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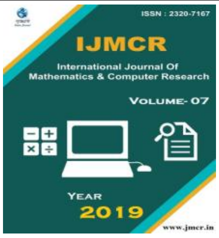

Gumbel - Fréchet Distribution and its Applications

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ABSTRACT

A new four - parameter distribution generated from Gumbel - X family called the Gumbel - Fréchet distribution is introduced and studied. The structural properties including ordinary and incomplete moments, quantiles and generating functions, probability weighted moments, mean deviation about mean and median, moments of residual and reversed residual life, Renyi and δ- entropies and order statistics are discussed in detail. The new density function can be expressed as a linear mixture of Fréchet densities. The maximum likelihood method is used to estimate the model parameters. The new distribution is applied to a real data set on survival life times to establish the flexibility of the newly developed model.

KEYWORDSAND PHRASES: Gumbel Distribution, Fréchet Distribution, Survival function, T-X family.

1. Introduction

A number of new distributions are being developed nowadays which are either more flexible and suitable for fitting a wide range of data sets from real world scenarios. This motivated the researchers for developing new generalized families of distributions. Many lifetime distributions have been constructed with a view for applications in several areas like survival analysis, reliability engineering, demography, actuarial study, hydrology and others.

Many new class of distributions are developed by generating the distribution from another distribution. Some examples are Beta generated distributions (Eugene et al.(2002)) and Kumaraswamy generated distributions (Jones (2009) and Cordeiro and de Castro (2011)). In both these cases range of the distribution varies from 0 to 1. Alzaatreh et al. (2013 b) introduced a new method as any non - negative continuous random variable can be used as the generator. Alzaatreh called these distributions as *Transformed - Transformer* or T - X family of distributions.

Let F (x) be the cumulative distribution function (cdf) of any random variable X and r(t) be the probability density function (pdf) of a random variable T defined on $[0,\infty)$. Then the cdf of the T-X family of distributions is given by

$$G(x) = \int_0^{U[F(x)]} r(t)dt \qquad (1)$$

where U[F(x)] is a function of the cumulative distribution function (cdf) of a random variable X such that U[F(x)] satisfies the following conditions:

- U[F(x)] $\in$[a,b],
- U[F(x)] is differentiable and monotonically non-decreasing, and
- U[F(x)] $\to a$, as x $\to -\infty$ and U[F(x)] $\to b$, as x $\to \infty$

Recently, Al-Aqtash (2013) and Al - Aqtash et al. (2014, 2015) proposed the Gumbel - X family by taking T as the Gumbel random variable and U[F(x)] = log $[\frac{F(x)}{\bar{F}(x)}]$ in (1). Then we get the cdf

$$G(x) = \exp\left[-e^{\mu/\sigma}\left(\frac{F(x)}{\bar{F}(x)}\right)^{-1/\sigma}\right],$$

where $-\infty < x < \infty$, $\sigma > 0$ and $-\infty < \mu < \infty$. By setting $\lambda = e^{\mu/\sigma}$, the cdf reduces to

$$G(x) = \exp\left[-\lambda\left(\frac{F(x)}{\bar{F}(x)}\right)^{-\frac{1}{\sigma}}\right] \tag{2}$$

Then the pdf is obtained as

$$g(x) = \frac{\lambda}{\sigma} f(x) \frac{(F(x))^{-\frac{1}{\sigma}-1}}{(\bar{F}(x))^{-\frac{1}{\sigma}+1}} \exp\left[-\lambda\left(\frac{F(x)}{\bar{F}(x)}\right)^{-\frac{1}{\sigma}}\right] \tag{3}$$

In this paper, a new distribution Gumbel - Fréchet distribution is introduced in section 2. The density function as a linear combination of Fréchet densities is discussed in section 3. Some of the mathematical properties of Gumbel - Fréchet distribution like quantile function, moments, incomplete moments, probability weighted moments, moment generating function, mean deviation about mean and median, moments of residual and reversed residual life, density function of order statistics, Renyi and δ - entropies are discussed in section 4. In section 5, the maximum likelihood estimation of the model parameters is explained. The application of this distribution to a real data set is presented in section 6. The conclusions are given in section 7.

2. Gumbel-Frechet Distribution

The Frechet distribution is a well - defined limiting distribution for the maximum of random variables with non - negative real support. So it is well suitable for characterizing random variables of large features. It has application ranging from accelerated life testing like earthquakes, floods, rainfall, wind speeds, sea waves, track race records, queues in supermarkets, cash flow (finance), etc. For more details one can refer Kotz and Nadarajah(2000). Recently, there have been a number of works related to the Frechet distribution. Some recent works are Marshall - Olkin extended Frechet distribution by Krishna et al. (2013 a, b), Weibull Frechet distribution by Afify et al. (2016), application of extended Frechet distribution by Zayed and Butt (2017), etc.

If X follows Frechet distribution with parameters α and β, the cdf is given by

$$F(x) = e^{-\left(\frac{\alpha}{x}\right)^{\beta}}, x > 0, \quad \alpha, \beta > 0 \tag{4}$$

and the corresponding pdf is

$$f(x) = \frac{\beta}{\alpha}\left(\frac{\alpha}{x}\right)^{\beta+1} e^{-\left(\frac{\alpha}{x}\right)^{\beta}}, x > 0, \quad \alpha, \beta > 0.$$

Substituting the cdf, pdf and survival function of Frechet distribution in (3), we get the cdf of Gumbel - Frechet (GuFr) distribution (for x > 0) which is obtained as

$$G(x) = exp\left\{-\lambda\left[\frac{e^{-\left(\frac{\alpha}{x}\right)^{\beta}}}{1-e^{-\left(\frac{\alpha}{x}\right)^{\beta}}}\right]^{-1/\sigma}\right\} \tag{5}$$

The pdf is

$$g(x) = \frac{\lambda\beta}{\alpha\sigma}\left(\frac{\alpha}{x}\right)^{\beta+1} e^{\left(\frac{\alpha}{x}\right)^{\beta}}\left[e^{\left(\frac{\alpha}{x}\right)^{\beta}}-1\right]^{\frac{1}{\sigma}-1} exp\left\{-\lambda\left[\frac{e^{-\left(\frac{\alpha}{x}\right)^{\beta}}}{1-e^{-\left(\frac{\alpha}{x}\right)^{\beta}}}\right]^{-1/\sigma}\right\} \tag{6}$$

where α is a scale parameter representing the characteristic life and β, λ and σ are the shape parameters of GuFr distribution. A random variable X following Gumbel-*Fréchet* distribution is denoted by $X \sim GuFr(x,\lambda,\sigma,\alpha,\beta)$.

The reliability function is

$$\bar{S}(x) = 1 - G(x) = 1 - exp\left\{-\lambda\left[\frac{e^{-\left(\frac{\alpha}{x}\right)^{\beta}}}{1-e^{-\left(\frac{\alpha}{x}\right)^{\beta}}}\right]^{-1/\sigma}\right\}$$

and the hazard function

$$h(x) = \frac{\frac{\lambda\beta}{\alpha\sigma}\left(\frac{\alpha}{x}\right)^{\beta+1} e^{\left(\frac{\alpha}{x}\right)^{\beta}}\left[e^{\left(\frac{\alpha}{x}\right)^{\beta}}-1\right]^{\frac{1}{\sigma}-1} exp\left\{-\lambda\left[\frac{e^{-\left(\frac{\alpha}{x}\right)^{\beta}}}{1-e^{-\left(\frac{\alpha}{x}\right)^{\beta}}}\right]^{-1/\sigma}\right\}}{1 - exp\left\{-\lambda\left[\frac{e^{-\left(\frac{\alpha}{x}\right)^{\beta}}}{1-e^{-\left(\frac{\alpha}{x}\right)^{\beta}}}\right]^{-1/\sigma}\right\}}$$

Figure 1 and 2 illustrates the possible shapes of density function and hazard function for selected values of λ and σ.

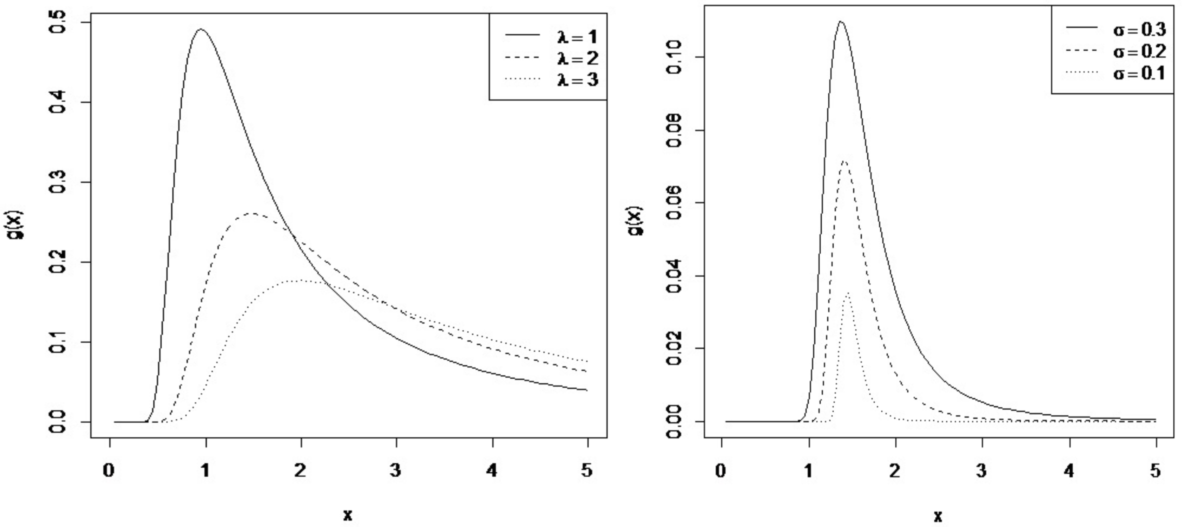

Fig.1: Graph of GuFrdensity function for various values of λ and σ

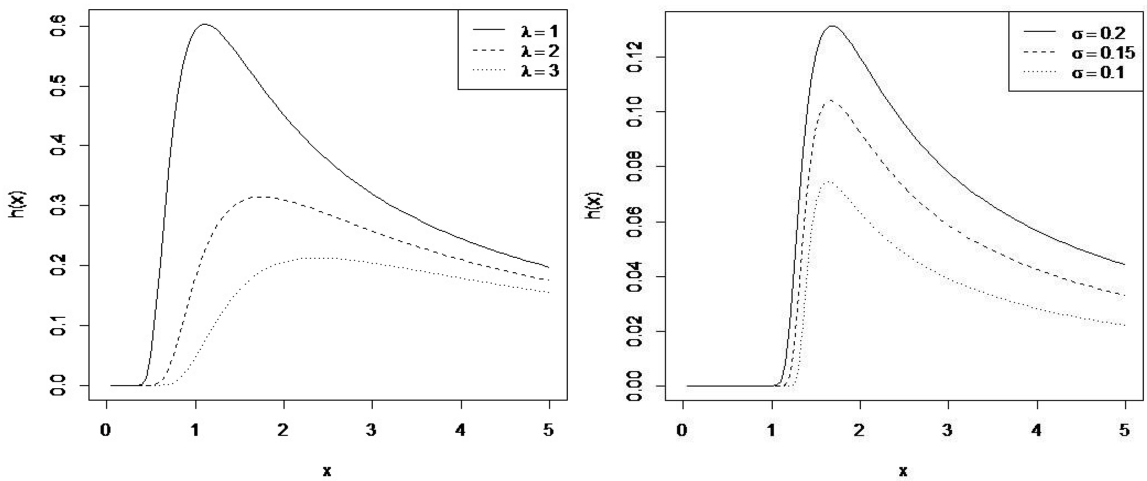

Fig.2: Graph of GuFrhazard function for various values of λ andσ

3. Mixture Representation

By substituting (4) in (3),

$$g(x) = \frac{\lambda\beta}{\alpha\sigma}(\frac{\alpha}{x})^{\beta+1} e^{\frac{1}{\sigma}(\frac{\alpha}{x})^{\beta}}[1-e^{-(\frac{\alpha}{x})^{\beta}}]^{\frac{1}{\sigma}-1} exp\left\{-\lambda[\frac{e^{-(\frac{\alpha}{x})^{\beta}}}{1-e^{-(\frac{\alpha}{x})^{\beta}}}]^{-1/\sigma}\right\} \tag{9}$$

Here the last expression in the above equation can be given as

$$exp\left\{-\lambda[\frac{e^{-(\frac{\alpha}{x})^{\beta}}}{1-e^{-(\frac{\alpha}{x})^{\beta}}}]^{-1/\sigma}\right\} = \sum_{k=0}^{\infty}\frac{(-1)^k\lambda^k}{k!}\frac{\exp[\frac{k}{\sigma}(\frac{\alpha}{x})^{\beta}]}{\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\}^{-k/\sigma}}$$

Substituting this expression in (9) and after some algebra, we get

$$g(x) = \frac{\beta}{\alpha\sigma}(\frac{\alpha}{x})^{\beta+1}\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^k}{k!}\exp[\frac{k+1}{\sigma}(\frac{\alpha}{x})^{\beta}][1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]]^{-(1-\frac{k+1}{\sigma})}$$

Using the relation

$$(1-z)^{-k} = \sum_{j=0}^{\infty}\frac{k^{(j)}}{j!}z^j, |z|<1, k>0$$

where, $k^{(j)} = \Gamma(k+j)/\Gamma k$ and after solving we get,

$$g(x) = \frac{\beta}{\alpha\sigma}(\frac{\alpha}{x})^{\beta+1}\sum_{j,k=0}^{\infty}\frac{(-1)^k\lambda^{k+1}[1-\frac{k+1}{\sigma}]^{(j)}}{j!\ k!}\exp\{-(j-\frac{k+1}{\sigma})(\frac{\alpha}{x})^{\beta}\}$$

$$= \sum_{j,k=0}^{\infty}\eta_{j,k}\frac{\beta}{\alpha}[j-\frac{k+1}{\sigma}](\frac{\alpha}{x})^{\beta+1}\exp\{-(j-\frac{k+1}{\sigma})(\frac{\alpha}{x})^{\beta}\},$$

where, $\eta_{j,k} = \frac{(-1)^k\lambda^{k+1}[1-\frac{k+1}{\sigma}]^{(j)}}{j!\,k!\sigma[j-\frac{k+1}{\sigma}]}.$

Thus we get,

$$g(x) = \sum_{j,k=0}^{\infty}\eta_{j,k}\,p(x), \tag{10}$$

where $p(x)$ is the density function of *Fréchet* distribution with scale parameter $\alpha[j-\frac{k+1}{\sigma}]^{1/\beta}$ and the shape parameter β. Thus the Gumbel - *Fréchet* distribution can be expressed as a linear combination of *Fréchet* densities. Thus, several of its structural properties can be obtained from (10).

From (10), we get the cdf of GuFr as

$$G(x) = \sum_{j,k=0}^{\infty}\eta_{j,k}\,P(x),$$

where $p(x)$ is the cdf of *Fréchet* distribution with parameters $\alpha[j-\frac{k+1}{\sigma}]^{1/\beta}$ and β.

4. Mathematical Properties

In this section, we study some mathematical properties of GuFr like quantile function, order statistics, ordinary and incomplete moments, mean deviation about mean and median, moments of residuals and reversed residual lifes, Mgf, Renyi and δ entropies etc.

4.1 Quantile Function

The quantile function of X is the real solution of the equation $G(x_p) = p$. Then by inverting (5), we obtain

$$x_p = \alpha\{\log[1+(-\frac{\log p}{\lambda})^{\sigma}]\}^{-1/\beta}$$

If p=0.5, we get the median of X.

4.2 Moments

From (10), the r[th] raw moment of X is given by

$$E(X^r) = \sum_{j,k=0}^{\infty}\eta_{j,k}\int_0^{\infty}x^r p(x)dx.$$

For $\gamma < \beta$, we get

$$E(X^r) = \sum_{j,k=0}^{\infty}\eta_{j,k}\alpha^r\left[j-\frac{k+1}{\sigma}\right]^{\frac{r}{\beta}}\Gamma(1-r/\beta). \tag{11}$$

The mean of GuFr distribution can be obtained from (11) by substituting r =1.

The n[th] central moment of X, say μ_n is

$$\mu_n = E(X-\mu)^n = \sum_{k=0}^{\infty}n(-1)^k nC_k\mu_1'^k\mu_{n-k}'$$

The cumulants κ_n of X can be obtained from

$$\kappa_n = \mu_n' - \sum_{k=0}^{n-1}n-1\,C_{k-1}\kappa_r\,\mu_{n-r}',$$

where $\kappa_1 = \mu_1'$.

The coefficients of skewness and kurtosis can be obtained from the ordinary moments using the well - known relationships.

4.3 Incomplete Moments

The s^th incomplete moment, say $\varphi_s(t)$of GuFr distribution is

$$\varphi_s(t) = \int_0^t x^s g(x)dx$$

$$= \sum_{j,k=0}^{\infty} \eta_{j,k} \int_0^t x^s p(x)dx$$

Then, for $s < \beta$ and using the lower incomplete gamma function, we get,

$$\varphi_s(t) = \alpha^s \sum_{j,k=0}^{\infty} \eta_{j,k} \left[j - \frac{k+1}{\sigma}\right]^{\frac{s}{\beta}} \Gamma(1 - \frac{s}{\beta}, [j - \frac{k+1}{\sigma}](\frac{\alpha}{t})^{\beta}) \tag{12}$$

The first incomplete moment can be obtained by substituting s = 1 in the last equation.

4.4 Probability Weighted Moments (PWMs)

The Probability Weighted Moments are expectations of certain functions of a random variable. They can be derived for any random variable whose ordinary moments exist. These moments have low variances and no severe biases, and they compare favorably with estimators obtained by the maximum likelihood method. The PWM approach can be used for estimating parameters of any distribution whose inverse form cannot be expressed explicitly and quantiles of generalized distributions.

The (r,s)^th PWM of X, say , where $r \geq 1, s \geq 0$ is defined by

$$\rho_{r,s} = E[X^r G(x)^s]$$

$$= \int_0^{\infty} x^r G(x)^s g(x)dx.$$

From (5),

$$G(x)^s = \exp\{-\lambda s[\frac{e^{-(\frac{\alpha}{x})^{\beta}}}{1-e^{-(\frac{\alpha}{x})^{\beta}}}]^{-1/\sigma}\} \tag{13}$$

Using (6) and (13), we can write,

$$\rho_{r,s} = \int_0^{\infty} x^r \frac{\lambda\beta}{\alpha\sigma} (\frac{\alpha}{x})^{\beta+1} e^{\frac{1}{\sigma}(\alpha/x)^{\beta}} [1 - e^{-(\frac{\alpha}{x})^{\beta}}]^{\frac{1}{\sigma}-1} \exp\{-\lambda s[\frac{e^{-(\frac{\alpha}{x})^{\beta}}}{1 - e^{-(\frac{\alpha}{x})^{\beta}}}]^{-1/\sigma}\}$$

Solving this as in the mixture representation we get,

$$\rho_{r,s} = \sum_{j,k=0}^{\infty} \frac{(-1)^k (1+s)^k \lambda^k [1 - \frac{k+1}{\sigma}]^{(j)}}{j!\, k!} \int_0^{\infty} x^r (\frac{\alpha}{x})^{\beta+1} \exp\{(\frac{k+1}{\sigma} - j)(\frac{\alpha}{x})^{\beta}\}dx$$

$$= \sum_{j,k=0}^{\infty} \frac{(-1)^k (1+s)^k \lambda^{k+1} [1 - \frac{k+1}{\sigma}]^{(j)}}{j!\, k!\, \sigma[\frac{k+1}{\sigma} - j]^{r/\beta}} \int_0^{\infty} x^r p(x)dx$$

Where $p(x)$ is the density function of the *Fréchet* distribution. Therefore, using (11) we can write

$$\rho_{r,s} = \sum_{j,k=0}^{\infty} \xi_{j,k} \alpha^r \left[\frac{k+1}{\sigma} - j\right]^{\frac{r}{\beta}} \Gamma(1 - r/\beta)$$

where,

$$\xi_{j,k} = \frac{(-1)^k (1+s)^k \lambda^{k+1} [1 - \frac{k+1}{\sigma}]^{(j)}}{j!\, k!\, \sigma[\frac{k+1}{\sigma} - j]^{r/\beta}}.$$

4.5 Mean Deviations

The amount of scatter in a population is measured to some extent by the totality of deviations from the mean and median. Let X be a random variable from GuFr distribution with mean μ and Median M.

The mean deviation from the mean is

$$E|X-\mu| = \int_0^\infty |X-\mu| g(x)dx$$
$$=2\int_0^\mu (\mu - x)g(x)dx + \int_0^\infty (x-\mu)g(x)dx$$
$$= 2\mu G(\mu) - 2\int_0^\mu xg(x)dx$$
$$=2\mu G(\mu) - 2\phi_1(\mu)$$

Similarly, the mean deviation from the median is

$$E|X-M| = \int_0^\infty |X-M| g(x)dx$$
$$=2\int_0^M (M - x)g(x)dx + \int_M^\infty (x-M)g(x)dx$$
$$= \mu - 2\int_0^M xg(x)dx$$
$$=\mu - 2\phi_1(M)$$

Here G(μ) can be determined from (5) and $\phi_1(k) = \int_0^k xg(x)dx$ is the incomplete moment obtained from (12) with s =1.

Application of these equations can be made to obtain the Bonferroni curve, $B(x) = \frac{\phi_1(x)}{E(X)}$ and the Lorenz curve, $L(x) = \frac{\phi_1(x)}{G(x)E(X)}$, where $\phi_1(x)$ is the incomplete moment from (12) with s = 1, G(x) is the cdf of GuFr distribution and E(X) is the mean.

These curves are very useful in economics, reliability, medicine, insurance and demography.

4.6 Moments of the Residual and Reversed Residual life

Several functions like failure rate function, mean residual life function and the left censored mean function are related to the residual life. These functions uniquely determines G(x) [Zoroa et al (1990)].

The moments of the residual life $m_n(t) = E[(X-t)^n | X > t], n = 1, 2, ...$uniquely determines G(x) [Navarro et al. (1998)].

We have,

$$m_n(t) = \frac{1}{1-G(t)}\int_t^\infty (x-t)^n dG(x)$$

$$= \frac{1}{1-G(t)}\sum_{r=0}^n nC_r(-t)^{n-r}\int_t^\infty x^r dG(x), \text{ for } r < \beta$$

$$= \frac{1}{\bar{S}(t)}\sum_{r=0}^n (-1)^{n-r}\alpha^r t^{n-r} nC_r \sum_{j,k=0}^\infty \eta_{jk}\left[j - \frac{k+1}{\sigma}\right]^{\frac{r}{\beta}} \Gamma(1-\frac{r}{\beta},(j-\frac{k+1}{\sigma})(\frac{\alpha}{t})^\beta)$$

where $\Gamma(p,q) = \int_q^\infty y^{p-1} e^{-y} dy$ is the incomplete gamma function.

The mean residual life function $m_n(t)$ represents the expected additional life length of a unit that is alive at age x.
In a similar manner, Navarro et. al (1998) proved that the reversed residual life $M_n(t)$ uniquely determines G(x).

We have,

$$M_n(t) = \frac{1}{G(t)}\int_0^t (t-x)^n dG(x)$$

For GuFr distribution, for $r < \beta$,

$$M_n(t) = \frac{1}{G(t)}\sum_{r=0}^n (-1)^r \alpha^r nC_r \sum_{j,k=0}^\infty \eta_{jk}\left[j - \frac{k+1}{\sigma}\right]^{\frac{r}{\beta}} \Gamma(1-\frac{r}{\beta},(j-\frac{k+1}{\sigma})(\frac{\alpha}{t})^\beta)$$

The mean reversed residual life (MRRL) function corresponding to $M_1(t)$ (by setting n=1 in the above equation) represents the waiting time elapsed for the failure of an item under the condition that this failure had occured in (0, t).

4.7 Moment Generating Function (MGF)

We can obtain the MGF of Frechet distribution by substituting $y = 1/x$ in Frechet density function

$$F(x) = \frac{\beta}{\alpha}(\frac{\alpha}{x})^{\beta+1}e^{-(\frac{\alpha}{x})^{\beta}}.$$

Then,

$$M_Y(t) = \int_0^{\infty} \beta\alpha^{\beta} e^{t/y}\, y^{\beta-1} e^{-(\alpha y)^{\beta}} dy$$

$$= \sum_{m=0}^{\infty} \frac{\alpha^m t^m}{m!}\Gamma(\frac{\beta-m}{\beta}) \qquad (14)$$

By using (10) and (14), the MGF of GuFr distribution is

$$M_X(t) = \sum_{j,k=0}^{\infty} \eta_{j,k} \sum_{m=0}^{\infty} \frac{\alpha^m t^m}{m!}\Gamma(\frac{\beta-m}{\beta}).$$

4.8 Renyi and δ – Entropies

The Renyi entropy(Alfred Renyi) of a random variable X represents a measure of variation of the uncertainty which is definedby

$$I_\delta(x) = \frac{1}{1-\delta} log \int_{-\infty}^{\infty} g^{\delta}(x)dx, \;\; \delta > 0 \text{ and} \delta \neq 1$$

Here

$$g^{\delta}(x) = (\frac{\lambda\beta}{\alpha\sigma})^{\delta}(\frac{\alpha}{x})^{\delta(\beta+1)} e^{\frac{\delta}{\sigma}(\frac{\alpha}{x})^{\beta}}[1 - e^{-(\frac{\alpha}{x})^{\beta}}]^{\delta(\frac{1}{\sigma}-1)} exp\left\{-\lambda\delta[\frac{e^{-(\frac{\alpha}{x})^{\beta}}}{1 - e^{-(\frac{\alpha}{x})^{\beta}}}]^{-1/\sigma}\right\}$$

After solving we get,

$$g^{\delta}(x) = \sum_{j,k=0}^{\infty} p_{j,k} x^{-\delta(\beta+1)} \exp\{-[j - \frac{\delta+k}{\sigma}](\frac{\alpha}{x})^{\beta}\}$$

where,

$$p_{j,k} = (\frac{\beta}{\sigma})^{\delta}\alpha^{\delta\beta}\frac{(-1)^k\delta^k\lambda^{k+\delta}[\delta - \frac{k+\delta}{\sigma}]^{(j)}}{j!\,k!}$$

Then the Renyi entropy becomes

$$I_\delta(x) = \frac{1}{1-\delta} log[\sum_{j,k=0}^{\infty} p_{j,k} \int_{-\infty}^{\infty} x^{-\delta(\beta+1)} \exp\{-[j - \frac{\delta+k}{\sigma}](\frac{\alpha}{x})^{\beta}\}\, dx]$$

$$= \frac{1}{1-\delta}\log[\Gamma(\frac{\delta(\beta+1)}{\beta})] \sum_{j,k=0}^{\infty} q_{j,k}$$

where,

$$q_{j,k} = \frac{(-1)^k\delta^k\lambda^{k+\delta}[\delta - \frac{k+\delta}{\sigma}]^{(j)}}{j!k!\sigma^{\delta}}(\frac{\beta}{\alpha})^{\delta-1}[j - \frac{\delta+k}{j}]^{(1-\delta(\beta+1))/\beta} \qquad (15)$$

The δ – entropy $\delta > 0$and $\delta \neq 1$, say $H_\delta(x)$, is defined as

$$H_\delta(x) = \frac{1}{\delta-1}\log[1 - \int_{-\infty}^{\infty} g^{\delta}(x)dx]$$

$$= \frac{1}{\delta-1}\log[1 - \{\Gamma(\frac{\delta(\beta+1)}{\beta})\; \sum_{j,k=0}^{\infty} q_{j,k}\}]$$

where $q_{j,k}$ is given in (15).

The Renyi entropy of converge to the Shannon entropy when $\delta \rightarrow 1$.

4.9 Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the GuFr distribution and $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ are the corresponding order statistics, then the pdf of i^{th} order statistic denoted by $g_{i:n}(x)$ is given by

$$g_{i:n}(x) = \frac{g(x)}{B(i,n-i+1)} \sum_{j=0}^{n-i} (-)^j \ (n-i)C_j G(x)^{i+j-1} \qquad (16)$$

where,

$$G(x)^{i+j-1} = \exp\{-\lambda(i+j-1)[\frac{e^{-(\frac{\alpha}{x})^\beta}}{1-e^{-(\frac{\alpha}{x})^\beta}}]^{-1/\sigma}\} \qquad (17)$$

Substituting (6) and (17) in (16), we get

$$g_{i:n}(x) = \sum_{r,k=0}^{\infty} \eta_{k,r} p(x),$$

where,

$$\eta_{k,r} = \sum_{j=0}^{n-i} \frac{(-1)^{k+j} \lambda^{k+1} (i+j)^k [1-\frac{k+1}{\sigma}]^{(r)}}{k!\, r!\, \sigma [j-\frac{k+1}{\sigma}]}$$

and $p(x)$ is the pdf of Frechet $[\alpha(j-\frac{k+1}{\sigma})^{1/\beta}, \beta]$.

5. Maximum Likelihood Estimation

The maximum likelihood estimation (mle) method is used for the parameter estimation of GuFr distribution. Let $X_1, X_2, \ldots, X_n$ be a random sample from Gumbel Frechet (GuFr) distribution. Also let $\Theta = (\lambda, \sigma, \alpha, \beta)$

The likelihood function for the GuFr distribution is given by

$$L(\Theta) = (\frac{\lambda\beta}{\sigma})^n \alpha^{n\beta} \prod_{i=1}^{n} x_i^{-(\beta+1)} \ e^{\frac{1}{\sigma}\sum_{i=1}^{n}(\frac{\alpha}{x_i})^\beta} \prod_{i=1}^{n} (1-s_i)^{\frac{1}{\sigma}-1} \ exp\left\{-\lambda \sum_{i=1}^{n} (\frac{s_i}{1-s_i})^{-1/\sigma}\right\}$$

where $s_i = e^{-\left(\frac{\alpha}{x_i}\right)^\beta}$.

The log likelihood for Θ is

$$logL(\Theta) = nlog\,\lambda + n\log\beta - n\log\sigma + n\beta log\alpha - (\beta+1)\sum_{i=1}^{n}\log x_i +$$

$$\frac{1}{\sigma}\sum_{i=1}^{n}\left(\frac{\alpha}{x_i}\right)^\beta + (\frac{1}{\sigma}-1)\sum_{i=1}^{n}\log(1-s_i) - \lambda\sum_{i=1}^{n}(\frac{s_i}{1-s_i})^{-\frac{1}{\sigma}}$$

The log likelihood can be maximized either by solving the non-linear equations to zero or using R software with *nlm* function. The components of the score vector $U(\Theta)$ are given by

$$U_\alpha = \frac{n\beta}{\alpha} + \frac{1}{\sigma}\sum_{i=1}^{n}\beta(\frac{\alpha}{x_i})^{\beta-1}\frac{1}{x_i} + (\frac{1}{\sigma}-1)\frac{\beta}{\alpha^{1-\beta}}\sum_{i=1}^{n}\left(\frac{s_i x_i^{-\beta}}{1-s_i}\right)$$

$$+\frac{\lambda\beta}{\sigma\alpha^{1-\beta}}\sum_{i=1}^{n}\frac{x_i^{-\beta} s_i^{-1/\sigma}}{(1-s_i)^{1-\frac{1}{\sigma}}}$$

Let $y_i = (\frac{\alpha}{x_i})^\beta \log(\frac{\alpha}{x_i})$. Then we get

$$U_\beta = \frac{n}{\beta} + nlog\alpha - \sum_{i=1}^{n} logx_i + \frac{1}{\sigma}\sum_{i=1}^{n} y_i + (\frac{1}{\sigma}-1)\sum_{i=1}^{n}\frac{s_i y_i}{1-s_i} + \frac{\lambda}{\sigma}\sum_{i=1}^{n}\frac{s_i^{-1/\sigma} y_i}{(1-s_i)^{1-1/\sigma}}$$

$$U_\lambda = \frac{n}{\lambda} - \sum_{i=1}^{n}\left(\frac{s_i}{1-s_i}\right)^{-\frac{1}{\sigma}}$$

$$U_\sigma = -\frac{n}{\sigma} - \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{\alpha}{x_i}\right)^\beta - 1/\sigma^2\sum_{i=1}^{n}\log(1-s_i) - \frac{\lambda}{\sigma^2}\sum_{i=1}^{n}\left(\frac{s_i}{1-s_i}\right)^{-\frac{1}{\sigma}}\log(\frac{s_i}{1-s_i})$$

6. Application to a real data set

In this section, we prove the flexibility of this distribution with a real data set. We compare the distribution with Weibull-Frechet distribution introduced by Afify et al. (2016), a member of Weibull-X family.

The density function of Weibull- Frechet distribution (x >0) is given by

$$g(x) = ab\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-b\left(\frac{\alpha}{x}\right)^{\beta}\right]\{1-exp[-\left(\frac{\alpha}{x}\right)^{\beta}]\}^{-b-1}$$
$$\exp\left(-a\left\{exp[\left(\frac{\alpha}{x}\right)^{\beta}\right]-1\}^{-b}\right)$$

For this, we consider the data from Bjerkedal (1960).This data set consists of 64 observations on survival times of injected guinea pigs with different doses of tubercle bacilli:

Dataset:34,38,38,43,44,48,52,53,54,54,55,56,57,58,58,59,60,60,60,60,61,62,63,65,65,67,68,70,70,72,73,75,76,76,81,83,84,85,87, 91,95,96,98,99,109,110,121,127, 129, 131, 143, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376.

To compare the performance of the distributions, we consider $-logl_n$, AIC (Akaike information criterion), *CAIC* (consistent Akaike information criterion), BIC (Bayesian information criterion) and HQIC (Hannan- Quinn information criterion). These statistics are given by

- K-S statistic $D_n = sup|F(x) - F_n(x)|$ where $F_n(x)$ is the empirical distribution function
- AIC $= -2logl_n + 2k$
- CAIC$=-2logl_n + 2kn/(n-k-1)$
- BIC $= -2logl_n + klog((n))$
- HQIC$=-2logl_n + 2klog[log(n)]$

where $logl_n$ denotes the log likelihood function evaluated at the mles, k is the number of model parameters and n is the sample size. The model with lowest values for these statistics could be chosen as the best model to fit the data.

Table 1 gives the estimates and Table 2 provides the values of the above statistics.

Table 1: MLEs for the fitted model

Model	Parameters
GumbelFrechet	λ=18.6036, σ=4.4786, α=18.4642, β=10.0648
WeibullFrechet	a=0.0009,b=1.4825, α=1.1964, β= 1.0187

Table 2: The statistics $-logl_n$, K-S, AIC,CAIC, BIC and HQIC for the data set

Distribution	$-logl_n$	K-S	AIC	CAIC	BIC	HQIC	p-value
GumbelFrechet	332.6554	0.0784	673.3108	673.9896	672.5355	667.3657	0.8327
WeibullFrechet	349.5509	0.1805	707.1	707.7798	706.3265	701.1559	0.03284

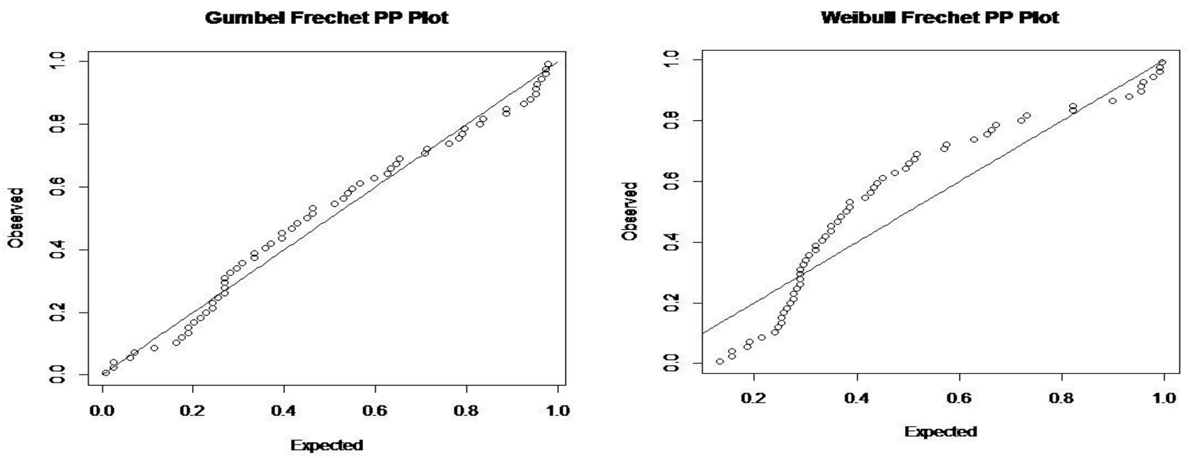

Fig.3: P-P plots for the dataset

7. Conclusion

In this paper, we introduced a new four-parameter model from Gumbel- X family called the Gumbel- *Fréchet* (GuFr) distribution. We provide some of its mathematical properties including quantile function, moments, incomplete moments, probability weighted moments, mean deviation about mean and median, moments of residual and reversed residual lifes, moment generating function and Renyi and δ entropies. The density function of GuFr distribution can be expressed as a mixture of *Fréchet* densities. We obtained the density function of order statistics. We discussed the maximum likelihood estimation of the model parameters. A real data set is used to illustrate the flexibility of the newly developed distribution and compare the result with an existing distribution.

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