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Some Threshold Theorems for a Prey-Predator Model with an Optimal Harvesting of the Prey

B. Ravindra Reddy
Department of Mathematics, JNTUHCEJ, Nachupally, Karimnagar District, India
Email: rbollareddy@gmail.com

Abstract

The present paper is devoted to derive some threshold theorems for a two species model comprising a prey and a predator. Predator is provided with a limited resource of food in addition to the prey and the prey is harvested under optimal conditions. In consonance with the principle of competitive exclusion Gauss, three theorems and ten lemmas has been derived. The model is characterized by a couple of first order non-linear ordinary differential equations.

1 Introduction

Ecology relates to the study of living beings in relation to their living styles. Research in the area of theoretical ecology was initiated by Lotka [1] and by Volterra [2]. Since then many mathematicians and ecologists contributed to the growth of this area of knowledge as reported in the treatises of Meyer [3], Kushing [4], Paul colinvaux [5], Kapur [6] etc. The ecological interactions can be broadly classified as Prey – predation, Competition, Commensalim, Ammensalism, Neutralism and so on. N.C.Srinivas [7] studied competitive eco-systems of two species and three species with limited and unlimited resources. Later, Lakshminarayan and Pattabhi Ramacharyulu [8] studied some threshold theorems for a Prey-predator model harvesting. Recently, the present author et al [9-12] investigated mutualism between two species.

2 Basic equations

The model equations for a two species prey-predator system are given by the following system of non-linear ordinary differential equations employing the following notation:

N_1 and N_2 are population of the prey and predator, a_1 and a_2 are the rates of natural growth of the prey and predator, α_{11} is rate of decrease of the prey due to insufficient food, α_{12} is rate of decrease of the prey due to successful attacks by the predator, α_{22} is rate of decrease of the predator due to insufficient food other than the prey, α_{21} is rate of increase of the predator due to successful attacks on the prey, q_1 is the catch ability co-efficient of the prey, E is the harvesting effort and $q_1 EN_1$ is the catch-rate function based on the CPUE (catch-per-unit-effort) hypothesis]. Further both the variables N_1 and N_2 are non-negative and the model parameters $a_1, a_2, \alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}, q_1,$ E and $\left(a_1 - q_1 E\right)$ are assumed to be non-negative constants. Employing the above terminology, the model equations for a two species prey-predator system is given by the following system of non-linear ordinary differential equations.

(i) Equation for growth rate of prey species (N_1):

$$\frac{dN_1}{dt} = a_1 N_1 - \alpha_{11} N_1^2 - \alpha_{12} N_1 N_2 - q_1 E N_1$$

$$\Rightarrow \frac{dN_1}{dt} = \left(a_1 - q_1 E\right) N_1 - \alpha_{11} N_1^2 - \alpha_{12} N_1 N_2$$

(2.1)

(ii) Equation for the growth rate of predator species (N_2):

$$\frac{dN_2}{dt} = a_2 N_2 - \alpha_{22} N_2^2 + \alpha_{21} N_1 N_2 \qquad (2.2)$$

3 Equilibrium states

The system under investigation has four equilibrium states:

I. The fully washed out state with the equilibrium point $\bar{N}_1 = 0; \bar{N}_2 = 0$ (3.1)

II. The state in which, only the predator survives and the prey is washed out. The equilibrium point

is $\bar{N}_1 = 0; \bar{N}_2 = \frac{a_2}{\alpha_{22}}$ (3.2)

III. The state in which, only the prey survives and the predator is washed out

The equilibrium point

is $\bar{N}_1 = \frac{(a_1 - q_1 E)}{\alpha_{11}}; \bar{N}_2 = 0$ (3.3)

IV. The co-existent state (**normal steady state**). The equilibrium point is

$$\bar{N}_1 = \frac{\alpha_{22}(a_1 - q_1 E) - a_2\alpha_{12}}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}};$$

$$\bar{N}_2 = \frac{a_2\alpha_{11} + \alpha_{21}(a_1 - q_1 E)}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}} \quad (3.4)$$

This state would exit only when $\alpha_{22}(a_1 - q_1 E) > a_2\alpha_{12}$

4 Threshold theorems

In consonance with the principle of competitive exclusion, Gauss [13] three Threshold theorems one for each of the above three not-fully washed equilibrium states has been deduced. The equations are:

$$\frac{dN_1}{dt} = \frac{(a_1 - q_1 E)N_1}{k_1}\{k_1 - N_1 - \beta_1 N_2\},$$

$$\frac{dN_2}{dt} = \frac{a_2 N_2}{k_2}\{k_2 - N_2 - \beta_2 N_1\} \quad (4.1)$$

where $k_1 = \frac{(a_1 - q_1 E)}{\alpha_{11}}; k_2 = \frac{a_2}{\alpha_{22}};$

$\beta_1 = \frac{\alpha_{12}}{(a_1 - q_1 E)}$ and $\beta_2 = -\frac{\alpha_{21}}{a_2}$

Theorem 1: *Principle of Competitive Exclusion for Equilibrium State II:*

$\bar{N}_1 = 0; \bar{N}_2 = \frac{a_2}{\alpha_{22}}$

When $k_1 > k_2$, then every solution $N_1(t), N_2(t)$ of (4.1) approaches the equilibrium solution $N_1 = k_1, N_2 = 0$ as t approaches infinity. In other words, if species 1 and 2 are nearly identical and the microcosm can support more members of species 1 than species 2, then species 2 will ultimately becomes extinct.

Proof: The first step in proving this is to show that $N_1(t)$ and $N_2(t)$ can never become negative. To this end, observe that

$$N_1(t) = \frac{k_1 N_1(0)}{N_1(0) + (k_1 - N_1(0))e^{-(a_1 - q_1 E)\,t}} \quad \text{and}$$

$$N_2(t) = 0 \quad (4.2)$$

is a solution of (4.1) for any choice of $N_1(0)$. The orbit of this solution in the $N_1 - N_2$ plane is the point (0, 0) for $N_1(0) = 0$; the line $0 < N_1 < k_1$, $N_2 = 0$ for $0 < N_1(0) < k_1$; the point $(k_1, 0)$ for $N_1(0) = k_1$; and the line $k_1 < N_1 < \infty$, $N_2 = 0$ for $N_1(0) > k_1$. Thus the N_1 axis, for $N_1 \geq 0$, is the union of four distinct orbits of (4.1).Similarly, the N_2 axis, for $N_2 \geq 0$, is the union of four distinct orbits of (4.1).This implies that $N_1(t), N_2(t)$ of (4.1) which start in the first quadrant $(N_1(t) > 0, N_2 > 0)$ of the $N_1 - N_2$ plane must remain there for all future time.

The second step is to split the first quadrant into regions in which both $\frac{dN_1}{dt}$ and $\frac{dN_2}{dt}$ have fixed signs. This is accomplished in the following manner.

Let l_1 and l_2 be the lines

$$(k_1 - N_1 - \beta_1 N_2) = 0 \quad (4.3)$$

and

$$(k_2 - N_2 - \beta_2 N_1) = 0 \quad (4.4)$$

These lines are non-parallel and non-intersecting in N_1-N_2 plane respectively (Ref.Fig.1). Observe that $\frac{dN_1}{dt}$ is negative if (N_1, N_2) lies above l_1 and positive if (N_1, N_2) lies below l_1. Similarly, $\frac{dN_2}{dt}$ is negative if (N_1, N_2) lies above l_2 and positive if (N_1, N_2) lies below l_2. Thus the two lines l_1 and l_2 split the first quadrant of the $N_1 - N_2$ plane into three regions in which both $\frac{dN_1}{dt}$ and $\frac{dN_2}{dt}$ have fixed signs. Both $N_1(t), N_2(t)$ increases with time (along any solution of (4.1) in region I; $N_1(t)$ increases and $N_2(t)$ decreases with time in region II; and both $N_1(t)$ and $N_2(t)$ decrease with time in region III. This is illustrated in Fig.1

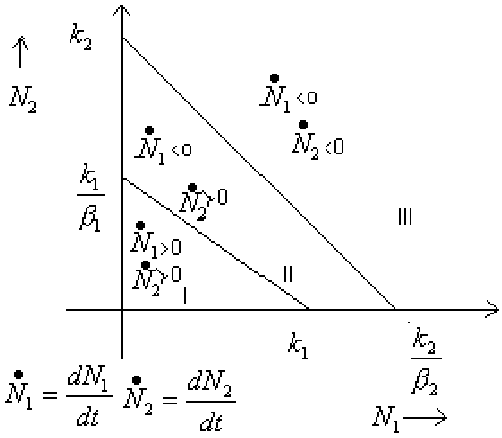

Fig.1

Finally we require the following three lemmas for establishing the threshold theorems.

Lemma 1: *Any solution of* $N_1(t), N_2(t)$ *of (4.1) which starts in region I at time* $t = t_0$ *must leave this region I at some latter instant of time (Fig..1).*

Proof: Suppose that a solution $N_1(t), N_2(t)$ of (4.1) remain in region I for all time $t \geq t_0$. This implies that both $N_1(t)$ and $N_2(t)$ are monotonic increasing functions of time for $t \geq t_0$, with $N_1(t)$ and $N_2(t)$ less than k_2. Consequently both $N_1(t)$ and $N_2(t)$ have limits ξ, η respectively, as t approaches infinity. This, in turn, implies that (ξ, η) is an equilibrium point of (4.1). Now the only equilibrium points of (4.1) are (0, 0), (k_1, 0), $(0, k_2)$ and (ξ, η) obviously cannot equal any of these three points. We conclude, therefore, that any solution $N_1(t), N_2(t)$ of (4.1) which starts in region I must leave this region at a later time.

Lemma 2: *Any solution of* $N_1(t), N_2(t)$ *of (4.1) which starts in region II at time* $t = t_0$ *will remain in this region for all future time* $t \geq t_0$*, and ultimately approach the equilibrium solution* $N_1 = k_1, N_2 = 0$ *(Fig.1).*

Proof: Suppose that a solution $N_1(t), N_2(t)$ of (4.1) leaves region II at time $t = t^*$. Then either $\frac{dN_1}{dt}(t^*)$ or $\frac{dN_2}{dt}(t^*)$ is zero, since the only way a solution of (4.1) can leave region II is by crossing l_1 or l_2. Assume that $\frac{dN_1}{dt}(t^*) = 0$. Differentiation both sides of the first equation of (4.1) with respect to t and setting $t = t^*$ gives

$$\frac{d^2 N_1(t^*)}{dt} = \frac{-(a_1 - q_1 E)\beta_1 N_1(t^*)}{k_1} \frac{dN_2(t^*)}{dt}$$

This quantity is positive. Hence $N_1(t)$ has a minimum at $t = t^*$. However, this is impossible, since $N_1(t)$ is increasing whenever a solution of $N_1(t), N_2(t)$ of (4.1) is in region *II*.

Similarly, if $\frac{dN_2}{dt}(t^*) = 0$,

then
$$\frac{d^2 N_2(t^*)}{dt} = \frac{-a_2 \beta_2 N_2(t^*)}{k_2} \frac{dN_1}{dt}(t^*). \tag{4.7}$$

This quantity is negative, implying that $N_2(t)$ has a maximum at $t = t^*$, but this is impossible, since $N_2(t)$ is decreasing whenever a solution $N_1(t), N_2(t)$ of (4.1) is in region *II*.

The previous argument shows that any solution $N_1(t), N_2(t)$ of (4.1) which starts in region *II* at time $t = t_0$ will remain in region *II* for all future time $t \geq t_0$. This implies that $N_1(t)$ is monotonic increasing and $N_2(t)$ is monotonic decreasing for $t \geq t_0$; with $N_1(t) < k_1$ and $N_2(t) > k_2$. Consequently, both $N_1(t)$ and $N_2(t)$ have limits ξ, η respectively, as t approaches infinity. This in turn, implies that (ξ, η) is an equilibrium point of (4.1). Now (ξ, η) obviously cannot equal (0, 0) or $(0, k_2)$. Consequently, $(\xi, \eta) = (k_1, 0)$ and this proves Lemma 2.

Lemma 3: Any solution of $N_1(t), N_2(t)$ of (4.1) which starts in region III at time $t \geq t_0$ and remains there for all future time must approach the equilibrium solution $N_1(t) = k_1, N_2(t) = 0$ as t approaches infinity *(Fig.1).*

Proof: If a solution $N_1(t), N_2(t)$ of (4.1) remains in region *III* for $t \geq t_0$, then both $N_1(t)$ and $N_2(t)$ are monotonic decreasing functions of time for $t \geq t_0$, with $N_1(t) > k_1$ and $N_2(t) > k_2$, consequently, both $N_1(t)$ and $N_2(t)$ have limits ξ, η respectively, as t approaches infinity. This, in turn implies that (ξ, η) is an equilibrium point of (4.1). Now, (ξ,η) obviously cannot equal (0, 0) or $(0,k_2)$. Consequently $(\xi,\eta)=(k_1,0)$.

Proof of Theorem: Lemmas 1 and 2 state that every solution ($N_1(t), N_2(t)$) of (4.1) which starts in region *I* or *II* at time $t=t_0$ must approach the equilibrium solution $N_1=k_1$, $N_2=0$ as t approaches infinity. Similarly, Lemma 3 shows that every solution ($N_1(t), N_2(t)$) of (4.1) which starts in region *III* at time $t=t_0$ and remains there for all future time must also approach equilibrium solution $N_1=k_1$, $N_2=0$. Next, observe that any solution ($N_1(t), N_2(t)$) of (4.1) which starts on l_1 or l_2 would soon enter region *II*. Finally, if a solution ($N_1(t), N_2(t)$) of (4.1) leaves region *III*, then it must crosses the line l_1 and immediately afterwards enters region *II*. Lemma 2 then forces the solution to approach the equilibrium solution $N_1=k_1$, $N_2=0$. This is illustrated in the Fig.2.

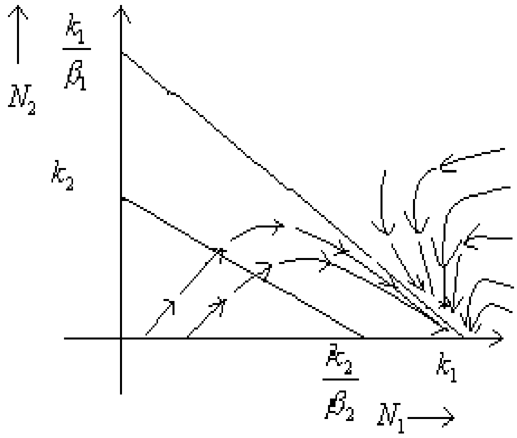

Fig.2

Theorem 2: *Principle of Competitive Exclusion for Equilibrium State III:*

$$\overline{N}_1=\frac{(a_1-q_1E)}{\alpha_{11}};\overline{N}_2=0$$

When $k_1<k_2$, then every solution $N_1(t), N_2(t)$ of (4.1) approaches the equilibrium solution $N_1=0$, $N_2=k_2$ as t approaches infinity. In other words, if species 1 and 2 are nearly identical and the microcosm can support more members of species 1 than species 2, then species 2 will ultimately becomes extinct.

Proof: The first step in our proof is to show that $N_1(t)$ and $N_2(t)$ can never become negative. To this end, we observe that

$$N_1=0 \text{ and } N_2(t)=\frac{k_2N_2(0)}{N_2(0)+\left(k_2-N_2(0)\right)e^{-a_2t}}$$

is a solution of (4.1) for any choice of $N_2(0)$. The orbit of this solution in the N_1-N_2 plane is the point (0, 0) for $N_2(0)=0$; the line $0<N_1<k_1$, $N_1(=0)$ for $0<N_2(0)<k_2$; the point $(0,k_2)$ for $N_2(0)=k_2$; and the line $k_2<N_2<\infty$, $N_1=0$ for $N_2(0)>k_2$. Thus the N_2 axis, for $N_2\geq 0$, is the union of four distinct orbits of (4.1). Similarly, the N_1 axis, for $N_1\geq 0$, is the union of four distinct orbits of (4.1). This implies that $N_1(t), N_2(t)$ of (4.1) which starts in the first quadrant $\left(N_1(t)>0, N_2>0\right)$ of the N_1-N_2 plane must remain there for all future time.

The second step in our proof is to split the first quadrant into regions in which both $\frac{dN_1}{dt}$ and $\frac{dN_2}{dt}$ have fixed signs. This is accomplished in the following manner.

Let l_1 and l_2 be the lines $\left(k_1-N_1-\beta_1N_2\right)=0$ and $\left(k_2-N_2-\beta_2N_1\right)=0$ respectively. Observe that $\frac{dN_1}{dt}$ is negative if (N_1,N_2) lies above l_1 and positive if (N_1,N_2) lies below l_1. Similarly, $\frac{dN_2}{dt}$ is negative if (N_1,N_2) lies above l_2 and positive if (N_1,N_2) lies below l_2. Thus the two parallel lines l_1 and l_2 split the first quadrant of the N_1-N_2 plane into three regions in which both $\frac{dN_1}{dt}$ and $\frac{dN_2}{dt}$ have fixed signs. Both $N_1(t), N_2(t)$ increases with time along any solution of (4.1) in region I; $N_1(t)$ increases and $N_2(t)$ decreases with time in region II; and both $N_1(t)$ and $N_2(t)$ decrease with time in region III (Ref. Fig.3). We require the following three lemmas.

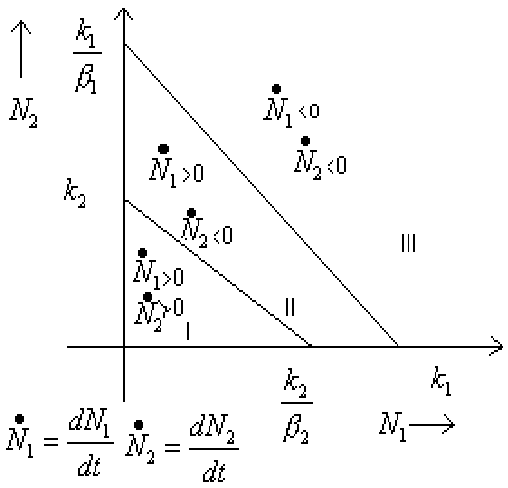

Fig.3

Lemma 4: *Any solution of* $N_1(t), N_2(t)$ *of .4.1) which starts in region I at time* $t = t_0$ *must leave this region I at some latter time. (Fig.3)*

Proof: Suppose that a solution $N_1(t), N_2(t)$ of (4.1) remain in region *I* for all time $t \ge t_0$. This implies that both $N_1(t)$ and $N_2(t)$ are monotonic increasing functions of time for $t \ge t_0$, with $N_1(t)$ and $N_2(t)$ less than k_2. Consequently both $N_1(t)$ and $N_2(t)$ have limits ξ, η respectively, as t approaches infinity. This, in turn, implies that (ξ, η) is an equilibrium point of (4.1). Now the only equilibrium points of (4.1) are (0, 0), (k_1,0), $(0, k_2)$ and obviously (ξ, η) cannot equal any of these three points. We conclude, therefore, that any solution $N_1(t), N_2(t)$ of (4.1) which starts in region *I* must leave this region at a later time.

Lemma 5: *Any solution of* $N_1(t), N_2(t)$ *of (4.1) which starts in region II at time* $t = t_0$ *will remain in this region for all future time* $t \ge t_0$, *and ultimately approach the equilibrium solution* $N_1 = 0, N_2 = k_2$ (Fig.3).

Proof: Suppose that a solution $N_1(t), N_2(t)$ of (4.1) leaves region *II* at time $t = t^*$. Then either $\frac{dN_1}{dt}(t^*)$ or $\frac{dN_2}{dt}(t^*)$ is zero, since the only way a solution of (4.1) can leave region *II* is by crossing l_1 or l_2. Assume that $\frac{dN_1}{dt}(t^*) = 0$. Differentiation both sides of the first equation of (4.1) with respect to *t* and setting $t = t^*$ gives

$$\frac{d^2N_1(t^*)}{dt} = \frac{-\left(a_1 - q_1E\right)\beta_1 N_1(t^*)}{k_1}\frac{dN_2(t^*)}{dt}$$

This quantity is positive. Hence $N_1(t)$ has a minimum at $t = t^*$. However, this is impossible, since $N_1(t)$ is increasing whenever a solution of $N_1(t), N_2(t)$ of (4.1) is in region *II*.

Similarly, if $\frac{dN_2}{dt}(t^*) = 0$,

then $\frac{d^2N_2(t^*)}{dt} = \frac{-a_2\beta_2 N_2(t^*)}{k_2}\frac{dN_1}{dt}(t^*)$.

This quantity is negative, implying that $N_2(t)$ has a maximum at $t = t^*$, but this is impossible, since $N_2(t)$ is decreasing whenever a solution $N_1(t), N_2(t)$ of (4.1) is in region *II*.

The previous argument shows that any solution $N_1(t), N_2(t)$ of (4.1) which starts in region *II* at time $t = t_0$ will remain in region *II* for all future time $t \ge t_0$. This implies that $N_1(t)$ is monotonic increasing and $N_2(t)$ is monotonic decreasing for $t \ge t_0$; with $N_1(t) < k_1$ and $N_2(t) > k_2$. Consequently, both $N_1(t)$ and $N_2(t)$ have limits ξ, η respectively, as *t* approaches infinity. This in turn, implies that (ξ, η) is an equilibrium point of (4.1). Now (ξ, η) obviously cannot equal (0, 0) or $(0, k_2)$. Consequently, $(\xi, \eta) = (0, k_2)$ and this proves Lemma 5.

Lemma 6: *Any solution of* $N_1(t), N_2(t)$ *of (4.1) which starts in region III at time* $t \ge t_0$ *and remains there for all future time must approach the equilibrium solution* $N_1(t) = 0$, $N_2(t) = k_2$ *as t approaches infinity* (Fig 3).

Proof: If a solution $N_1(t), N_2(t)$ of (4.1) remains in region *III* for $t \ge t_0$, then both $N_1(t)$ and $N_2(t)$ are monotonic decreasing functions of time for $t \ge t_0$, with $N_1(t) > k_1$ and $N_2(t) > k_2$, consequently, both $N_1(t)$ and $N_2(t)$ have limits ξ, η respectively, as t approaches infinity. This, in turn implies that (ξ, η) is an equilibrium point of (4.1). Now, (ξ, η) obviously cannot equal (0, 0) or $(k_1, 0)$. Consequently $(\xi, \eta) = (0, k_2)$.

Proof of Theorem: Lemmas 4 and 5 state that every solution $N_1(t), N_2(t)$ of (4.1) which starts in region *I* or *II* at time $t = t_0$ must approach the equilibrium solution $N_1 = 0$, $N_2 = k_2$ as *t* approaches infinity. Similarly, Lemma 6 shows that every solution $N_1(t), N_2(t)$ of (4.1) which starts in region *III* at time $t = t_0$ and remains there for all future time must also approach equilibrium solution $N_1 = 0$, $N_2 = k_2$. Next, observe that any solution $N_1(t), N_2(t)$ of (4.1) which starts on l_1 or l_2 must immediately afterwards enter region *II*. Finally, if a solution $N_1(t), N_2(t)$ of (4.1) leaves region *III*, then it must cross the line l_1 and immediately afterwards enter region *II*. Lemma 5 then forces the solution to approach the equilibrium solution $N_1 = 0$, $N_2 = k_2$.This is illustrated in Fig.4.

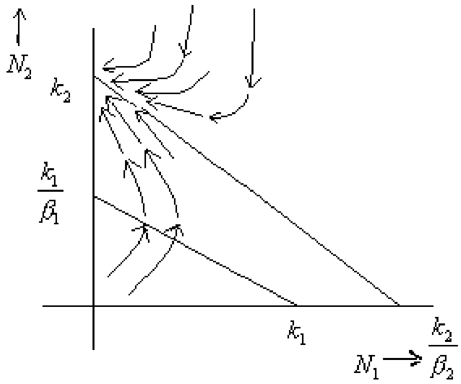

Fig.4

Theorem 3: *Principle of Competitive Exclusion for Equilibrium State IV:*

$$\overline{N}_1 = \frac{(a_1 - q_1E)\alpha_{22} - a_2\alpha_{12}}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}};$$

$$\overline{N}_2 = \frac{a_2\alpha_{11} + (a_1 - q_1E)\alpha_{21}}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}$$

When $\frac{k_1}{\beta_1} > k_2$ and $\frac{k_2}{\beta_2} > k_1$, then every solution of $N_1(t), N_2(t)$ of (4.1) approaches the equilibrium solution $N_1(t) = \overline{N}_1(\neq 0)$ and $N_2(t) = \overline{N}_2(\neq 0)$ as t approaches infinity. In other words, if species 1 and 2 are nearly identical and the microcosm can support both the members of species 1 and 2 depending up on the initial conditions.

Proof: The first step in our proof is to show that $N_1(t)$ and $N_2(t)$ can never become negative. To this end, observe that

$$N_1(t) = \overline{N}_1 = \frac{(a_1 - q_1E)\alpha_{22} - a_2\alpha_{12}}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}} \text{ and}$$

$$N_2(t) = \overline{N}_2 = \frac{a_2\alpha_{11} + (a_1 - q_1E)\alpha_{21}}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}$$

is a solution of (4.1) for any choice of $N_1(0)$. The orbit of this solution in the $N_1 - N_2$ plane is the point (0, 0) for $N_1(0) = 0$; the line $0 < N_1 < k_1$, $N_2 = 0$ for $0 < N_1(0) < k_1$; the point $(k_1, 0)$ for $N_1(0) = k_1$; and the line $k_1 < N_1 < \infty$, $N_2 = 0$ for $N_1(0) > k_1$. Thus the N_1 axis, for $N_1 \geq 0$ is the union of four distinct orbits of (4.1). Similarly the N_2 axis, for $N_2 \geq 0$, is the union of four distinct orbits of (4.1). This implies that all solutions $N_1(t), N_2(t)$ of (4.1) which start in the first quadrant $\left(N_1(t) > 0, N_2 > 0\right)$ of the $N_1 - N_2$ plane must remain there for all future time.

The second step in our proof is to split the first quadrant into regions in which both $\frac{dN_1}{dt}$ and $\frac{dN_2}{dt}$ have fixed signs. This is accomplished in the following manner.

Let l_1 and l_2 be the lines $\left(k_1 - N_1 - \beta_1 N_2\right) = 0$ and $\left(k_2 - N_2 - \beta_2 N_1\right) = 0$ respectively and the point of their intersection, is $(\overline{N}_1, \overline{N}_2)$. Observe that $\frac{dN_1}{dt}$ is negative if (N_1, N_2) lies above the line l_1 and positive if (N_1, N_2) lies below l_1. Similarly, $\frac{dN_2}{dt}$ is negative if (N_1, N_2) lies above l_2 and positive if (N_1, N_2) lies below l_2. Thus the two lines l_1 and l_2 split the first quadrant of the $N_1 - N_2$ plane into four regions in which both $\frac{dN_1}{dt}$ and $\frac{dN_2}{dt}$ have fixed signs.

$N_1(t), N_2(t)$ both increase with time along any solution of (4.1) in region *I*;

$N_1(t)$ increases and $N_2(t)$ decreases with time in region *II*;

$N_1(t)$ decreases and $N_2(t)$ increases with time in region *III*

and both $N_1(t)$ and $N_2(t)$ decrease with time in region *IV*. In this region both the prey predator compete with each other but do not flourish and at the same time do not get extinct as shown in Fig.5.

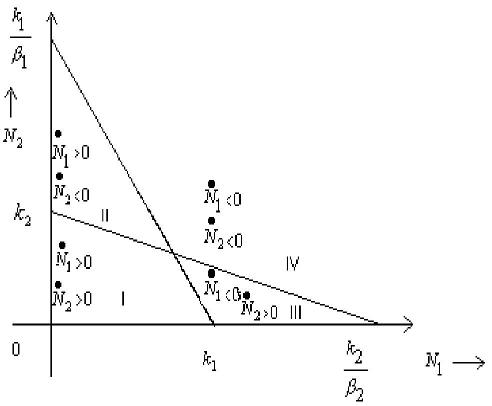

Fig.5

Finally we require the following four lemmas.

Lemma 7: *Any solution of* $N_1(t), N_2(t)$ *of (4.1) which starts in region I at time* $t = t_0$ *will remain in this region for all future time* $t \geq t_0$*, and ultimately approach the equilibrium solution* $N_1(t) = \bar{N}_1, N_2(t) = \bar{N}_2$ (Fig 5).

Proof: Suppose that a solution $N_1(t), N_2(t)$ of (4.1) leaves region *I* at time $t = t^*$. Then either $\frac{dN_1}{dt}(t^*)$ or $\frac{dN_2}{dt}(t^*)$ is zero, since the only way a solution of (4.1) can leave region *I* is by crossing l_1 or l_2. Assume that $\frac{dN_1}{dt}(t^*) = 0$. Differentiation both sides of the first equation of (4.1) with respect to *t* and setting $t = t^*$ gives

$$\frac{d^2N_1(t^*)}{dt} = \frac{-(a_1 - q_1E)\beta_1 N_1(t^*)}{k_1}\frac{dN_2(t^*)}{dt}$$

< 0

Hence $N_1(t)$ is monotonic increasing and it has maximum whenever a solution of $N_1(t), N_2(t)$ of (4.1) is in region *I*.

Similarly, if $\frac{dN_2}{dt}(t^*) = 0$, then $\frac{d^2N_2(t^*)}{dt} = \frac{-a_2\beta_2 N_2(t^*)}{k_2}\frac{dN_1}{dt}(t^*) < 0$ implies that $N_2(t)$ is monotonic increasing and it has maximum whenever a solution $N_1(t), N_2(t)$ of (4.1) is in region *I*.

If a solution $N_1(t), N_2(t)$ of (4.1) remains in region *I* for $t \geq t_0$, then both $N_1(t)$ and $N_2(t)$ are monotonic increasing functions of time for $t \geq t_0$, with $N_1(t) < k_1$ and $N_2(t) < k_2$, consequently, both $N_1(t)$ and $N_2(t)$ have limits ξ, η respectively, as t approaches infinity. This, in turn implies that (ξ, η) is an equilibrium point of (4.1). Now, (ξ, η) obviously cannot equal (0, 0); $(k_1, 0)$ or $(0, k_2)$. Consequently $(\xi, \eta) = (\bar{N}_1, \bar{N}_2)$.

Lemma 8: *Any solution of* $N_1(t), N_2(t)$ *of (4.1) which starts in region II at time* $t = t_0$ *will remain in this region for all future time* $t \geq t_0$*, and ultimately approach the equilibrium solution* $N_1(t) = \bar{N}_1, N_2(t) = \bar{N}_2$ (Fig 5).

Proof: Suppose that a solution $N_1(t), N_2(t)$ of (4.1) leaves region *II* at time $t = t^*$. Then either $\frac{dN_1(t^*)}{dt}$ or $\frac{dN_2(t^*)}{dt}$ is zero, since the only way a solution of (4.1) can leave region *II* is by crossing l_1 or l_2. Assume that $\frac{dN_1(t^*)}{dt}$ =0. Differentiation both sides of the first equation of (4.1) with respect to *t* and setting $t = t^*$ gives

$$\frac{d^2N_1(t^*)}{dt} = \frac{-(a_1 - q_1E)\beta_1 N_1(t^*)}{k_1}\frac{dN_2(t^*)}{dt}$$

This quantity is positive. Hence $N_1(t)$ has a minimum at $t = t^*$. However, this is impossible, since $N_1(t)$ is increasing whenever a solution of $N_1(t), N_2(t)$ of (4.1) is in region *II*.

Similarly, if $\frac{dN_2(t^*)}{dt}$ =0,

then $\frac{d^2N_2(t^*)}{dt} = \frac{-a_2\beta_2 N_2(t^*)}{k_2}\frac{dN_1}{dt}(t^*)$

This quantity is negative, implying that $N_2(t)$ has a maximum at $t = t^*$, but this is impossible, since $N_2(t)$ is decreasing whenever a solution $N_1(t), N_2(t)$ of (4.1) is in region *II*.

The previous argument shows that any solution $N_1(t), N_2(t)$ of (4.1) which starts in region *II* at time $t = t_0$ will remain in region *II* for all future time $t \geq t_0$. This implies that $N_1(t)$ is monotonic increasing and $N_2(t)$ is monotonic decreasing for $t \geq t_0$; with $N_1(t) < k_1$ and $N_2(t) > k_2$. Consequently, both $N_1(t)$ and $N_2(t)$ have limits ξ, η respectively, as *t* approaches infinity. This in turn, implies that (ξ, η) is an equilibrium point of (4.1). Now (ξ, η) obviously cannot equal (0, 0); $(0, k_1)$ or $(0, k_2)$.

Consequently, $(\xi,\eta)=(\overline{N}_1,\overline{N}_2)$ and this proves Lemma 8.

Lemma 9: *Any solution of* $N_1(t),N_2(t)$ *of (4.1) which starts in region III at time* $t=t_0$ *will remain in this region for all future time* $t\geq t_0$*, and ultimately approach the equilibrium solution* $N_1(t)=\overline{N}_1, N_2(t)=\overline{N}_2$ (Fig 5).

Proof: Suppose that a solution $N_1(t),N_2(t)$ of (4.1) leaves region *III* at time $t=t^*$. Then either $\frac{dN_1(t^*)}{dt}$ or $\frac{dN_2(t^*)}{dt}$ is zero, since the only way a solution of (4.1) can leave region *II* is by crossing l_1 or l_2. Assume that $\frac{dN_1(t^*)}{dt}$ =0. Differentiation both sides of the first equation of (4.1) with respect to *t* and setting $t=t^*$ gives

$$\frac{d^2N_1(t^*)}{dt}=\frac{-\left(a_1-q_1E\right)\beta_1N_1(t^*)}{k_1}\frac{dN_2(t^*)}{dt}$$

This quantity is negative. Hence $N_1(t)$ has a maximum at $t=t^*$. However, this is impossible, since $N_1(t)$ is decreasing whenever a solution of $N_1(t),N_2(t)$ of (4.1) is in region *II*.

Similarly, if $\frac{dN_2(t^*)}{dt}$ =0,

then $\frac{d^2N_2(t^*)}{dt}=\frac{-a_2\beta_2N_2(t^*)}{k_2}\frac{dN_1}{dt}(t^*)$

This quantity is positive, implying that $N_2(t)$ has a minimum at $t=t^*$, but this is impossible, since $N_2(t)$ is increasing whenever a solution $N_1(t),N_2(t)$ of (4.1) is in region *III*.

The previous argument shows that any solution $N_1(t),N_2(t)$ of (4.1) which starts in region *III* at time $t=t_0$ will remain in region *III* for all future time $t\geq t_0$. This implies that $N_1(t)$ is monotonic increasing and $N_2(t)$ is monotonic decreasing for $t\geq t_0$; with $N_1(t)>k_1$ and $N_2(t)<k_2$. Consequently, both $N_1(t)$ and $N_2(t)$ have limits ξ,η respectively, as t approaches infinity. This in turn, implies that (ξ,η) is an equilibrium point of (4.1). Now (ξ,η) obviously cannot equal (0, 0);($0,k_1$) or $(0,k_2)$. Consequently, $(\xi,\eta)=(\overline{N}_1,\overline{N}_2)$ and this proves Lemma 9.

Lemma 10: *Any solution of* $N_1(t),N_2(t)$ *of (4.1) which starts in region VI at time* $t=t_0$ *will remain in this region for all future time* $t\geq t_0$*, and ultimately approach the equilibrium solution* $N_1(t)=\overline{N}_1, N_2(t)=\overline{N}_2$ (Fig 5).

Proof: Suppose that a solution $N_1(t),N_2(t)$ of (4.1) leaves region *VI* at time $t=t^*$. Then either $\frac{dN_1(t^*)}{dt}$ or $\frac{dN_2(t^*)}{dt}$ is zero, since the only way a solution of (4.1) can leave region *I* is by crossing l_1 or l_2. Assume that $\frac{dN_1(t^*)}{dt}$ =0. Differentiation both sides of the first equation of (4.1) with respect to *t* and setting $t=t^*$ gives

$$\frac{d^2N_1(t^*)}{dt}=\frac{-\left(a_1-q_1E\right)\beta_1N_1(t^*)}{k_1}\frac{dN_2(t^*)}{dt}$$

This quantity is positive. Hence $N_1(t)$ is monotonic decreasing and it has minimum whenever a solution of $N_1(t),N_2(t)$ of (4.1) is in region *VI*.

Similarly, if $\frac{dN_2(t^*)}{dt}$ =0,

then $\frac{d^2N_2(t^*)}{dt}=\frac{-a_2\beta_2N_2(t^*)}{k_2}\frac{dN_1}{dt}(t^*)$.

This quantity is positive, implying that $N_2(t)$ is monotonic decreasing and it has minimum whenever a solution $N_1(t),N_2(t)$ of (4.1) is in region *VI*.

If a solution $N_1(t),N_2(t)$ of (4.1) remains in region *VI* for $t\geq t_0$, then both $N_1(t)$ and $N_2(t)$ are monotonic decreasing functions of time for $t\geq t_0$, with $N_1(t)>k_1$ and $N_2(t)>k_2$, consequently, both $N_1(t)$ and $N_2(t)$ have limits ξ,η respectively, as t approaches infinity. This, in turn implies that (ξ,η) is an equilibrium point of (4.1). Now, (ξ,η) obviously cannot equal (0, 0); $(k_1,0)$ or $(0,k_2)$. Consequently $(\xi,\eta)=(\overline{N}_1,\overline{N}_2)$.

Proof of Theorem: Lemmas 7,8,9and 10 state that every solution $N_1(t),N_2(t)$ of (4.1) which starts in region *I* ,*II III* or *VI* at time $t=t_0$ and remains there for all future time must also approach equilibrium solution $N_1(t)=\overline{N}_1$, $N_2(t)=\overline{N}_2$ as *t* approaches infinity. Next, observe that any solution $N_1(t), N_2(t)$ of (4.1) which starts on l_1 or l_2 must immediately afterwards enter regions *I*, *II*, *III* or *VI*. Finally the solution approaches the equilibrium solution $N_1(t) = \bar{N}_1, N_2(t) = \bar{N}_2$. This is illustrated in Fig. 6.

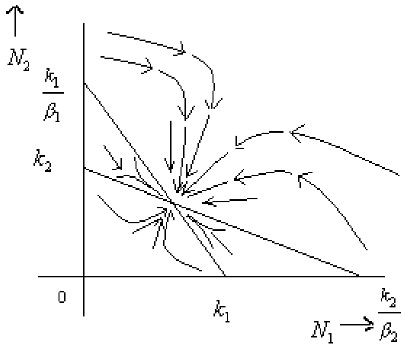

Fig.6

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