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Controllability of a Class of Nonlinear Fractional Composite Dynamical Systems of Order $1<\alpha\leq 2$

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ABSTRACT

In this paper, controllability of nonlinear fractional composite dynamical systems of order in finite-dimensional spaces is investigated. Solution represents of linear and nonlinear fractional composite dynamical systems are defined. The method used in this paper is Mittag-Leffler matrix function and iterative technique. An example is provided to illustrate effectiveness of the main result.

KEYWORDS: Controllability, Fractional composite dynamical systems, Mittag-Leffler matrix function

Introduction

Recent years, fractional calculus has attracted attention of many authors, see, e.g. [1]-[7]. Fractional differential equations have been proved to be important tools in modeling of many scientific problems. Zhang et al.[1] investigated the stability of n-dimensional nonlinear fractional order systems with order of $1<\alpha<2$. Wang et al. [2] proposed the generalized Caputo fractional derivative, and sufficient conditions of stability were proved by comparison principles. Qin et al. [3] considered the approximate controllability and optimal controls of fractional systems of order $1<\alpha<2$ in Banach space. Gao et al. [4] presented observer-based fuzzy control for nonlinear fractional-order dynamical systems with order $1<\alpha<2$. K. Balachandran et al. [5] provide a computational procedure for state and control for the following nonlinear fractional systems of order $1<\alpha\leq 2$

$$\begin{cases} {}^{C}D^{\alpha}x(t)+A^2x(t)=Bu(t)+f\left(t,x(t),\int_0^t h(t,s,x(s))ds\right), & t\in J \\ x(0)=x_0, \ x'(0)=y_0 \end{cases} \quad (1.1)$$

K. Balachandran et al. [6] derived sufficient conditions for controllability of nonlinear fractional order
$1<\alpha<2$ in finite dimensional spaces

$$\begin{cases} {}^{C}D^{\alpha}x(t)+A^2x(t)=Bu(t)+f\left(t,x(t),u(t)\right), & t\in J \\ x(0)=x_0, \ x'(0)=y_0 \end{cases} \quad (1.2)$$

So the research of fractional calculus theory becomes a rapidly growing filed in mathematical theory and engineering applications. Since scientific problems are better characterized using a no-integer order model, and inspired by the above works, consider nonlinear fractional composite dynamical systems of order $1<\alpha\leq 2$ of the following type

$$\begin{cases} {}^{C}D^{\alpha}x(t)+A^2x'(t)=Bu(t)+f\left(t,x(t),{}^{C}D^{\beta}x(t),\int_0^t h(t,s,x(s))ds\right), & t\in J \\ x(0)=x_0, \ x'(0)=x_0' \end{cases} \quad (1.3)$$

Where state $x\in\mathbf{R}^n$, control $u\in\mathbf{R}^m$, $1<\alpha\leq 2$, $0<\beta<1$, A is a $n\times n$ matrix, and B is a $n\times m$ matrix, $f: J\times\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$ is continuous function. Solution represent of nonlinear fractional composite dynamical systems will be defined. And sufficient conditions of controllability of nonlinear fractional composite systems (1.3) in finite-dimensional spaces will be established.

1. Preliminaries

Let $\mathbf{R}^+ = [0,+\infty)$, $\mathbf{R}_+ = (0,+\infty)$, $\mathbf{R}^n$ be the n-dimensional Euclidean space.

Definition 2.1 Fractional order integral is defined as following

$$(I_{0+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds \quad (2.1)$$

where $n-1 < \alpha \le n$, $n \in \mathbf{N}$. Caputo fractional derivative

$$({}^C D_{0+}^{\alpha} f)(t) = (I_{0+}^{n-\alpha} D^n f)(t) \quad (2.2)$$

For brevity, Caputo fractional derivative ${}^C D_{0+}^{\alpha}$ is taken as ${}^C D^{\alpha}$.

Definition 2.2 The Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ z \in \mathbf{C}, \ \alpha, \beta > 0 \qquad (2.3)$$

$$E_{\alpha,1}(z) = E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \ z \in \mathbf{C}, \ \alpha, > 0 \qquad (2.4)$$

For an arbitrary $n \times n$ matrix A, the Mittag-Leffler matrix function is

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0 \qquad (2.5)$$

The Laplace transform of Caputo fractional derivative

$$L({}^C D^{\alpha} f(t))(s) = s^{\alpha} F(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+) s^{\alpha-1-k} \ (2.6)$$

If $0 < \alpha < 1$, $L({}^C D^{\alpha} f(t))(s) = s^{\alpha} F(s) - f(0^+) s^{\alpha-1}$. And for $1 < \alpha \le 2$, $L({}^C D^{\alpha} f(t))(s) = s^{\alpha} F(s) - f(0^+) s^{\alpha-1} - f'(0^+) s^{\alpha-2}$

Lemma 2.1 Consider the following linear fractional composite dynamical systems

$$\begin{cases} {}^C D^{\alpha} x(t) + A^2 x'(t) = Bu(t), \ t \in J \\ x(0) = x_0, \ x'(0) = x'_0 \end{cases} \quad (2.7)$$

with $x \in \mathbf{R}^n$, A is a $n \times n$ matrix and B is a $n \times m$ matrix, $u \in L^2(J, \mathbf{R}^m)$, It solution is given by the following integral equation

$$\begin{aligned} x(t) &= E_{\alpha-1,1}(-A^2 t^{\alpha-1}) x_0 + t^{\alpha-1} E_{\alpha-1,\alpha}(-A^2 t^{\alpha-1}) A^2 x_0 + t E_{\alpha-1,2}(-A^2 t^{\alpha-1}) x'_0 \\ &+ \int_0^t (t-s)^{\alpha-1} E_{\alpha-1,\alpha}(-A^2 (t-s)^{\alpha-1}) Bu(s) ds \end{aligned} \quad (2.8)$$

Proof: In order to establish the solution represent of system (2.7), taking Laplace transform on both sides, we obtain

$$s^{\alpha} X(s) - s^{\alpha-1} x_0 - s^{\alpha-2} x'_0 + A^2 s X(s) - A^2 x_0 = BU(s) \ (2.9)$$

Then

$$s^{\alpha} X(s) + A^2 s X(s) = BU(s) + s^{\alpha-1} x_0 + s^{\alpha-2} x'_0 + A^2 x_0 \ (2.10)$$

That is

$$(s^{\alpha} + A^2 s) X(s) = s^{\alpha-1} x_0 + A^2 x_0 + s^{\alpha-2} x'_0 + BU(s) \ (2.11)$$

So

$$X(s) = \frac{s^{\alpha-1}}{(s^{\alpha} + A^2 s)} x_0 + \frac{A^2}{(s^{\alpha} + A^2 s)} x_0 + \frac{s^{\alpha-2}}{(s^{\alpha} + A^2 s)} x'_0 + \frac{BU(s)}{(s^{\alpha} + A^2 s)} \ (2.12)$$

It is easy to get that

$$X(s) = \frac{s^{\alpha-2}}{(s^{\alpha-1} + A^2)} x_0 + \frac{s^{-1} A^2}{(s^{\alpha-1} + A^2)} x_0 + \frac{s^{\alpha-3}}{(s^{\alpha-1} + A^2)} x'_0 + \frac{s^{-1} BU(s)}{(s^{\alpha-1} + A^2)} \ (2.13)$$

Apply inverse Laplace transform on both sides of (2.13), one can get that the solution of (2.7) as

$$\begin{aligned} x(t) &= E_{\alpha-1,1}(-A^2 t^{\alpha-1}) x_0 + t^{\alpha-1} E_{\alpha-1,\alpha}(-A^2 t^{\alpha-1}) A^2 x_0 + t E_{\alpha-1,2}(-A^2 t^{\alpha-1}) x'_0 \\ &+ \int_0^t (t-s)^{\alpha-1} E_{\alpha-1,\alpha}(-A^2 (t-s)^{\alpha-1}) Bu(s) ds \end{aligned} \quad (2.14)$$

Definition 2.3 The system (1.3) or (2.7) is said to be controllable on J if, for each x_0, x'_0, $x_1 \in \mathbf{R}^n$, there exists a control function $u \in L^2(J, \mathbf{R}^m)$ such that the solution of (1.3) or (2.7) with $x(0) = x_0$ satisfies $x(T) = x_1$.

For brevity, define

$$n_1 = \max_{t \in J} \| (T-t)^{\alpha-1} B^* E_{\alpha-1,\alpha}((-A^2)^*(T-t)^{\alpha-1}) W^{-1} \|$$

$$n_2 = \max_{s \in J} \| (T-s)^{\alpha-1} E_{\alpha-1,\alpha}(-A^2 (T-s)^{\alpha-1}) \|$$

$$m_1 = \max_{t \in J} \| E_{\alpha-1,1}(-A^2 t^{\alpha-1}) + t^{\alpha-1} E_{\alpha-1,\alpha}(-A^2 t^{\alpha-1}) A^2 - I \| \| x_0 \|$$

$$m_2 = \max_{t \in J} \| t E_{\alpha-1,2}(-A^2 t^{\alpha-1}) \| \| x'_0 \|$$

$$N = \max_{s,t \in J} \| (t-s)^{\alpha-1} E_{\alpha-1,\alpha}(-A^2 (t-s)^{\alpha-1}) \|$$

$$p = \| B \|, \quad q = \| y_1 \|$$

$$d = \max_{t,s \in J} \| (t-s)^{\alpha-2} E_{\alpha-1,\alpha-1}\left(-A^2 (t-s)\right)^{\alpha-1} \|$$

Lemma 2.2 The linear system (2.7) is controllable on J iff the controllability Gramian matrix

$$W = \int_0^T (T-s)^{2\alpha-2} E_{\alpha-1,\alpha}(-A^2 (T-s)^{\alpha-1}) B B^* E_{\alpha-1,\alpha}((-A^2)^* (T-s)^{\alpha-1}) ds \qquad (2.15)$$

is nonsingular, where * denotes the transpose of matrix.

Proof: Defined the control function $u(t)$ as following

$$u(t) = (T-t)^{\alpha-1} B^* E_{\alpha-1,\alpha}((-A^2)^* (T-t)^{\alpha-1}) \times W^{-1} y_1 \quad (2.16)$$

where $y_1 = x_1 - E_{\alpha-1,1}(-A^2 T^{\alpha-1}) x_0 - T^{\alpha-1} E_{\alpha-1,\alpha}(-A^2 T^{\alpha-1}) A^2 x_0 - T E_{\alpha-1,2}(-A^2 T^{\alpha-1}) x'_0$. Then the solution of (2.7) at $t = T$, satisfies

$$\begin{aligned}
x(T) &= E_{\alpha-1,1}(-A^2 T^{\alpha-1}) x_0 + T^{\alpha-1} E_{\alpha-1,\alpha}(-A^2 T^{\alpha-1}) A^2 x_0 + T E_{\alpha-1,2}(-A^2 T^{\alpha-1}) x'_0 \\
&\quad + \int_0^T (T-s)^{\alpha-1} E_{\alpha-1,\alpha}(-A^2 (T-s)^{\alpha-1}) B u(s) ds \\
&= E_{\alpha-1,1}(-A^2 T^{\alpha-1}) x_0 + T^{\alpha-1} E_{\alpha-1,\alpha}(-A^2 T^{\alpha-1}) A^2 x_0 + T E_{\alpha-1,2}(-A^2 T^{\alpha-1}) x'_0 \\
&\quad + \int_0^T (T-s)^{2\alpha-2} E_{\alpha-1,\alpha}(-A^2 (T-s)^{\alpha-1}) B B^* E_{\alpha-1,\alpha}((-A^2)^* (T-t)^{\alpha-1}) W^{-1} y_1 ds \\
&= E_{\alpha-1,1}(-A^2 T^{\alpha-1}) x_0 + T^{\alpha-1} E_{\alpha-1,\alpha}(-A^2 T^{\alpha-1}) A^2 x_0 + T E_{\alpha-1,2}(-A^2 T^{\alpha-1}) x'_0 \\
&\quad + \int_0^T (T-s)^{2\alpha-2} E_{\alpha-1,\alpha}(-A^2 (T-s)^{\alpha-1}) B (T-t)^{\alpha-1} B^* E_{\alpha-1,\alpha}((-A^2)^* (T-t)^{\alpha-1}) W^{-1} y_1 ds \\
&= y_1 + E_{\alpha-1,1}(-A^2 T^{\alpha-1}) x_0 + T^{\alpha-1} E_{\alpha-1,\alpha}(-A^2 T^{\alpha-1}) A^2 x_0 + T E_{\alpha-1,2}(-A^2 T^{\alpha-1}) x'_0 = x_1
\end{aligned}$$

Hence the system (2.7) is controllable on J.

2. Main Results

Let $\mathbf{X} = \{x : x' \in C(J, \mathbf{R}^n)$ and ${}^C D^\beta x \in C(J, \mathbf{R}^n)\}$ with norm $\| x \|_X = \max\{\| x \|, \|{}^C D^\beta x \|\}$. Assume the following hypotheses:

(H1) For $t \in J$, $x \in \mathbf{R}^n$, there exists a positive constant $M > 0$ such that

$$\left\| f\left(t, x(t), {}^C D^\beta x(t), \int_0^t h(t,s,x(s)) ds\right) \right\| \le M \qquad (3.1)$$

(H2) For $t \in J$, $x, y, u, v, w, z \in \mathbf{R}^n$, there exist functions $\gamma_i \in C(J, \mathbf{R}^+)$ $(i = 1,2)$ such that

$$\| f(t,x,u,w) - f(t,y,v,z) \| \le \gamma_1(t) \| x - y \| + \gamma_2(t) \| w - z \| \quad (3.2)$$

(H3) For $t,s \in J$, $x,y \in \mathbf{R}^n$, there exists a function $\eta \in C(J,\mathbf{R}^+)$ such that

$$\| h(t,s,x) - h(t,s,y)\| \le \eta_1(t) \| x-y \| \qquad (3.3)$$

Theorem 3.1 Suppose that linear system (2.7) is controllable, and nonlinear term *f* satisfies (H1) ,(H2) and (H3). Then the nonlinear fractional composite dynamical system (1.3) is controllable on J.

Proof: Apply iterative technique to prove the controllability result. Define

$$x_0(t) = x_0$$

$$x_{n+1}(t) = E_{\alpha-1,1}(-A^2t^{\alpha-1})x_0 + t^{\alpha-1}E_{\alpha-1,\alpha}(-A^2t^{\alpha-1})A^2x_0 + tE_{\alpha-1,2}(-A^2t^{\alpha-1})x_0' + \int_0^t (t-s)^{\alpha-1}E_{\alpha-1,\alpha}(-A^2(t-s)^{\alpha-1})\left[Bu_n(s) + f\left(s, x_n(s), {}^CD^\beta x_n(s), \int_0^s h(s,r,x_n(r))dr\right)\right]ds \qquad (3.4)$$

where

$$u_n(t) = (T-t)^{\alpha-1}B^*E_{\alpha-1,\alpha}((-A^2)^*(T-t)^{\alpha-1}) \times W^{-1}\left(y_1 - \int_0^T (T-s)^{\alpha-1}E_{\alpha-1,\alpha}(-A^2(T-s)^{\alpha-1})f\left(s, x_n(s), {}^CD^\beta x_n(s), \int_0^s h(s,r,x_n(r))dr\right)ds\right) \qquad (3.5)$$

with $y_1 = x_1 - E_{\alpha-1,1}(-A^2T^{\alpha-1})x_0 - T^{\alpha-1}E_{\alpha-1,\alpha}(-A^2T^{\alpha-1})A^2x_0 - TE_{\alpha-1,2}(-A^2T^{\alpha-1})x_0'$.

Note that the sequence of the function $\{x_n(t)\}$ is known, we shall show that $\{x_n(t)\}$ is a Cauchy sequence in $\mathbf{X}$. Noting that $x_{n+1} = x_0 + \sum_{j=0}^{n}(x_{j+1} - x_j)$, we only need to prove that the series $\sum_{j=0}^{n}(x_{j+1} - x_j)$ converges uniformly with respect to $t \in J$. It is easy to see that

$$\begin{aligned} \| u_n(t) \| &\le \| (T-t)^{\alpha-1}B^*E_{\alpha-1,\alpha}((-A^2)^*(T-t)^{\alpha-1})W^{-1} \| \| y_1 \| + \| (T-t)^{\alpha-1}B^*E_{\alpha-1,\alpha}((-A^2)^*(T-t)^{\alpha-1})W^{-1} \| \\ &\quad \times \int_0^T \| (T-s)^{\alpha-1}E_{\alpha-1,\alpha}(-A^2(T-s)^{\alpha-1}) \| \| f\left(s, x_n(s), {}^CD^\beta x_n(s), \int_0^s h(s,r,x_n(r))dr\right) \| ds \\ &\le n_1q + n_1n_2MT \end{aligned}$$

and

$$\begin{aligned} &\| u_n(t) - u_{n-1}(t) \| \\ &\le \| (T-t)^{\alpha-1}B^*E_{\alpha-1,\alpha}((-A^2)^*(T-t)^{\alpha-1})W^{-1} \| \\ &\quad \times \int_0^T \| (T-s)^{\alpha-1}E_{\alpha-1,\alpha}(-A^2(T-s)^{\alpha-1}) \| \\ &\quad \times \| f\left(s, x_n(s), {}^CD^\beta x_n(s), \int_0^s h(s,r,x_n(r))dr\right) - f\left(s, x_{n-1}(s), {}^CD^\beta x_{n-1}(s), \int_0^s h(s,r,x_{n-1}(r))dr\right) \| ds \\ &\le n_1n_2T[\gamma_1(t) \| x_n(t) - x_{n-1}(t) \| + \gamma_2(t)\int_0^t \| h(t,r,x_n(r)) - h(t,r,x_{n-1}(r)) \| dr] \end{aligned}$$

From (H2), there exist constants $M_1, M_2, M_3 > 0$ such that

$$\| u_n(t) - u_{n-1}(t) \| \le n_1n_2T[M_1 \| x_n(t) - x_{n-1}(t) \| + M_2M_3T \| x_n(t) - x_{n-1}(t) \|]$$

Then

$$\begin{aligned} &\| x_{n+1}(t) - x_n(t) \| \\ &\le \int_0^t \| (t-s)^{\alpha-1}E_{\alpha-1,\alpha}(-A^2(t-s)^{\alpha-1}) \| [\| B \| \| u_n(s) - u_{n-1}(s) \| \\ &\quad + \| f\left(s, x_n(s), {}^CD^\beta x_n(s), \int_0^s h(s,r,x_n(r))dr\right) - f\left(s, x_{n-1}(s), {}^CD^\beta x_{n-1}(s), \int_0^s h(s,r,x_{n-1}(r))dr\right) \| ds] \\ &\le Np\int_0^t \| u_n(s) - u_{n-1}(s) \| ds \\ &\quad + N\int_0^t \| f\left(s, x_n(s), {}^CD^\beta x_n(s), \int_0^s h(s,r,x_n(r))dr\right) - f\left(s, x_{n-1}(s), {}^CD^\beta x_{n-1}(s), \int_0^s h(s,r,x_{n-1}(r))dr\right) \| ds \end{aligned}$$

$$\leq Np\int_0^t \| u_n(s)-u_{n-1}(s)\| ds + N\int_0^t \gamma_1(s)\| x_n(s)-x_{n-1}(s)\| ds$$

$$+N\int_0^t \gamma_2(s)\left\| \int_0^s h(s,r,x_n(r))dr - \int_0^s h(s,r,x_{n-1}(r))dr \right\| ds$$

$$\leq Np\int_0^t \| u_n(s)-u_{n-1}(s)\| ds + NM_1\int_0^t \| x_n(s)-x_{n-1}(s)\| ds + NM_2M_3T\int_0^t \| x_n(s)-x_{n-1}(s)\| ds$$

Further, we have

$$\| x_1(t)-x_0(t)\|$$

$$\leq \| E_{\alpha-1,1}(-A^2t^{\alpha-1})+t^{\alpha-1}E_{\alpha-1,\alpha}(-A^2t^{\alpha-1})A^2 - I \|\| x_0\| + \| tE_{\alpha-1,2}(-A^2t^{\alpha-1})\|\| x_0'\|$$

$$+\int_0^t \|(t-s)^{\alpha-1}E_{\alpha-1,\alpha}(-A^2(t-s)^{\alpha-1})\| [\| B\|\| u_0(s)\|$$

$$+\left\| f\left(s,x_1(s),{}^CD^\beta x_1(s),\int_0^s h(s,r,x_1(r))dr\right)\right\| ds]$$

$$\leq m_1+m_2+N[pn_1q+n_1n_2MT+M]T\leq LT, L>0$$

$$\| u_1(t)-u_0\|$$

$$\leq \|(T-t)^{\alpha-1}B^*E_{\alpha-1,\alpha}((-A^2)^*(T-t)^{\alpha-1})W^{-1}\|$$

$$\times\int_0^T \|(T-s)^{\alpha-1}E_{\alpha-1,\alpha}(-A^2(T-s)^{\alpha-1})\|$$

$$\times\left\| f\left(s,x_1(s),{}^CD^\beta x_1(s),\int_0^s h(s,r,x_1(r))dr\right) - f\left(s,x_0(s),{}^CD^\beta x_0(s),\int_0^s h(s,r,x_0(r))dr\right)\right\| ds$$

$$\leq 2n_1n_2MT\leq RT, R>0$$

After simple calculation， it is easy to obtain

$$\| x_{n+1}(t)-x_n(t)\| \leq NpL\frac{T^{n+1}}{(n+1)!}+NM_1R\frac{T^{n+1}}{(n+1)!}+NM_2M_3TR\frac{T^{n+1}}{(n+1)!}$$

and

$$\|{}^CD^\beta x_{n+1}(t) - {}^CD^\beta x(t)\|$$

$$\leq \left\| \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\left(\int_0^s (s-\tau)^{\alpha-2}E_{\alpha-1,\alpha-1}\left(-A^2(s-\tau)\right)^{\alpha-1}[B(u_n(\tau)-u(\tau))\right.\right.$$

$$\left.\left.+\left(f\left(\tau,x_n(\tau),{}^CD^\beta x_n(\tau),\int_0^\tau h(\tau,r,x_n(r))dr\right)-f\left(\tau,x(\tau),{}^CD^\beta x(\tau),\int_0^\tau h(\tau,r,x(r))dr\right)\right)\right\| d\tau ds$$

$$=\left\| \int_0^t (t-s)^{\alpha-2}E_{\alpha-1,\alpha-1}\left(-A^2(t-s)\right)^{\alpha-1}\times[B(u_n(s)-u(s))\right.$$

$$\left.+\left(f\left(s,x_n(s),{}^CD^\beta x_n(s),\int_0^s h(s,r,x_n(r))dr\right)-f\left(s,x(s),{}^CD^\beta x(s),\int_0^s h(s,r,x(r))dr\right)\right)ds]\right\|$$

$$\leq pd\int_0^t \| u_n(s)-u(s)\| ds + dM_1T\| x_n(t)-x(t)\| + dM_2M_3T^2\| x_n(t)-x(t)\|$$

Then it implies that $x_n(t)\to x(t)$, ${}^CD^\beta x_{n+1}(t)\to {}^CD^\beta x(t)$, $n\to\infty$, that is， $\{x_n(t)\}$ is a Cauchy sequence in X, and $\{x_n(t)\}$ converges uniformly to a continuous function $x(t)$ on J. Therefore, we have

$$x(t)=E_{\alpha-1,1}(-A^2t^{\alpha-1})x_0+t^{\alpha-1}E_{\alpha-1,\alpha}(-A^2t^{\alpha-1})A^2x_0+tE_{\alpha-1,2}(-A^2t^{\alpha-1})x_0'$$

$$+\int_0^t (t-s)^{\alpha-1}E_{\alpha-1,\alpha}(-A^2(t-s)^{\alpha-1})\left[Bu(s)+f\left(s,x(s),{}^CD^\beta x(s),\int_0^s h(s,r,x(r))dr\right)\right]ds$$

where

$$u(t)$$

$$=(T-t)^{\alpha-1}B^*E_{\alpha-1,\alpha}((-A^2)^*(T-t)^{\alpha-1})W^{-1}\left(y_1-\int_0^T (T-s)^{\alpha-1}f\left(s,x_n(s),{}^CD^\beta x_n(s),\int_0^s h(s,r,x_n(r))dr\right)ds\right)$$

By taking limit as $n \to \infty$ on both sides. Then $x(T) = x_1$ which means that the control function $u(t)$ steers the system from the initial state x_0 to x_1 in times T provided that the system (1.3) is controllable on J.

3. Example

Consider the following fractional composite dynamical system of order $1 < \alpha < 2$

$$\begin{cases} {}^C D^{\frac{3}{2}} x(t) + A^2 x'(t) = Bu(t) + f(t, x(t), {}^C D^{\frac{1}{2}} x(t)), \quad t \in J = [0,1] \\ x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ x'(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases} \tag{4.1}$$

where $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ and $x'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix}, A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$f = \begin{pmatrix} [\exp(-2t)(|x_1(t)| + |\int_0^t x_1(s)ds|)]/(1 + |{}^C D^\beta x_1(t)|) \\ [\exp(-2t)(|x_2(t)| + |\int_0^t x_2(s)ds|)]/(1 + |{}^C D^\beta x_2(t)|) \end{pmatrix}$$

After simple matrix calculation one can get

$$W = \int_0^1 (1-s) E_{1/2,3/2}(-A^2(1-s)^{1/2}) BB^* E_{1/2,3/2}\left((-A^2)^*(1-s)^{1/2}\right) ds = \begin{bmatrix} 0.1294 & 0 \\ 0 & 0.1294 \end{bmatrix}$$

is positive definite for any $T > 0$. Further, the nonlinear function f is continuous and satisfies the hypotheses of Theorem 3.1. Observe that the control defined by

$$u(t) = (1-t)^{1/2} B^* E_{1/2,3/2}((-A^2)^*(1-t)^{1/2}) W^{-1}$$
$$\times \left(y_1 - \int_0^1 (1-s)^{1/2} E_{1/2,3/2}(-A^2(1-s)^{1/2}) f\left(s, x(s), {}^C D^{1/2} x(s), \int_0^s h(s, r, x_n(r)) dr \right) ds \right)$$

steers the system (4.1) from x_0 to x_1. Hence the system (4.1) is controllable on $[0,1]$.

4. Acknowledgement

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