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Exact traveling wave solutions of the fifth-order KdV equation via the new approach of generalized (G'/G)-expansion method

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Abstract

Mathematical modeling of numerous physical phenomena often leads to high-dimensional partial differential equations and thus the higher dimensional nonlinear evolution equations come into further attractive in many branches of physical sciences. In this works, we construct the traveling wave solutions involving parameters of the fifth-order KdV equation by using the new approach of generalized (G'/G)-expansion method. Abundant traveling wave solutions with arbitrary parameters are successfully obtained by this method and the wave solutions are expressed in terms of the hyperbolic, trigonometric, and rational functions. It is shown that the new approach of generalized (G'/G)-expansion method is a powerful and concise mathematical tool for solving nonlinear partial differential equations.

Keywords: New generalized (G'/G) -expansion method, the fifth-order KdV equation, homogeneous balance, traveling wave solutions, nonlinear evolution equations.

Mathematics Subject Classification: 35C07, 35C08, 35P99.

1. Introduction

The importance of nonlinear evolution equations is now well established, because these equations arise in various areas of science and engineering, especially in fluid mechanics, solid-state physics, biophysics, chemical kinematics, geochemistry, electricity, propagation of shallow water waves, plasma physics, highenergy physics, condensed matter physics, quantum mechanics, optical fibers, elastic media and so on. As a key problem, finding their analytical solutions is of great interest, and is actually performed through various powerful and prolific methods such as the homogeneous balance method [1], the tanh-function method [2], the extended tanh-function method [3, 4], the Exp-function method [5, 6], the sine-cosine method [7], the modified Exp-function method [8], the generalized Riccati equation [9], the Jacobi elliptic function expansion method [10, 11], the Hirota's bilinear method [12], the Miura transformation [13], the (G'/G)-expansion method [14-18], the novel (G'/G)-expansion method [19, 20], the modified simple equation method [21, 22], the improved (G'/G)-expansion method [23], the inverse scattering transform [24], the Jacobi elliptic function expansion method [25, 26], the new generalized (G'/G)-expansion method [27-31], the Adomian decomposition method [32, 33], the method of bifurcation of planar dynamical systems [34, 35], the wave of translation method [36], the ansatz method [37, 38], the Cole-Hopf transformation [39] and so on.

The objective of this article is to apply the new generalized (G'/G) expansion method to construct the exact solutions for nonlinear evolution equations in mathematical physics via the renowned fifth-order KdV equation.

The outline of this paper is organized as follows: In Section 2, we give the description of the new generalized (G'/G) expansion method. In Section 3, we apply this method to the fifth-order KdV equation. In Section 4, Discussions are given. Conclusions are given in Section 5.

2. Description of the new generalized (G'/G)-expansion method

Let us consider a general nonlinear PDE in the form

$$P(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \cdots) = 0, \qquad (1)$$

where $u = u(x,t)$ is an unknown function, P is a polynomial in $u(x,t)$ and its derivatives in which highest order derivatives and nonlinear terms are involved and the subscripts stand for the partial derivatives.

Step 1: We combine the real variables x and t by a complex variable ξ,

$$u(x,t) = u(\xi), \quad \xi = x \pm V t, \qquad (2)$$

where V is the speed of the traveling wave. The traveling wave transformation (2) converts Eq. (1) into an ordinary differential equation (ODE) for $u = u(\xi)$:

$$Q(u, u', u'', u''', \cdots) = 0, \qquad (3)$$

where Q is a polynomial of u and it derivatives and the superscripts indicate the ordinary derivatives with respect to ξ.

Step 2: According to possibility, Eq. (3) can be integrated term by term one or more times, yields constant(s) of integration. The integral constant may be zero for simplicity.

Step 3: Suppose the traveling wave solution of Eq. (3) can be expressed as follows:

$$u(\xi) = \sum_{i=0}^{N} a_i (d + H)^i + \sum_{i=1}^{N} b_i (d + H)^{-i}, \qquad (4)$$

where either a_N or b_N may be zero, but both a_N and b_N could be zero at a time, a_i $(i = 0, 1, 2, \cdots, N)$ and b_i $(i = 1, 2, \cdots, N)$ and d are arbitrary constants to be determined later and $H(\xi)$ is given by

$$H(\xi) = (G'/G) \qquad (5)$$

where $G = G(\xi)$ satisfies the following auxiliary nonlinear ordinary differential equation:

$$AGG'' - BGG' - EG^2 - C\,(G')^2 = 0 \qquad (6)$$

where the prime stands for derivative with respect to ξ; A, B, C and E are real parameters.

Step 4: To determine the positive integer N, taking the homogeneous balance between the highest order nonlinear terms and the derivatives of the highest order appearing in Eq. (3).

Step 5: Substitute Eq. (4) and Eq. (6) including Eq. (5) into Eq. (3) with the value of N obtained in Step 4, we obtain polynomials in $(d + H)^N$ $(N = 0, 1, 2, \cdots)$ and $(d + H)^{-N}$ $(N = 0, 1, 2, \cdots)$. Then, we collect each coefficient of the resulted polynomials to zero yields a set of algebraic equations for a_i $(i = 0, 1, 2, \cdots, N)$ and b_i $(i = 1, 2, \cdots, N)$, d and V.

Step 6: Suppose that the value of the constants a_i $(i = 0, 1, 2, \cdots, N)$, b_i $(i = 1, 2, \cdots, N)$, d and V can be found by solving the algebraic equations obtained in Step 5. Since the general solution of Eq. (6) is well known to us, inserting the values of a_i $(i = 0,1,2,\cdots,N)$, b_i $(i = 1,2,\cdots,N)$, d and V into Eq. (4), we obtain more general type and new exact traveling wave solutions of the nonlinear partial differential equation (1).

Using the general solution of Eq. (6), we have the following solutions of Eq. (5):

Family 1: When $B \neq 0,\ \psi = A - C$ and $\Omega = B^2 + 4E(A-C) > 0,$

$$H(\xi) = \left(\frac{G'}{G}\right) = \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi} \frac{C_1 \sinh\left(\frac{\sqrt{\Omega}}{2A}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\Omega}}{2A}\xi\right)}{C_1 \cosh\left(\frac{\sqrt{\Omega}}{2A}\xi\right) + C_2 \sinh\left(\frac{\sqrt{\Omega}}{2A}\xi\right)} \tag{7}$$

Family 2: When $B \neq 0,\ \psi = A - C$ and $\Omega = B^2 + 4E(A-C) < 0,$

$$H(\xi) = \left(\frac{G'}{G}\right) = \frac{B}{2\psi} + \frac{\sqrt{-\Omega}}{2\psi} \frac{-C_1 \sin\left(\frac{\sqrt{-\Omega}}{2A}\xi\right) + C_2 \cos\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)}{C_1 \cos\left(\frac{\sqrt{-\Omega}}{2A}\xi\right) + C_2 \sin\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)} \tag{8}$$

Family 3: When $B \neq 0,\ \psi = A - C$ and $\Omega = B^2 + 4E(A-C) = 0,$

$$H(\xi) = \left(\frac{G'}{G}\right) = \frac{B}{2\psi} + \frac{C_2}{C_1 + C_2\xi} \tag{9}$$

Family 4: When $B = 0,\ \psi = A - C$ and $\Delta = \psi E > 0,$

$$H(\xi) = \left(\frac{G'}{G}\right) = \frac{\sqrt{\Delta}}{\psi} \frac{C_1 \sinh(\frac{\sqrt{\Delta}}{A}\xi) + C_2 \cosh(\frac{\sqrt{\Delta}}{A}\xi)}{C_1 \cosh(\frac{\sqrt{\Delta}}{A}\xi) + C_2 \sinh(\frac{\sqrt{\Delta}}{A}\xi)} \tag{10}$$

Family 5: When $B = 0,\ \psi = A - C$ and $\Delta = \psi E < 0,$

$$H(\xi) = \left(\frac{G'}{G}\right) = \frac{\sqrt{-\Delta}}{\psi} \frac{-C_1 \sin(\frac{\sqrt{-\Delta}}{A}\xi) + C_2 \cos(\frac{\sqrt{-\Delta}}{A}\xi)}{C_1 \cos(\frac{\sqrt{-\Delta}}{A}\xi) + C_2 \sin(\frac{\sqrt{-\Delta}}{A}\xi)} \tag{11}$$

3. Application of the method

In this section, we will employ the new generalized (G'/G)-expansion method to look for the exact solutions and then the solitary wave solutions to fifth-order KdV equation. Let us consider fifth-order KdV equation,

$$u_t + 30u^2 u_x + 20u_x u_{xx} + 10u u_{xxx} + u_{xxxxx} = 0, \qquad (12)$$

where $u = u(x,t)$ is a differentiable function. Eq. (12) is a special case (Lax case) of the standard fifth-order KdV equation and it can be written as:

$$u_t + 10\,(u^3)_x + 10\,(uu_{xx})_x + 5((u_x)^2)_x + u_{5x} = 0 \qquad (13)$$

Using the wave variable $\xi = x - Vt$, Eq. (13) is carried to an ODE. Then integrating it once, we obtain

$$K - Vu + 10\,u^3 + 10\,u\,u'' + 5(u')^2 + u^{(4)} = 0, \qquad (14)$$

where K is an integral constant which is to be determined.

Taking homogeneous balance between $u^{(4)}$ and $u\,u''$ in Eq. (14), we obtain $N = 2$. Therefore, the solution of Eq. (15) is of the form

$$u(\xi) = a_0 + a_1(d+H) + a_2(d+H)^2 + b_1(d+H)^{-1} + b_2(d+H)^{-2}, \qquad (15)$$

where a_0, a_1, a_2, b_1, b_2 and d are constants to be determined.

Substituting Eq. (15) together with Eqs. (5) and (6) into Eq. (14), the left-hand side is converted into polynomials in $(d+H)^N$ $(N = 0,1,2,\cdots)$ and $(d+H)^{-N}$ $(N = 1,\ 2,\cdots)$. We collect each coefficient of these resulted polynomials to zero, yields a set of simultaneous algebraic equations (for simplicity which are not presented here) for a_0, a_1, a_2, b_1, b_2, d, K and V. Solving these algebraic equations with the help of symbolic computation software, such as, Maple, we obtain following:

Case 1:

$$K = -\frac{1}{2A^6}(64A^3E^3 + B^6 + 192\,AE^3C^2 + 12EB^4A - 12EB^4C + 48E^2B^2C^2 - 192\,A^2E^3C$$
$$+ 48A^2E^2B^2 - 64E^3C^3 - 96E^2B^2CA),$$

$$V = \frac{7}{2A^4}(B^4 + 8B^2E\psi + 16E^2\psi^2),\ \ d = d,\ \ b_1 = 0,\ \ b_2 = 0,\ \ a_2 = -\frac{6\psi^2}{A^2}, \qquad (16)$$

$$a_1 = \frac{6}{A^2}(B\psi + 2d\psi^2),\ a_0 = -\frac{1}{2A^2}(12d^2\psi^2 + 12BdA - 8E\psi + B^2).$$

where $\psi = A - C,\ \ a_0 = a_0,\ A,\ B,\ C$ and E are free parameters.

Case 2:

$$K = -\frac{1}{2A^6}(64A^3E^3 + B^6 + 192AE^3C^2 + 12EB^4A - 12EB^4C + 48E^2B^2C^2 - 192A^2E^3C$$
$$+ 48A^2E^2B^2 - 64E^3C^3 - 96E^2B^2CA),$$

$$V = \frac{7}{2A^4}(B^4 + 8B^2E\psi + 16E^2\psi^2),\ \ d = d,\ \ a_1 = 0,\ \ a_2 = 0,$$

$$b_2 = -\frac{6}{A^2}(d^4\psi^2 + E^2 + 2d^3B\psi + B^2d^2 - 2EBd - 2Ed^2\psi), \qquad (17)$$

$$b_1 = \frac{6}{A^2}(3d^2B\psi - 2Ed\psi + 2d^3\psi^2 + B^2d - EB),\ a_0 = -\frac{1}{2A^2}(12d^2\psi^2 + 12BdA - 8E\psi + B^2).$$

where $\psi = A - C,\ a_0 = a_0,\ A,\ B,\ C$ and E are free parameters.

Case 3:

$$K = -\frac{11}{2A^6}(64A^3E^3 + B^6 + 192AE^3C^2 + 12EB^4A - 12EB^4C + 48E^2B^2C^2 - 192A^2E^3C$$
$$+ 48A^2E^2B^2 - 64E^3C^3 - 96E^2B^2CA),$$

$$V = \frac{21}{A^4}(B^4 + 8B^2E\psi + 16E^2\psi^2),\ \ d = -\frac{B}{2\psi},\ \ a_1 = 0,\ \ a_2 = -\frac{6\psi^2}{A^2},\ \ b_1 = 0, \qquad (18)$$

$$b_2 = -\frac{1}{8A^2\psi^2}(B^4 + 8B^2E\psi + 16E^2\psi^2),\ a_0 = \frac{4E\psi + B^2}{A^2}.$$

where $\psi = A - C,\ \ a_0 = a_0,\ \ A,\ \ B,\ \ C$ and E are free parameters.

Case 4:

$$K = -\frac{32}{A^6}(64A^3E^3 + B^6 + 192AE^3C^2 + 12EB^4A - 12EB^4C + 48E^2B^2C^2 - 192A^2E^3C$$
$$+ 48A^2E^2B^2 - 64E^3C^3 - 96E^2B^2CA),$$

$$V=\frac{56}{A^4}(B^4+8B^2E\psi+16E^2\psi^2),\ d=-\frac{B}{2\psi},\ a_1=0,\ a_2=-\frac{6\psi^2}{A^2},\ b_1=0, \tag{19}$$

$$b_2=-\frac{3}{8A^2\psi^2}(B^4+8B^2E\psi+16E^2\psi^2),\ a_0=\frac{4E\psi+B^2}{A^2}.$$

Where $\psi=A-C,\ a_0=a_0,\ A,\ B,\ C$ and E are free parameters.

Case 5:

$$K=-\frac{11}{2A^6}(64A^3E^3+B^6+192AE^3C^2+12EB^4A-12EB^4C+48E^2B^2C^2-192A^2E^3C$$
$$+48A^2E^2B^2-64E^3C^3-96E^2B^2CA),$$

$$V=\frac{21}{A^4}(B^4+8B^2E\psi+16E^2\psi^2),\ d=-\frac{B}{2\psi},\ a_1=0,\ a_2=-\frac{6\psi^2}{A^2},\ b_1=0, \tag{20}$$

$$b_2=-\frac{3}{8A^2\psi^2}(B^4+8B^2E\psi+16E^2\psi^2),\ a_0=\frac{4E\psi+B^2}{A^2}.$$

Where $\psi=A-C,\ a_0=a_0,\ A,\ B,\ C$ and E are free parameters.

Case 6:

$$K=\frac{4}{A^6}(5{a_0}^3A^6-20{a_0}^2A^5E-16a_0E^2A^4+20{a_0}^2A^4CE-5{a_0}^2A^4B^2+32a_0E^2A^3C-8a_0$$
$$EB^2A^3+64A^3E^3+48A^2E^2B^2+8a_0EB^2A^2C-192A^2E^3C-16a_0E^2A^2C^2-B^4a_0A^2+$$
$$192AE^3C^2-96E^2B^2CA+12EB^4A-12EB^4C-64E^3C^3+48E^2B^2C^2+B^6),$$

$$V=\frac{2}{A^4}(15{a_0}^2A^4-40a_0EA^3-10a_0B^2A^2+40a_0CEA^2+48E^2A^2+24B^2EA-96E^2CA$$
$$+3B^4+48C^2E^2-24B^2CE), \tag{21}$$

$$d=-\frac{B}{2\psi},\ a_1=0,\ a_2=-\frac{2\psi^2}{A^2},\ b_1=0,\ b_2=-\frac{1}{8A^2\psi^2}(B^4+8B^2E\psi+16E^2\psi^2),\ a_0=a_0.$$

Where $\psi=A-C,\ a_0=a_0,\ A,\ B,\ C$ and E are free parameters.

For case 1, substituting Eq. (16) into Eq. (15), along with Eq. (7) and simplifying, our traveling wave solutions become (if $C_1=0$ but $C_2\neq 0$; $C_2=0$ but $C_1\neq 0$) respectively:

$$u_{1_1}(\xi)=\frac{1}{2A^2}((2B^2-3\Omega\coth^2(\frac{\sqrt{\Omega}}{2A}\xi))+12Bd(\psi-d)+8E\psi),$$

$$u_{1_2}(\xi)=\frac{1}{2A^2}((2B^2-3\Omega\tanh^2(\frac{\sqrt{\Omega}}{2A}\xi))+12Bd(\psi-d)+8E\psi),$$

Substituting Eqs. (16) into Eq. (15), along with Eq. (8) and simplifying, yields exact solutions (if $C_1=0$ but $C_2\neq 0$; $C_2=0$ but $C_1\neq 0$) respectively:

$$u_{1_3}=\frac{1}{2A^2}((2B^2+3\Omega\cot^2(\frac{\sqrt{-\Omega}}{2A}\xi))+12Bd(\psi-d)+8E\psi),$$

$$u_{1_4}(\xi)=\frac{1}{2A^2}((2B^2+3\Omega\tan^2(\frac{\sqrt{-\Omega}}{2A}\xi))+12Bd(\psi-d)+8E\psi),$$

Substituting Eqs. (16) into Eq. (15), along with Eq. (9) and simplifying, our obtained solution becomes:

$$u_{1_5}(\xi)=\frac{1}{2A^2}((2B^2-3\Omega(\frac{C_2}{C_1+C_2\xi})^2+12Bd(\psi-d)+8E\psi),$$

Substituting Eq. (16) into Eq. (15), together with Eq. (10) and simplifying, yields following traveling wave solutions (if $C_1=0$ but $C_2\neq 0$; $C_2=0$ but $C_1\neq 0$) respectively:

$$u_{1_6}(\xi)=\frac{1}{2A^2}(12Bd(\psi-1)+8E\psi+B(12\sqrt{\Delta}\coth(\frac{\sqrt{\Delta}}{A}\xi)-B))-12\Delta\coth^2(\frac{\sqrt{\Delta}}{A}\xi)),$$

$$u_{1_7}(\xi)=\frac{1}{2A^2}(12Bd(\psi-1)+8E\psi+B(12\sqrt{\Delta}\tanh(\frac{\sqrt{\Delta}}{A}\xi)-B))-12\Delta\tanh^2(\frac{\sqrt{\Delta}}{A}\xi)),$$

Substituting Eqs. (16) into Eq. (15), along with Eq. (11) and simplifying, our exact solutions become (if $C_1=0$ but $C_2\neq 0$; $C_2=0$ but $C_1\neq 0$) respectively:

$$u_{1_8}(\xi)=\frac{1}{2A^2}(12Bd(\psi-1)+8E\psi+B(12i\sqrt{\Delta}\cot(\frac{\sqrt{-\Delta}}{A}\xi)-B))+12\Delta\cot^2(\frac{\sqrt{-\Delta}}{A}\xi)),$$

$$u_{1_9}(\xi)=\frac{1}{2A^2}(12Bd(\psi-1)+8E\psi+B(12i\sqrt{\Delta}\tan(\frac{\sqrt{-\Delta}}{A}\xi)-B))-12\Delta\tan^2(\frac{\sqrt{-\Delta}}{A}\xi)),$$

where $\xi=x-(\frac{7}{2A^4}(B^4+8B^2E\psi+16E^2\psi^2))t.$

Similarly, For case 2, substituting Eq. (17) into Eq. (15), together with Eq. (7) and simplifying, yields following traveling wave solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{2_1}(\xi) = a_0 + b_1(d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\coth(\frac{\sqrt{\Omega}}{2A}\xi))^{-1} + b_2(d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\coth(\frac{\sqrt{\Omega}}{2A}\xi))^{-2},$$

$$u_{2_2}(\xi) = a_0 + b_1(d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\tanh(\frac{\sqrt{\Omega}}{2A}\xi))^{-1} + b_2(d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\tanh(\frac{\sqrt{\Omega}}{2A}\xi))^{-2},$$

Substituting Eqs. (17) into Eq. (15), along with Eq. (8) and simplifying, we obtain following solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{2_3}(\xi) = a_0 + b_1(d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\cot(\frac{\sqrt{\Omega}}{2A}\xi))^{-1} + b_2(d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\cot(\frac{\sqrt{\Omega}}{2A}\xi))^{-2},$$

$$u_{2_4}(\xi) = a_0 + b_1(d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\tan(\frac{\sqrt{\Omega}}{2A}\xi))^{-1} + b_2(d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\tan(\frac{\sqrt{\Omega}}{2A}\xi))^{-2},$$

Substituting Eqs. (17) into Eq. (15), along with Eq. (9) and simplifying, our obtained solution becomes:

$$u_{2_5}(\xi) = a_0 + b_1(d + \frac{B}{2\psi} + \frac{C_2}{C_1 + C_2\xi})^{-1} + b_2(d + \frac{B}{2\psi} + \frac{C_2}{C_1 + C_2\xi})^{-2},$$

Substituting Eq. (17) into Eq. (15), along with Eq. (10) and simplifying, yields following exact traveling wave solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{2_6}(\xi) = a_0 + b_1(d + \frac{\sqrt{\Delta}}{\psi}\coth(\frac{\sqrt{\Delta}}{A}\xi))^{-1} + b_2(d + \frac{\sqrt{\Delta}}{\psi}\coth(\frac{\sqrt{\Delta}}{A}\xi))^{-2},$$

$$u_{2_7}(\xi) = a_0 + b_1(d + \frac{\sqrt{\Delta}}{\psi}\tanh(\frac{\sqrt{\Delta}}{A}\xi))^{-1} + b_2(d + \frac{\sqrt{\Delta}}{\psi}\tanh(\frac{\sqrt{\Delta}}{A}\xi))^{-2},$$

substituting Eqs. (17) into Eq. (15), along with Eq. (11) and simplifying, our obtained exact solutions become (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{2_8}(\xi) = a_0 + b_1(d + \frac{\sqrt{-\Delta}}{\psi}\cot(\frac{\sqrt{-\Delta}}{A}\xi))^{-1} + b_2(d + \frac{\sqrt{-\Delta}}{\psi}\cot(\frac{\sqrt{-\Delta}}{A}\xi))^{-2},$$

$$u_{2_9}(\xi) = a_0 + b_1 (d - \frac{\sqrt{-\Delta}}{\psi}\tan(\frac{\sqrt{-\Delta}}{A}\xi))^{-1} + b_2 (d - \frac{\sqrt{-\Delta}}{\psi}\tan(\frac{\sqrt{-\Delta}}{A}\xi))^{-2},$$

where $\xi = x - (\frac{7}{2A^4}(B^4 + 8B^2 E\psi + 16E^2\psi^2))t.$

Similarly, For case 3, substituting Eq. (18) into Eq. (15), together with Eq. (7) and simplifying, yields following traveling wave solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{3_1}(\xi) = a_0 - \frac{3\Omega}{2A^2}\coth^2(\frac{\sqrt{\Omega}}{2A}\xi) + \frac{4b_2\psi^2}{\Omega}\tanh^2(\frac{\sqrt{\Omega}}{2A}\xi),$$

$$u_{3_2}(\xi) = a_0 - \frac{3\Omega}{2A^2}\tanh^2(\frac{\sqrt{\Omega}}{2A}\xi) + \frac{4b_2\psi^2}{\Omega}\coth^2(\frac{\sqrt{\Omega}}{2A}\xi),$$

Substituting Eqs. (18) into Eq. (15), along with Eq. (8) and simplifying, we obtain following solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{3_3}(\xi) = a_0 + \frac{3\Omega}{2A^2}\cot^2(\frac{\sqrt{-\Omega}}{2A}\xi) - \frac{4b_2\psi^2}{\Omega}\tan^2(\frac{\sqrt{\Omega}}{2A}\xi),$$

$$u_{3_4}(\xi) = a_0 + \frac{3\Omega}{2A^2}\tan^2(\frac{\sqrt{-\Omega}}{2A}\xi) - \frac{4b_2\psi^2}{\Omega}\cot^2(\frac{\sqrt{-\Omega}}{2A}\xi),$$

Substituting Eqs. (18) into Eq. (15), along with Eq. (9) and simplifying, our obtained solution becomes:

$$u_{3_5}(\xi) = a_0 - \frac{6\psi^2}{A^2}(\frac{C_2}{C_1 + C_2\xi})^2 + b_2(\frac{C_2}{C_1 + C_2\xi})^{-2},$$

Substituting Eq. (18) into Eq. (15), along with Eq. (10) and simplifying, yields following exact traveling wave solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{3_6}(\xi) = a_0 - \frac{6\psi^2}{A^2}(\frac{-B}{2\psi} + \frac{\sqrt{\Delta}}{\psi}\coth(\frac{\sqrt{\Delta}}{A}\xi))^2 + b_2(\frac{-B}{2\psi} + \frac{\sqrt{\Delta}}{\psi}\coth(\frac{\sqrt{\Delta}}{A}\xi))^{-2},$$

$$u_{3_7}(\xi) = a_0 - \frac{6\psi^2}{A^2}(\frac{-B}{2\psi} + \frac{\sqrt{\Delta}}{\psi}\tanh(\frac{\sqrt{\Delta}}{A}\xi))^2 + b_2(\frac{-B}{2\psi} + \frac{\sqrt{\Delta}}{\psi}\tanh(\frac{\sqrt{\Delta}}{A}\xi))^{-2},$$

Substituting Eqs. (18) into Eq. (15), along with Eq. (11) and simplifying, our obtained exact solutions become (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{3_8}(\xi) = a_0 - \frac{6\psi^2}{A^2}(\frac{-B}{2\psi} + \frac{\sqrt{-\Delta}}{\psi}\cot(\frac{\sqrt{-\Delta}}{A}\xi))^2 + b_2(\frac{-B}{2\psi} + \frac{\sqrt{-\Delta}}{\psi}\cot(\frac{\sqrt{-\Delta}}{A}\xi))^{-2},$$

$$u_{3_9}(\xi) = a_0 - \frac{6\psi^2}{A^2}(\frac{-B}{2\psi} - \frac{\sqrt{-\Delta}}{\psi}\tan(\frac{\sqrt{-\Delta}}{A}\xi))^2 + b_2(\frac{-B}{2\psi} - \frac{\sqrt{-\Delta}}{\psi}\tan(\frac{\sqrt{-\Delta}}{A}\xi))^{-2},$$

where $\xi = x - (\frac{21}{A^4}(B^4 + 8B^2E\psi + 16E^2\psi^2))t.$

Similarly, For case 4, substituting Eq. (19) into Eq. (15), together with Eq. (7) and simplifying, yields following traveling wave solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{4_1}(\xi) = a_0 - \frac{3\Omega}{2A^2}\coth^2(\frac{\sqrt{\Omega}}{2A}\xi) + \frac{4b_2\psi^2}{\Omega}\tanh^2(\frac{\sqrt{\Omega}}{2A}\xi),$$

$$u_{4_2}(\xi) = a_0 - \frac{3\Omega}{2A^2}\tanh^2(\frac{\sqrt{\Omega}}{2A}\xi) + \frac{4b_2\psi^2}{\Omega}\coth^2(\frac{\sqrt{\Omega}}{2A}\xi),$$

Substituting Eqs. (19) into Eq. (15), along with Eq. (8) and simplifying, we obtain following solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{4_3}(\xi) = a_0 + \frac{3\Omega}{2A^2}\cot^2(\frac{\sqrt{-\Omega}}{2A}\xi) - \frac{4b_2\psi^2}{\Omega}\tan^2(\frac{\sqrt{\Omega}}{2A}\xi),$$

$$u_{4_4}(\xi) = a_0 + \frac{3\Omega}{2A^2}\tan^2(\frac{\sqrt{-\Omega}}{2A}\xi) - \frac{4b_2\psi^2}{\Omega}\cot^2(\frac{\sqrt{-\Omega}}{2A}\xi),$$

Substituting Eqs. (19) into Eq. (15), along with Eq. (9) and simplifying, our obtained solution becomes:

$$u_{4_5}(\xi) = a_0 - \frac{6\psi^2}{A^2}(\frac{C_2}{C_1 + C_2\xi})^2 + b_2(\frac{C_2}{C_1 + C_2\xi})^{-2},$$

Substituting Eq. (19) into Eq. (15), along with Eq. (10) and simplifying, yields following exact traveling wave solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{4_6}(\xi)=a_0-\frac{6\psi^2}{A^2}(\frac{-B}{2\psi}+\frac{\sqrt{\Delta}}{\psi}\coth(\frac{\sqrt{\Delta}}{A}\xi))^2+b_2(\frac{-B}{2\psi}+\frac{\sqrt{\Delta}}{\psi}\coth(\frac{\sqrt{\Delta}}{A}\xi))^{-2},$$

$$u_{4_7}(\xi)=a_0-\frac{6\psi^2}{A^2}(\frac{-B}{2\psi}+\frac{\sqrt{\Delta}}{\psi}\tanh(\frac{\sqrt{\Delta}}{A}\xi))^2+b_2(\frac{-B}{2\psi}+\frac{\sqrt{\Delta}}{\psi}\tanh(\frac{\sqrt{\Delta}}{A}\xi))^{-2},$$

Substituting Eqs. (19) into Eq. (15), along with Eq. (11) and simplifying, our obtained exact solutions become (if $C_1=0$ but $C_2\neq 0$; $C_2=0$ but $C_1\neq 0$) respectively:

$$u_{4_8}(\xi)=a_0-\frac{6\psi^2}{A^2}(\frac{-B}{2\psi}+\frac{\sqrt{-\Delta}}{\psi}\cot(\frac{\sqrt{-\Delta}}{A}\xi))^2+b_2(\frac{-B}{2\psi}+\frac{\sqrt{-\Delta}}{\psi}\cot(\frac{\sqrt{-\Delta}}{A}\xi))^{-2},$$

$$u_{4_9}(\xi)=a_0-\frac{6\psi^2}{A^2}(\frac{-B}{2\psi}-\frac{\sqrt{-\Delta}}{\psi}\tan(\frac{\sqrt{-\Delta}}{A}\xi))^2+b_2(\frac{-B}{2\psi}-\frac{\sqrt{-\Delta}}{\psi}\tan(\frac{\sqrt{-\Delta}}{A}\xi))^{-2},$$

where $\xi=x-(\frac{56}{A^4}(B^4+8B^2E\psi+16E^2\psi^2))t.$

Similarly, For case 5, substituting Eq. (20) into Eq. (15), together with Eq. (7) and simplifying, yields following traveling wave solutions (if $C_1=0$ but $C_2\neq 0$; $C_2=0$ but $C_1\neq 0$) respectively:

$$u_{5_1}(\xi)=a_0-\frac{3\Omega}{2A^2}\coth^2(\frac{\sqrt{\Omega}}{2A}\xi)+\frac{4b_2\psi^2}{\Omega}\tanh^2(\frac{\sqrt{\Omega}}{2A}\xi),$$

$$u_{5_2}(\xi)=a_0-\frac{3\Omega}{2A^2}\tanh^2(\frac{\sqrt{\Omega}}{2A}\xi)+\frac{4b_2\psi^2}{\Omega}\coth^2(\frac{\sqrt{\Omega}}{2A}\xi),$$

Substituting Eqs. (20) into Eq. (15), along with Eq. (8) and simplifying, we obtain following solutions (if $C_1=0$ but $C_2\neq 0$; $C_2=0$ but $C_1\neq 0$) respectively:

$$u_{5_3}(\xi)=a_0+\frac{3\Omega}{2A^2}\cot^2(\frac{\sqrt{-\Omega}}{2A}\xi)-\frac{4b_2\psi^2}{\Omega}\tan^2(\frac{\sqrt{\Omega}}{2A}\xi),$$

$$u_{5_4}(\xi)=a_0+\frac{3\Omega}{2A^2}\tan^2(\frac{\sqrt{-\Omega}}{2A}\xi)-\frac{4b_2\psi^2}{\Omega}\cot^2(\frac{\sqrt{-\Omega}}{2A}\xi),$$

Substituting Eqs. (20) into Eq. (15), along with Eq. (9) and simplifying, our obtained solution becomes:

$$u_{5_5}(\xi)=a_0-\frac{6\psi^2}{A^2}(\frac{C_2}{C_1+C_2\xi})^2+b_2(\frac{C_2}{C_1+C_2\xi})^{-2},$$

Substituting Eq. (20) into Eq. (15), along with Eq. (10) and simplifying, yields following exact traveling wave solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{5_6}(\xi) = a_0 - \frac{6\psi^2}{A^2}\left(\frac{-B}{2\psi} + \frac{\sqrt{\Delta}}{\psi}\coth\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right)^2 + b_2\left(\frac{-B}{2\psi} + \frac{\sqrt{\Delta}}{\psi}\coth\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right)^{-2},$$

$$u_{5_7}(\xi) = a_0 - \frac{6\psi^2}{A^2}\left(\frac{-B}{2\psi} + \frac{\sqrt{\Delta}}{\psi}\tanh\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right)^2 + b_2\left(\frac{-B}{2\psi} + \frac{\sqrt{\Delta}}{\psi}\tanh\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right)^{-2},$$

Substituting Eqs. (20) into Eq. (15), along with Eq. (11) and simplifying, our obtained exact solutions become (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{5_8}(\xi) = a_0 - \frac{6\psi^2}{A^2}\left(\frac{-B}{2\psi} + \frac{\sqrt{-\Delta}}{\psi}\cot\left(\frac{\sqrt{-\Delta}}{A}\xi\right)\right)^2 + b_2\left(\frac{-B}{2\psi} + \frac{\sqrt{-\Delta}}{\psi}\cot\left(\frac{\sqrt{-\Delta}}{A}\xi\right)\right)^{-2},$$

$$u_{5_9}(\xi) = a_0 - \frac{6\psi^2}{A^2}\left(\frac{-B}{2\psi} - \frac{\sqrt{-\Delta}}{\psi}\tan\left(\frac{\sqrt{-\Delta}}{A}\xi\right)\right)^2 + b_2\left(\frac{-B}{2\psi} - \frac{\sqrt{-\Delta}}{\psi}\tan\left(\frac{\sqrt{-\Delta}}{A}\xi\right)\right)^{-2},$$

where $\xi = x - \left(\frac{21}{A^4}(B^4 + 8B^2E\psi + 16E^2\psi^2)\right)t.$

Similarly, For case 6, substituting Eq. (21) into Eq. (15), together with Eq. (7) and simplifying, yields following traveling wave solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{6_1}(\xi) = a_0 - \frac{3\Omega}{2A^2}\coth^2\left(\frac{\sqrt{\Omega}}{2A}\xi\right) + \frac{4b_2\psi^2}{\Omega}\tanh^2\left(\frac{\sqrt{\Omega}}{2A}\xi\right),$$

$$u_{6_2}(\xi) = a_0 - \frac{3\Omega}{2A^2}\tanh^2\left(\frac{\sqrt{\Omega}}{2A}\xi\right) + \frac{4b_2\psi^2}{\Omega}\coth^2\left(\frac{\sqrt{\Omega}}{2A}\xi\right),$$

Substituting Eqs. (21) into Eq. (15), along with Eq. (8) and simplifying, we obtain following solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{6_3}(\xi) = a_0 + \frac{3\Omega}{2A^2}\cot^2\left(\frac{\sqrt{-\Omega}}{2A}\xi\right) - \frac{4b_2\psi^2}{\Omega}\tan^2\left(\frac{\sqrt{\Omega}}{2A}\xi\right),$$

$$u_{6_4}(\xi) = a_0 + \frac{3\Omega}{2A^2}\tan^2\left(\frac{\sqrt{-\Omega}}{2A}\xi\right) - \frac{4b_2\psi^2}{\Omega}\cot^2\left(\frac{\sqrt{-\Omega}}{2A}\xi\right),$$

Substituting Eqs. (21) into Eq. (15), along with Eq. (9) and simplifying, our obtained solution becomes:

$$u_{6_5}(\xi) = a_0 - \frac{6\psi^2}{A^2}(\frac{C_2}{C_1 + C_2\xi})^2 + b_2(\frac{C_2}{C_1 + C_2\xi})^{-2},$$

Substituting Eq. (21) into Eq. (15), along with Eq. (10) and simplifying, yields following exact traveling wave solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{6_6}(\xi) = a_0 - \frac{6\psi^2}{A^2}(\frac{-B}{2\psi} + \frac{\sqrt{\Delta}}{\psi}\coth(\frac{\sqrt{\Delta}}{A}\xi))^2 + b_2(\frac{-B}{2\psi} + \frac{\sqrt{\Delta}}{\psi}\coth(\frac{\sqrt{\Delta}}{A}\xi))^{-2},$$

$$u_{6_7}(\xi) = a_0 - \frac{6\psi^2}{A^2}(\frac{-B}{2\psi} + \frac{\sqrt{\Delta}}{\psi}\tanh(\frac{\sqrt{\Delta}}{A}\xi))^2 + b_2(\frac{-B}{2\psi} + \frac{\sqrt{\Delta}}{\psi}\tanh(\frac{\sqrt{\Delta}}{A}\xi))^{-2},$$

Substituting Eqs. (21) into Eq. (15), along with Eq. (11) and simplifying, our obtained exact solutions become (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{6_8}(\xi) = a_0 - \frac{6\psi^2}{A^2}(\frac{-B}{2\psi} + \frac{\sqrt{-\Delta}}{\psi}\cot(\frac{\sqrt{-\Delta}}{A}\xi))^2 + b_2(\frac{-B}{2\psi} + \frac{\sqrt{-\Delta}}{\psi}\cot(\frac{\sqrt{-\Delta}}{A}\xi))^{-2},$$

$$u_{6_9}(\xi) = a_0 - \frac{6\psi^2}{A^2}(\frac{-B}{2\psi} - \frac{\sqrt{-\Delta}}{\psi}\tan(\frac{\sqrt{-\Delta}}{A}\xi))^2 + b_2(\frac{-B}{2\psi} - \frac{\sqrt{-\Delta}}{\psi}\tan(\frac{\sqrt{-\Delta}}{A}\xi))^{-2},$$

where $\xi = x - (\frac{2}{A^4}(15a_0{}^2A^4 - 40a_0EA^3 - 10a_0B^2A^2 + 40a_0CEA^2 + 48E^2A^2 + 24B^2EA - 96E^2CA$
$+ 3B^4 + 48C^2E^2 - 24B^2CE))t.$

4. Discussions

The advantages and validity of the method over the basic (G'/G)-expansion method have been discussed in the following:

Advantages: The crucial advantage of the new approach against the basic (G'/G)-expansion method is that the method provides more general and large amount of new exact traveling wave solutions with several free parameters. The exact solutions have its great importance to expose the inner mechanism of the physical phenomena. Apart from the physical application, the close-form solutions of nonlinear evolution equations assist the numerical solvers to compare the accuracy of their results and help them in the stability analysis.

Comparison: In Ref. [40] Gao and Zhao used the linear ordinary differential equation as auxiliary equation and traveling wave solutions presented in the form $u(\xi)=\sum_{i=0}^{m} a_i (G'/G)^i$, where $a_m \neq 0$. It is noteworthy to point out that some of our solutions are coincided with already published results, if parameters taken particular values which authenticate our solutions. Moreover, in Ref. [40] Gao and Zhao investigated the well-established fifth-order KdV equation to obtain exact solutions via the basic (G'/G)-expansion method and achieved only four solutions (A.1)-(A.4) (see appendix). Moreover, in this article fifty four solutions of the well-known fifth-order KdV equation are constructed by applying the new approach of generalized (G'/G)-expansion method.

Conclusion: In this article, the new generalized (G'/G)-expansion method is used to find the exact traveling wave solutions of the fifth-order KdV equation. Abundant traveling wave solutions with arbitrary parameters are successfully obtained by this method which are expressed in terms of hyperbolic, trigonometric and rational functions. This study shows that the new generalized (G'/G)-expansion method is quite efficient and practically well suited to be used in finding exact solutions of NLEEs. Also, we observe that the new generalized (G'/G)-expansion method is straightforward and can be applied to many other nonlinear evolution equations.

Appendix: Gao and Zhao solutions [40]

Gao and Zhao [40] established exact solutions of the well-known the fifth-order KdV equation by using the basic (G'/G)-expansion method which are as follows:

When $\lambda^2-4\mu>0$,

$$u_1=-\frac{3}{2}(\lambda^2-4\mu)\left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi+C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi+C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi}\right)^2+\lambda^2-4\mu, \quad \text{(A. 1)}$$

where $\xi=x-(\frac{7}{2}\lambda^4-28\lambda^2\mu+56\mu^2)t$ and C_1, C_2 are arbitrary constants.

For $C_2\neq 0$, $C_1{}^2<C_2{}^2$ above, then the solutions (A. 1) turns into

$$u_1=-\frac{3}{2}(\lambda^2-4\mu)\tanh^2(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}+\xi_0)+\lambda^2-4\mu,\quad \xi_0=\tanh^{-1}(\frac{C_1}{C_2}),$$

$$u_2=-\frac{1}{2}(\lambda^2-4\mu)(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi})^2+a_0+\frac{\lambda^2}{2},\quad \text{(A. 2)}$$

where $\xi=x-(10a_0\lambda^2+12\lambda^2\mu+\lambda^4+80a_0\mu+30a_0{}^2+56\mu^2)t$ and C_1,C_2 are arbitrary constants.

For $C_2\neq 0,\ C_1{}^2<C_2{}^2$ above, then the solutions (A. 2) turns into

$$u_2=-\frac{1}{2}(\lambda^2-4\mu)\tanh^2(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}+\xi_0)+a_0+\frac{\lambda^2}{2},\quad \xi_0=\tanh^{-1}(\frac{C_1}{C_2}),$$

When $\lambda^2-4\mu<0$,

$$u_3=-\frac{3}{2}(4\mu-\lambda^2)(\frac{-C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi})^2+\lambda^2-4\mu,\quad \text{(A. 3)}$$

where $\xi=x-(\frac{7}{2}\lambda^4-28\lambda^2\mu+56\mu^2)t$ and C_1,C_2 are arbitrary constants.

For $C_2\neq 0,\ C_1{}^2<C_2{}^2$ above, then the solutions (A. 3) turns into

$$u_3=-\frac{3}{2}(4\mu-\lambda^2)\cot^2(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}+\xi_0)+\lambda^2-4\mu,\quad \xi_0=\tanh^{-1}(\frac{C_1}{C_2}),$$

$$u_4=-\frac{1}{2}(4\mu-\lambda^2)(\frac{-C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi})^2+a_0+\frac{\lambda^2}{2},\quad \text{(A. 4)}$$

where $\xi=x-(10a_0\lambda^2+12\lambda^2\mu+\lambda^4+80a_0\mu+30a_0{}^2+56\mu^2)t$ and C_1,C_2 are arbitrary constants.

For $C_2\neq 0,\ C_1{}^2<C_2{}^2$ above, then the solutions (A. 4) turns into

$$u_4=-\frac{1}{2}(4\mu-\lambda^2)\cot^2(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}+\xi_0)+a_0+\frac{\lambda^2}{2},\quad \xi_0=\tanh^{-1}(\frac{C_1}{C_2}),$$

When $\lambda^2 - 4\mu = 0$,

$$u_5 = \frac{-6C_2^{\ 2}}{(C_1 + C_2 x)^2},$$

where C_1, C_2 are arbitrary constants.

$$u_6 = \frac{-2C_2^{\ 2}}{(C_1 + C_2 \xi)^2} + a_0 + \frac{\lambda^2}{2},$$

where $\xi = x - (120a_0\mu + 30a_0^{\ 2} + 120\mu^2)t$ and C_1, C_2 are arbitrary constants.

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