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Existence of Positive Solutions for Second-Order Nonlinear Impulsive singular questions with integral Boundary Conditions in Banach Spaces

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ABSTRACT

In this paper, we concern the existence of positive solutions for second-order nonlinear impulsive singular equations with integral boundary conditions in Banach spaces. By applying the Schauder fixed point theorem and constructing a closed set, a new result on the existence of positive solutions for a boundary value problem of second-order nonlinear impulsive singular integro-differential equations of mixed type in Banach space is obtained. An example is also given to demonstrate the application of the main result.

KEYWORDS: Impulsive singular second-order differential equation, Integral boundary conditions, Schauder fixed point theorem, Banach space

INTRODUCTION

The theory of nonlinear differential equations has become one of the most important areas of investigation in recent years. It is much richer than the corresponding theory of linear differential equations, because the structure of its emergence has deep physical background, see [1-8]. The theory of nonlinear impulsive differential equations describes processes which include a sudden change of their state at certain moments. Processes with such a character arise naturally and often, especially in the phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics.

On the other hand, boundary-value problems with integral boundary conditions constitute a very interesting and important class of problems. They arise in different areas, for example, heat condition, chemical engineering, underground water flow, thermo-elasticity, and plasma physics can be reduced to the nonlocal problems with integral boundary conditions, see [8]

In the last years, the theory of ordinary differential equations with singularities in abstract spaces has become an important new branch; see [3-7]. In a late work, by using the Schauder fixed point theorem, Zhang [**7**] obtained the existence of positive solution for second-order nonlinear impulsive singular differential equations of mixed type in Banach spaces

$$\begin{cases} x'' = f(t,x,x',Tx,Sx), \quad t \epsilon I_1', \\ \Delta x|_{t=t_k} = I_{0k}\big(x(t_k),x'(t_k)\big), \\ \Delta x'|_{t=t_k} = I_{1k}\big(x(t_k),x'(t_k)\big), \ k = 1,2,3,\cdots,p, \\ x(0) = x_0, x'(0) = x_1, \end{cases}$$

where $I = [0,1], I_1 = (0,1), 0 < t_1 < t_2 < \cdots t_p < a, \quad I_1' = I_1 \backslash \{t_1, \cdots, < t_p\}$, f may be singular at $t = 0, a$ and $x = \theta$ or $x' = \theta$, $I_{ik}(i = 0,1)$may be singular at $x = \theta$or $x' = \theta$, θdenotes the zero element of Banach space$E$.

More recently, Zhang and Feng[8] considered boundary value problem with integral boundary conditions for second-order nonlinear impulsive integro-differential equation of mixed type in real Banach space. However, as we know, up to now, few papers have considered the existence of positive solutions for second-order impulsive singular integro-differential equation of mixed type with integral boundary conditions in abstract spaces.

Inspired by the above work, in this paper, we shall use the Schauder fixed point theorem to investigate the existence of positive solutions for second-order nonlinear impulsive singular boundary value problem with integral boundary conditions in Banach spaces. Boundary conditions considered in this paper are also different from many known results; this is another reason why we study this problem.

Consider the following boundary value problem with integral boundary conditions for second-order nonlinear impulsive singular integro-differential equation of mixed type in Banach space

$$\begin{cases} u''(t)+f\big(t,u(t),(Tu)(t),(Su)(t)\big)=\theta,\ t\epsilon J'_{+},\\ \Delta u|_{t=t_k}=I_{0k}\big(u(t_k),u'(t_k)\big),\\ \Delta u'|_{t=t_k}=-I_{1k}\big(u(t_k),u'(t_k)\big),(k=1,2,3,\cdots,m),\\ u(0)=u^{*},u'(1)=\beta\int_0^1 q(s)\,u'(s)ds+v^{*}, \end{cases} \tag{1.1}$$

where $J=[0,1], J_{+}=(0,1), 0<t_1<t_2<\cdots t_m<1, J'_{+}=J_{+}\setminus\{t_1,\cdots,t_m\}, \beta\geq 0, f$ may be singular at $t=0, a$ and $x=\theta$ or $x'=\theta$, $I_{ik}(i=0,1)$ may be singular at $x=\theta$ or $x'=\theta$, θ denotes the zero element of Banach space E, $u^{*},v^{*}\epsilon E, q\epsilon L^{1}[0,1]$ is nonnegative. T and S are the linear operators defined as follows

$$(Tu)(t)=\int_0^t k(t,s)u(s)ds,\ (Su)(t)=\int_0^1 h(t,s)u(s)ds, \tag{1.2}$$

Where $k\in C[D,R_{+}]$, $h\in C[D_0,R_{+}], D=\{(t,s)\in J\times J: t\geq s\}$, $D_0=\{(t,s)\in J\times J: 0\leq t,s\leq 1\}, R_{+}=[0,+\infty)$, $R^{+}=(0,+\infty)$, $\Delta u|_{t=t_k}$ denotes the jump of $u(t)$ at $t=t_k$, i.e., $\Delta u|_{t=t_k}=u(t_k^{+})-u(t_k^{-})$, where $u(t_k^{+}), u(t_k^{-})$ represent the right and left limits of $t=t_k$ respectively.

1. Preliminaries and lemmas

In this section, we present some preliminaries and lemmas that are useful to the proof of our main results.

Let $PC[J,E]=\{u: u$ is a map from J into E such that $u(t)$ is continuous at $t\neq t_k$, left continuous at $t=t_k$, and $u(t_k^{+})$ exists, $k=1,2,3,\cdots,m$. Obviously $PC[J,E]$ is a Banach space with the norm $\|u\|_{PC}=\sup_{t\epsilon J}\|u(t)\|$, Let $PC^1[J,E]=\{u: u$ is a map from J into E such that $u(t)$ is continuously differentiable at $t\neq t_k$, left continuous at $t=t_k$, and $u(t_k^{+}) u'(t_k^{+}), u'(t_k^{-})$ exists$\}$. By virtue of the mean value theorem

$$u(t_k)-u(t_k-h)\in h\,\overline{co}\{u'(t): t_k-h<t<t_k\}(h>0).$$

It is easy to say that the left derivative $u'_{-}(t_k)$ exists and

$$u'_{-}(t_k)=\lim_{h\to 0^{+}} h^{-}[u(t_k)-u(t_k-h)]=u'(t_k^{-}).$$

In (1.1) and in the following, $u'(t_k)$ is understood as $u'_{-}(t_k)$, Evidently, $PC^1[J,E]$ is a Banach space with norm $\|u\|_{PC^1}=\max\{\|u\|_{PC},\|u'\|_{PC}\}$. Let P be a normal cone in E with normal constant N which defines a partial ordering in E by $x\leq y$. If $x\leq y$ and $x\neq y$, we write $x<y$.

Let $P_{+}=P\setminus\{\theta\}$. So, $u\epsilon P_{+}$ if and only if $u>\theta$. For details on cone theory. In what follows, we always assume that $u^{*}\geq w_0$, $v^{*}\geq w_1$, $\omega_i\in P_{+}(i=0,1)$. Let $P_{i\lambda}=\{u\in P: u\geq\lambda w_i\}(\lambda>0, i=0,1)$. Obviously, $P_{i\lambda}\subset P_{+}(i=0,1)$, for any $\lambda>0$. When $\lambda=1$, we write $P_i=P_{i1}$, i.e. $P_i=\{u\in P: u\geq w_i\}(i=0,1)$. A map $u\in PC^1[J,E]\cap C^2[J'_{+},E]$ is called a positive solution of (1.1) if $u^{(i)}(t)>\theta$ for $t\in J$ and $u(t)$ satisfies (1.1).

Let α, α_{PC^1} denote the Kuratowski measure of noncompactness in E and $PC^1[J,E]$, respectively.

Throughout this paper, we set

$$\alpha_1=\frac{1}{1-\beta\int_0^1 q(s)ds}, \alpha_2=1+\frac{\beta\int_0^1 q(s)ds}{1-\beta\int_0^1 q(s)ds}, \alpha_3=\int_0^1\int_s^1 q(s)d\tau ds, \alpha_4=\int_0^1 q(s)ds.$$

For the sake of convenience, we first give the following assumptions:

(H_1) $f\in C[J_{+}\times P_{0\lambda}\times P_{1\lambda}\times P\times P,P]$, for any $\lambda>0$ and there exist $a,b\in L[J_{+},R_{+}], c\in L[J_{+},R^{+}]$ and $g\in C[R^{+}\times R^{+}\times R_{+}\times R+,R+$ such that

$$\|f(t,x,y,z,w)\|\leq a(t)+b(t)g(\|x\|,\|y\|,\|z\|,\|w\|),\forall t\in J_{+},x\in P_0,y\in P_1,z,w\in P$$

and

$$\frac{\|f(t,x,y,z,w)\|}{c(t)(\|x\|+\|y\|+\|z\|+\|w\|)}\to 0,$$

as

$$x\in P_0,y\in P_1,z,w\in P,\|x\|+\|y\|+\|z\|+\|w\|\to\infty,$$

uniformly for $t\in J_{+}$, and

$$\int_0^1 a(t)dt=a^{*}<\infty,\int_0^1 b(t)dt=b^{*}<\infty,\int_0^1 c(t)dt=c^{*}<\infty.$$

(H_2) $I_{ik}\in C[P_{0\lambda}\times P_{1\lambda},P]$ for any $\lambda>0(i=0,1;k=1,2,\cdots,m)$ and there exist $F_i\in C[R^{+}\times R^{+},R_{+}]$ and positive constants $\eta_{ik},\gamma_{ik}(i=0,1;k=1,2,\cdots,m)$ such that

$$\|I_{ik}(x,y)\|\leq\eta_{ik}F_i(\|x\|,\|y\|),\forall x\in P_0,y\in P_1,$$

and

$$\frac{\|I_{ik}(x,y)\|}{\gamma_{ik}(\|x\|+\|y\|)}\to 0$$

as

$$x\in P_0, y\in P_1, \| x \| + \| y \| \to \infty,$$

uniformly for $k=1,2,\cdots,m(i=0,1)$, and

$$\eta_i^* = \sum_{k=1}^{m} \eta_{ik}, \quad \gamma_i^* = \sum_{k=1}^{m} \gamma_{ik}.$$

(H_3) For any $t\in J_+$ and $R>0, f(t,P_{0R}^*,P_{1R}^*,P_R^*,P_R^*)=\{f(t,x,y,z,w): x\in P_{0R}^*, y\in P_{1R}^*, z,w\in P_R^*\}$ and $I_{ik}(P_{0R}^*,P_{1R}^*)=\{I_{ik}(x,y): x\in P_{0R}^*, y\in P_{1R}^*\}(i=0,1;k=1,2,\cdots,m)$ arerelatively compact in E, where $P_{iR}^*=\{u\in P: u\geq w_i, \| u \| \leq R$ and $PR*=x\in P:\|u\|\leq R$.

$$(\mathrm{H}_4) 0\leq \beta\int_0^1 q(s)ds < 1.$$

Remark1.1 since $u\geq w_i$ implies $\| u \| \geq N^{-1} \| w_i \|$, condition($H_3$) is satisfied automatically when E is finite dimensional.

Set $Q=\{u\in PC^1[J,P]: u^{(i)}(t)\geq w_i, t\in J, i=0,1\}$ and $I_k=(t_{k-1},t_k](k=1,2,\cdots,m)$,evidently, Q is a closed convex set in $PC^1[J,P]$.

Lemma 1.1[6] Let H be a countable set of strongly measurable function $x: I\to E$ suchthat there exists an $M\in L[I,R_+]$ such that $\| x \| \leq M(t) a.e. t\in I \quad \forall x\in H$. Then $\alpha(H(t))\in L[I,R_+]$ and

$$\alpha(\{\textstyle\int_I x(t)dt : x\in H\}) \leq 2\int_I \alpha(H(t))dt.$$

Lemma 1.2[6] If $H\subset PC^1[I,E]$ is bounded and the elements of H' are equicontinuous oneach $I_k(k=0,1,\cdots,p)$. Then

$$\alpha_{PC^1}(H)=\max\left\{\sup_{t\in I}\alpha(H(t)), \sup_{t\in I}\alpha(H'(t))\right\}.$$

We shall reduce boundary value problem (1.1) to an impulsive integral equation in E. To this end, we first consider operator A defined by

$$\begin{aligned}(Au)(t) &= u^* + \alpha_1 v^* t + \alpha_1\beta t\int_0^1\int_s^1 q(s) f\left(\tau,u(\tau),u'(\tau),(Tu)(\tau),(Su)(\tau)\right)d\tau ds \\ &+ \left[\int_0^1 t f\left(s,u(s),u'(s),(Tu)(s),(Su)(s)\right)ds - \int_0^t (t-s) f\left(s,u(s),u'(s),(Tu)(s),(Su)(s)\right)ds\right] \\ &+\alpha_1\beta t\left[\sum_{k=1}^{m} I_{1k}\left(u(t_k),u'(t_k)\right)\int_0^1 q(s)ds - \int_0^1 q(s)\sum_{0<t_k<s} I_{1k}\left(u(t_k),u'(t_k)\right)ds\right] \\ &+\left[t\sum_{k=1}^{m} I_{1k}\left(u(t_k),u'(t_k)\right) - \sum_{0<t_k<t}(t-t_k) I_{1k}\left(u(t_k),u'(t_k)\right)\right] + \sum_{0<t_k<t} I_{0k}\left(u(t_k),u'(t_k)\right).\end{aligned}$$

2. Main Results

Theorem 2.1. Assume that $(H_1)-(H_4)$ hold, then(1.1) has a positive solution $\bar{u}\in PC^1[J,E]\cap C^2[J_+',E]$ satisfying $(\bar{u})^i(t)\geq w_i$ for $t\in J(i=0,1)$.

Proof. Assume that(H_4)holds. $u\in PC^1[J,E]\cap C^2[J_+',E]$ is a solution of (1.1) if and only if $u\in PC^1[J,E]$ is a solution of the following impulsive integral equation, i.e., u is a fixed point of operator A in $PC^1[J,E]$.Assume that $(H_1)-(H_4)$ hold,then the operator A is continuous operator from Q into Q.we need only to show that A has a fixed point $\bar{x}$ in Q. Choose

$$R > 4\|u^*\| + 4\alpha_1\|v^*\| + 4(\alpha_1\alpha_3\beta+2)a^* + 4(\alpha_1\alpha_3\beta M+2M)b^* + 4(2\alpha_1\alpha_4\beta M_1+4M_1)\eta_1^* + 4M_1\eta_0^*, \quad (2.1)$$

and let $Q_1=\{u\in Q: \|u\|_{PC^1}\leq R\}$. Obviously, Q_1 is a bounded closed convex set in the space $PC^1[J,E]$. It is easy to see that Q_1 is not empty since $\omega(t)=u^*+tv^*\in Q_1$. It follows that $x\in Q_1$ implies $Ax\in Q_1$, i.e., A maps Q_1 into Q_1. Now, we arein position to show that $A(Q_1)$ is relatively compact. Let $V=\{u_n: n=1,2,\cdots\}\subset Q_1$.Then $\|u\|_{PC^1}\leq R$. We have

$$\left\|(Ax_n)'(t')-(Ax_n)'(t)\right\|$$
$$\leq \int_t^{t'}\left\|f(s,u(s),u'(s),(Tx)(s),(Sx)(s))\right\|ds + \sum_{t<t_k<t'}\left\|I_{1k}\left(u(t_k),u'(t_k)\right)\right\|$$

$$\leq \varepsilon_0(2+k^*+h^*)R\int_t^{t'} c(s)+a(s)+Mb(s)ds$$
$$+\frac{1}{16(\alpha_1\beta+1)(\alpha_4+1)}R+\eta_1^* M_1.$$

which implies that $\{(Ax_n)'(t)\}(n=1,2,3,\cdots)$ is equicontinuous on each $I_k(k=1,2,\cdots,m)$.It is clear that, $\{(Ax_n)'\}(n=1,2,3,\cdots)\subset Q_1\subset PC^1[J,E]$ is bounded. we have

$$\alpha_{PC^1}(AV)=max\{\sup_{t\in J}\alpha((AV)^{(i)}(t))\}, \text{i}=0,1, \quad (2.2)$$

where $AV=\{Ax_n: n=1,2,\cdots\}$, and $(AV)^{(i)}(t)=\{(Ax_n)^{(i)}(t): n=1,2,\cdots\}$. Since,for fixed $s\in J, V^{(i)}(s)\subset P^*_{i\bar{R}}(i=0,1)$ and $(TV)(s)\subset P^*_{\bar{R}}, (SV)(s)\subset P^*_{\bar{R}}$, where $\bar{R}=max\{R,k^*R,h^*R\}$, we have, by condition (H_3)

$$\alpha(f\big(s,V(s),V'(s),(TV)(s),(SV)(s)\big)=0, \forall s\in J_1, \quad (2.3)$$

and

$$\alpha(I_{ik}(V(t_k),V'(t_k)))=0,(i=0,1;k=1,2,\cdots,m), \quad (2.4)$$

it is easy to proof

$$\alpha((AV)^i(t))=0, t\in J, i=0,1. \quad (2.5)$$

Hence, by (2.2) we know that $\alpha_{PC^1}(AV)=0$and the relative compactness of $A(Q_1)$isproved. Finally, Schauder fixed point theorem guarantees that$A$has a fixed point $\bar{x}$ inQ_1

Remark 2.1 Since the impulsive BVPs with integral boundary conditions is singular, It is different to construct a closed set and the corresponding impulsive integral operator maps this bounded closed convex set into itself. The results obtained in this paper is important and interesting.

3. An Example

In this section, we construct an example to demonstrate the application of our main
result obtained in section 3.

$$\begin{cases} x_n''(t)=\frac{4}{n\sqrt[4]{t}}\left(2+x_n(t)+x_{2n+1}'(t)+\frac{1}{x_n'(t)+[x_{n+1}(t)]^2}-\cos x_{n+2}'(t)\right)^{\frac{1}{2}}. \\ \quad +\frac{1}{\sqrt{nt}}\left(\int_0^t(1+ts)e^{-s}x_{n+3}(s)ds\right)^{\frac{1}{3}}+\frac{1}{n\sqrt{t}(1+t)}\left(\int_0^1\frac{x_{3n}(s)ds}{(1+t+s)^2}\right)^{\frac{1}{5}}, \\ \quad \forall 0<t<a, t\neq t_k, k=1,2,\cdots,p, \\ \Delta x_n|_{t=t_k}=\frac{1}{n^5}\left(\frac{1}{x_n(t_k)+x_{2n}'(t_k)}\right)^{\frac{1}{2}}, k=1,2,\cdots,p, \\ \Delta x_n'|_{t=t_k}=\frac{1}{n^9}\left(x_{n+1}(t_k)+\frac{1}{x_{n+2}'(t_k)}\right)^{\frac{1}{3}}, k=1,2,\cdots,p, \\ x_n(0)=\frac{1}{\sqrt{n}}, x_n'(1)=\int_0^1\frac{1}{2}x'(s)ds+\frac{1}{n}, n=1,2,\cdots. \end{cases} \quad (3.1)$$

Proposition 1. Infinite system (3.1) has a positive solution $\{x_n(t)\}(n=1,2,\cdots)$suchthat$x_n(\text{t})\geq\frac{1}{\sqrt{n}}, x_n'(t)\geq\frac{1}{n}$ for $0\leq \text{t}\leq 1$.

Proof. Let$E=c_0=\{x=(x_1,\cdots,x_n,\cdots): x_n\to 0\}$with the norm$\|x\|=sup_n|x_n|andP=\{x=(x_1,\cdots,x_n,\cdots)\in c_0: x_n>0, n=$ *1,2,3⋯*.Then *P*is a normal cone in*E*and infinite system (4.1) can be regarded as a BVP of the form (1.1). In this situation, $x=(x_1,\cdots,x_n,\cdots), y=(y_1,\cdots,y_n,\cdots), z=(z_1,\cdots,z_n,\cdots), w=(w_1,\cdots,w_n,\cdots), k(t,s)=(1+ts)e^{-s}, h(t,s)=\frac{1}{(1+t+s)^2}, u^*=\left(1,\frac{1}{\sqrt{2}}\cdots,\frac{1}{\sqrt{n}},\cdots\right), v^*=\left(1,\frac{1}{2}\cdots,\frac{1}{n},\cdots\right), f=(f_1\cdots,f_n,\cdots)$,and$I_{ik}=(I_{ik1},\cdots,I_{ikn},\cdots),(i=0,1)$,in which

$$f_n(t,x,y,z,w)=\frac{4}{n^4\sqrt{t}}\left(2+x_n+y_{2n+1}+\frac{1}{y_n+x_{n+1}^2}-\cos y_{n+2}\right)^{\frac{1}{2}} \quad (3.2)$$
$$+\frac{1}{\sqrt{nt}}z_{n+3}^{\frac{1}{3}}+\frac{1}{n\sqrt{t}(1+t)}w_{3n}^{\frac{1}{5}},$$

$$I_{0kn}=\frac{1}{n^5}\left(\frac{1}{x_n+y_{2n}}\right)^{\frac{1}{2}}, \quad (3.3)$$

$$I_{1kn}=\frac{1}{n^9}\left(x_{n+1}+\frac{1}{y_{n+2}}\right)^{\frac{1}{3}}, \quad (3.4)$$

Let$w_0=u^*, w_1^*=v^*$.Then $P_{0\lambda}=\left\{x=(x_1,\cdots,x_n,\cdots): x_n\geq\frac{\lambda}{\sqrt{n}}, n=1,2,\cdots\right\}$,

$P_{1\lambda}=\left\{x=(x_1,\cdots,x_n,\cdots): x_n\geq\frac{\lambda}{n}, n=1,2,\cdots\right\}$ for λ > 0. It is clear that, $f\in C[I_1\times P_{0\lambda}\times P_{1\lambda}\times P\times P, P], I_{ik}\in \text{C}[P_{0\lambda}\times$ *P1λ,P*for anyλ*>0*i=0,1;*k=1,2,⋯,p*.For*t∈J1,x∈P0,y∈P1,z,w∈P*(by definition*P0=P01,P1=P11*), we have, by (4.2),

$$\|f(t,x,y,z,w)\| \le \frac{4}{\sqrt{t}}\left\{\left(\frac{7}{2}+\|x\|+\|y\|\right)^{\frac{1}{2}}+\|z\|^{\frac{1}{3}}+\|w\|^{\frac{1}{5}}\right\}. \qquad (3.5)$$

So, $(H_1)$is satisfied for$a(t) = 0, b(t) = c(t) = \frac{4}{\sqrt{t}}$and

$$g(u_0,u_1,u_2,u_3) = \left(\frac{7}{2}+u_0+u_1\right)^{\frac{1}{2}}+u_2^{\frac{1}{3}}+u_3^{\frac{1}{5}}.$$

On the other hand, for$x \in P_0, y \in P_1,$we have

$$\|I_{0k}(x,y)\| \le (1+\|y\|)^{\frac{1}{2}},$$
$$\|I_{1k}(x,y)\| \le (1+\|x\|)^{\frac{1}{3}},$$

which imply that condition$(H_2)$is satisfied for

$$F_0(u_0,u_1) = (1+u_1)^{\frac{1}{2}}, F_1(u_0,u_1) = (1+u_0)^{\frac{1}{3}}$$

and$\eta_{0k} = \eta_{1k} = \gamma_{0k} = \gamma_{1k} = 1.$ Now, we check condition (H_3). Let $t \in I_1$ and $R > 0$ be given and $\{u^{(m)}\}$ be any sequence in$f(t, P_{0R}^*, P_{1R}^*, P_R^*, P_R^*,)$,where$u^{(m)} = (u_1^{(m)}, \cdots, u_n^{(m)}, \cdots)$.By (3.2), we have

$$0 \le u_n^{(m)} \le \frac{4}{n^4\sqrt{t}}\left(\frac{7}{2}+2R\right)^{\frac{1}{2}}+\frac{1}{\sqrt{nt}}R^{\frac{1}{3}}+\frac{1}{n\sqrt{t}(1+t)}R^{\frac{1}{5}}(n,m=1,2,\cdots). \qquad (3.6)$$

So, $\{u^{(m)}\}$is bounded and, by the diagonal method, we can choose a subsequence$\{m_i\} \subset \{m\}$such that

$$u_n^{(m_i)} \to \bar{u}_n \text{as} i \to \infty (n = 1,2,\cdots) \qquad (3.7)$$

which implies by virtue of (3.6) that

$$0 \le \bar{u}_n \le \frac{4}{n^4\sqrt{t}}\left(\frac{7}{2}+2R\right)^{\frac{1}{2}}+\frac{1}{\sqrt{nt}}R^{\frac{1}{3}}+\frac{1}{n\sqrt{t}(1+t)}R^{\frac{1}{5}}(n=1,2,\cdots). \qquad (3.8)$$

Hence$\bar{u} = (\bar{u}_1, \cdots, \bar{u}_n, \cdots) \in c_0$.It is easy to see from (3.6)-(3.8) that
$\|u^{(m_i)} - \bar{u}\| = \sup_n |u_n{}^{(m_i)} - \bar{u}_n| \to 0 \text{as} i \to \infty.$
Thus, we have proved that$f(t, P_{0R}^*, P_{1R}^*, P_R^*, P_R^*,)$is relatively compact in$c_0$.In a similar way, by (3.3) and (3.4), we can show that $I_{ik}(P_{0R}^*, P_{1R}^*)$(i = 0,1; $k = 1,2,\cdots,p$) are relativelycompact in c_0. Therefore, (H_3) is satisfied. on the other hand, $0 < \beta\int_0^1 q(s)ds = \int_0^1 \frac{1}{2}ds = \frac{1}{2} < 1$,So,$(H_4)$is satisfied. Our conclusion follows from Theorem 3.1.

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REFERENCES

1. V .Vijayakumar, Hernán R. Henríquez, Existence of global solutions for a class of abstract second order nonlocal Cauchy problem with impulsive conditions in Banach spaces, Numerical Functional Analysis & Optimization, 3 (2018) 1-33.
2. Xingqiu Zhang, Positive solutions for a second-Order nonlinear impulsive singular integro-differential equation with integral conditions in Banach spaces, Journal of Mathematical Research with Applications, 5 (2012) 99-114.
3. ChunmeiMiao, WeigaoGe, Existence of positive solutions for singular impulsive differential equations with integral boundary conditions, Mathematical Methods in the Applied Sciences,38 (2015) 1146-1157.
4. Xuemei Zhang, Yafang Tian, Sharp conditions for the existence of positive solutions for a second-order singular impulsive differential equation, Applicable Analysis, (2017) 1-13.
5. Yansheng Liu, Multiple positive solutions of nonlinear singular boundary value problem for fourth-order equations, Applied Mathematics Letters,17 (2004) 747-757.
6. Ping Kang, Zhongli Wei, JuanjuanXu, Positive solutions to fourth-order singular boundary value problems with integral boundary conditions in abstract spaces, Applied Mathematics and Computation,206 (2008) 245-256.
7. Xingqiu Zhang, Existence of positive solution for second-order nonlinear impulsive singular differential equations of mixed type in Banach spaces, Nonlinear Analysis: Theory, Methods& Applications,70 (2009) 1620-1628.
8. Xuemei Zhang, MeiqiangFeng, WeigaoGe, Existence of solutions of boundary value problems with integral boundary conditions for second-order impulsive integro-differential equations in Banach spaces, Journal of Computational and Applied Mathematics,233 (2010) 1915-1926.