S(y)))$$

for every $x,y \in X$.

Theorem 3.4. Let (X,d) be a complex valued metric space. Suppose that the mappings $T,S: X \to X$ satisfy

$$d(T(x),T(y)) \preccurlyeq \varphi(d(S(x),S(y))) \quad (3.1)$$

for all $x,y \in X$, where φ is a φ –mapping . If $T(X) \subseteq S(X)$, T and S are weakly compatible, and $T(X)$ or $S(X)$ is complete, then T and S have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. Since $T(X) \subseteq S(X)$, we can choose $x_1 \in X$ such that $S(x_1) = T(x_0)$. Proceeding in this way, after choosing $x_n \in X$, we can choose $x_{n+1} \in X$ such that $S(x_{n+1}) = T(x_n)$. Since φ is nondecreasing , it follows from condition (3.1) that

$$\begin{aligned} d\big(T(x_{n+1}),T(x_n)\big) &\preccurlyeq \varphi\Big(d\big(S(x_{n+1}),S(x_n)\big)\Big) \\ &= \varphi(d(T(x_n),T(x_{n-1}))) \\ &\preccurlyeq \varphi^2(d(S(x_n),S(x_{n-1}))) \\ &\cdot \\ &\cdot \\ &\cdot \\ &\preccurlyeq \varphi^{n}(d(T(x_1),T(x_0))) \end{aligned} \quad (3.2)$$

Let $c \in \mathbb{C}$ with $0 \prec c$, then by (φ_2) , $\varphi(c) \prec c$, that is, $0 \prec c-\varphi(c)$. By applying (φ_3), we have

$$\lim_{n\to\infty} \varphi^n\Big(d\big(T(x_1),T(x_0)\big)\Big) = 0.$$

So, one can find $n_0 \in \mathbb{N}$ such that

$$\varphi^{m}\Big(d\big(T(x_1),T(x_0)\big)\Big) \prec c-\varphi(c), \text{ for all } m > n_0.$$

From (3.2), we have

$$d\big(T(x_m),T(x_{m+1})\big) \prec c-\varphi(c), \textit{ for all } m > n_0.$$

We show that

$$d(T(x_m),T(x_{n+1})) \prec c \quad (3.3)$$

for a fixed $m > n_0$ and $n \geq m$.
This is true for $n = m$. We assume that (3.3) holds for some $n \geq m$. Then,

$$\begin{aligned} d(T(x_m),T(x_{n+2})) &\preccurlyeq d\big(T(x_m),T(x_{m+1})\big) + d\big(T(x_{m+1}),T(x_{n+2})\big) \\ &\preccurlyeq d\big(T(x_m),T(x_{m+1})\big) + \varphi(d\big(S(x_{m+1}),S(x_{n+2})\big)) \\ &= d\big(T(x_m),T(x_{m+1})\big) + \varphi(d\big(T(x_m),T(x_{n+1})\big) \\ &\prec c-\varphi(c)+\varphi(c) \\ &= c\ . \end{aligned}$$

Therefore, by induction (3.3) holds for a fixed $m > n_0$ and $n \geq m$. Thus, $(T(x_n))$ is a Cauchy sequence in $T(X)$. Suppose that $T(X)$ is a complete subspace of X. So there exists $u \in T(X) \subseteq S(X)$ such that $T(x_n) \to u$ and also $S(x_n) \to u$. If $S(X)$ is complete, then this holds with $u \in S(X)$. Let $v \in X$ be such that $S(v) = u$.

For $0 \prec c$, there exists a natural number n_0 such that $d\big(u, T(x_n)\big) \prec \frac{c}{2}$ and $d\big(S(x_n), S(v)\big) \prec \frac{c}{2}$ for all $n > n_0$. Then,

$$\begin{aligned} d(u,T(v)) &\precsim d\big(u,T(x_n)\big) + d(T(x_n),T(v)) \\ &\precsim d\big(u,T(x_n)\big) + \varphi(d\big(S(x_n),S(v)\big)) \\ &\precnsim d\big(u,T(x_n)\big) + d\big(S(x_n),S(v)\big) \end{aligned}$$

$$\prec \frac{c}{2} + \frac{c}{2} = c$$

So, it must be the case that,

$$d(u,T(v)) \prec c.$$

Therefore, it follows that

$$d(u,T(v)) \prec \frac{c}{m}, \text{ for every } m \in \mathbb{N}.$$

Thus, $\big|\, d\big(u,T(v)\big) \,\big| < \left|\frac{c}{m}\right|$ for every $m \in \mathbb{N}$. Taking $m \to \infty$, we have $\big|\, d\big(u,T(v)\big) \,\big| = 0$,

which implies that $d(u,T(v)) = 0$ and so, $u = T(v) = S(v)$. Since T and S are weakly compatible, it follows that $T(u) = T(S(v)) = S(T(v)) = S(u)$. If $S(u) \neq u$, then

$$d(T(u),T(v)) \precsim \varphi\,(d(S(u),S(v))) \precnsim d(S(u),S(v)) = d(T(u),T(v)),$$

which is a contradiction. Therefore, $T(u) = S(u) = u$ and so u is a common fixed point of T and S.

For uniqueness, let there exists another point $w(\neq u) \in X$ such that $T(w) = S(w) = w$. Then,

$$\begin{aligned} d(u,w) &= d(T(u),T(w)) \\ &\precsim \varphi\,(d\big(S(u),S(w)\big)) \\ &= \varphi\,(d(u,w)) \\ &\precnsim d(u,w), \end{aligned}$$

which implies that $u = w$.

As an application of Theorem 3.4, we have the following results.

Corollary 3.5. Let (X,d) be a complete complex valued metric space and let $T : X \to X$ be a mapping satisfying

$$d(T(x),T(y)) \precsim \varphi(d(x,y))$$

for all $x, y \in X$. Then T has a unique fixed point in X.

Proof. The proof follows from Theorem 3.4 by taking $S = I$, the identity map.

Corollary 3.6. Let *(X, d)* be a complete complex valued metric space and let $S : X \to X$ be an onto mapping satisfying

$$d(x,y) \precsim \varphi(d(S(x),S(y)))$$

for all $x, y \in X$. Then S has a unique fixed point in X.

Proof. The proof can be obtained from Theorem 3.4 by considering $T = I$.

Corollary 3.7. Let (X, d) be a complete complex valued metric space and let $T : X \to X$ satisfies

$$d(T(x), T(y)) \precsim kd(x, y)$$

for all $x, y \in X$, where $0 \leq k < 1$. Then T has a unique fixed point in X.

Proof. Taking $S = I$ and $\varphi : P \to P$ by $\varphi(z) = kz$ where $0 \leq k < 1$, the conclusion of the Corollary follows from Theorem 3.4.

Corollary 3.8. Let (X, d) be a complete complex valued metric space and let $S : X \to X$ be an onto expansive mapping. Then S has a unique fixed point in X.

Proof. Putting $T = I$ and $\varphi : P \to P$ by $\varphi(z) = \frac{1}{k}z$ where $k > 1$, the proof follows from Theorem 3.4.

We conclude with an example.

Example 3.9. Let $X = \mathbb{R}$ and $d: X \times X \to \mathbb{C}$ be given by

$$d(x, y) = |x - y| + i|x - y|, \text{ for all } x, y \in X .$$

Then (X, d) is a complete complex valued metric space. Define $T, S : X \to X$ by

$T(x) = 2x - 1$, $S(x) = 3x - 2$, for all $x \in X$. Also, define $\varphi : P \to P$ by $\varphi(z) = \frac{3}{4}z$, for all $z \in P$.
Now,

$$\begin{aligned} d(T(x), T(y)) &= |T(x) - T(y)| + i|T(x) - T(y)| = 2[|x - y| + i|x - y|] \\ &= \frac{2}{3}[|3x - 3y| + i|3x - 3y|] \\ &= \frac{2}{3}[|S(x) - S(y)| + i|S(x) - S(y)|] \\ &= \frac{2}{3}d(S(x), S(y)) \\ &\prec \frac{3}{4}d(S(x), S(y)) \\ &= \varphi\big(d(S(x), S(y))\big), \quad \text{for all } x, y \in X. \end{aligned}$$

Thus, condition *(3.1)* of Theorem *3.4* is satisfied. Clearly, T and S are weakly compatible and $T(X) = S(X) = X$. We see that 1 is the unique common fixed point of T and S in X.

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