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Co-equitable Resolving Sets of a Graph

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ABSTRACT
The concept of resolving sets and minimum resolving sets has appeared in the name of locating sets and reference sets [6,7]. Independently, F. Harary and R.A. Melter discovered these concepts but used the term metric dimension rather than location number. G. Chartrand and others introduced the term resolvability in graphs and the metric dimension in their paper in 2000. Several results have been found out in resolvability. Equitability was first introduced in coloring by W.Meyer[8]. Degree Equitability in graphs was proposed by E. Sampathkumar. Later, this concept was used in Domination[2]. An ordered subset $S = \{u_1, u_2, \dots, u_k\}$ of $V(G)$ of a connected graph G is called a resolving set if the representation of v with respect to S by $(d(v,u_1), d(v,u_2), \dots, d(v,u_k))$ is different for different v. The minimum cardinality of a resolving set in a connected graph G is called the metric dimension of G and is denoted by $dim(G)$. A resolving set S is called a complementary equitable resolving set if $V-S$ is degree equitable in G. This concept is introduced and studied in this paper.

KEYWORDS: Co-equitable Resolving sets.
Classification: 05C12, 05C69

Introduction 1.1 *Let $G = (V,E)$ be a simple graph. Let $S = \{u_1, u_2, \dots, u_k\}$ be an ordered subset of $V(G)$. To each v in $V(G)$, associate the k-vector $r(v \backslash S) = (d(v,u_1), d(v,u_2), \dots, d(v,u_k))$. If the associated k-vectors are distinct for distinct v, then S is called a resolving set. If $V-S$ is degree equitable in G (that is, $|deg(x) - deg(y)| \le 1$ for any $x, y \in V-S$, where the degree is with respect to G), then S is called a complementary equitable resolving set (or co-equitable resolving set). The minimum cardinality of a co-equitable resolving set of a graph G is called the co-equitable metric dimension of G and is denoted by $coeqdim(G)$. A study of this parameter and the concept is initiated in this paper.*

Definition 1.2 *Let $G = (V,E)$ be a connected graph. A Subset S of $V(G)$ is called a co-equitable resolving set of G if S is resolving and $V-S$ is degree equitable(That is, for any $u, v \in V-S, |deg(u) - deg(v)| \le 1$). The minimum cardinality of a co-equitable resolving set of G is called the co-equitable dimension of G and is denoted by $coeqdim(G)$.*

Remark 1.3 *(i) $V(G)$ is a co-equitable resolving set of G.*
(ii) The property of co-equitable resolvability is super hereditary.

Definition 1.4 *A double star is a graph obtained by taking two stars and joining the centers with an edge. If the stars are $K_{1,r}$ and $K_{1,s}$, then the double star obtained by joining the centers of $K_{1,r}$ and $K_{1,s}$ is denoted by $D_{r,s}$.*

$coeqdim(G)$ for some well known graphs

1. $coeqdim(K_n) = n-1$
2. $coeqdim(K_{1,n}) = n$
3. $coeqdim(P_n) = 1$
4. $coeqdim(C_n) = 2$
5. $coeqdim(K_{m,n}) = \begin{cases} m+n-2 & \text{if } |m-n| \le 1 \\ m+n-1 & \text{otherwise} \end{cases}$

6. $coeqdim(K_m(a_1,a_2,\dots,a_m))=\begin{cases}1 & \text{if } m=2, a_1=1, a_2=1\\ a_1+a_2+\cdots+a_m & otherwise\end{cases}$
7. $coeqdim(D_{r,s})=\begin{cases} r + s & if\ r\ or\ s \geq 2\\ 1 & if\ r = s = 1\end{cases}$
8. $coeqdim(K_{n_1,n_2,\dots,n_r})=n_1+n_2+\cdots+n_r$ if K of $n_1,n_2,\dots,n_r$ are equitable
9. $coeqdim(P) = dim(P) = 4$
10. $coeqdim(W_n)=\begin{cases}3\ if\ n=4\\ 4\ if\ n = 5\\ 3\ if\ n\geq 6\end{cases}$

Remark 1.5 $dim(G) \leq coeqdim(G)$

Observation 1.6 $coeqdim(G) = 1$ *if and only if* $G = P_n$.

Proof. Suppose $coeqdim(G) = 1$. Then $dim(G) = 1$. Therefore, $G = P_n$. The converse is obvious.

Theorem 1.7 *Let* G *be a connected graph of order* $n \geq 4$. $dim(G) = n-1$ *or* $n-2$. *Then* $coeqdim(G) = n-1$ *if and only if* $G = K_n$ *or* $K_{m,n}$ *or* $K_s + (K_1 \cup K_t)$ *or* $K_s + \overline{K_t}$.

Proof. Let G be a connected graph of order $n \geq 4$. Let $coeqdim(G) = n-1$. Let S be a minimum co-equitable resolving set of G. Then $|S| = n-1$. Suppose $dim(G) = n-1$. Then $G = K_n$. Otherwise $dim(G) \leq n-2$. Suppose $dim(G) = n-2$. Then $G = K_{m,n}$ or $K_s + \overline{K_t}(s \geq 1, t \geq 2)$ or $K_s + (K_1 \cup K_t), s,t \geq 1$. When $G = Km,n$, $coeqdim(G) = m+n-1$ if and only if $|m-n| \geq 2$. When $G = K_s + \overline{K_t}$. Then $coeqdim(G) = s+t-1$ if and only if $t \geq 3$. When $G = K_s + (K_1 \cup K_t)$, $coeqdim(G) = dim(G) = s+t-1$. When $dim(G) = n-2, coeqdim(G) = n-1$ if and only if $G = K_{m,n}$ with $|m-n| \geq 2$ or $G = K_s + \overline{K_t}$ with $t \geq 3, s \geq 1$.

Theorem 1.8 *Let* H *be a connected graph. Thencoeqdim*$(H) \leq coeqdim(H \square K_2) \leq 2coeqdim(H)$.

Proof. Let $G = H \square K_2$. Let $V(H) = \{u_1,u_2,\dots,u_n\}$.
Let $V(K_2) = \{V_1,V_2\}$. Let $V(G) = \{(u_1,v_1),(u_2,v_1),\dots,(u_n,v_1),(u_1,v_2),(u_2,v_2),\dots,(u_n,v_n)\}$.
Let $S = \{w_1,w_2,\dots,w_k\}$ be a minimum co-equitable resolving set of H.
Let $S_1 = \{(w_1,v_1),(w_2,v_1),\dots,(w_k,v_1),(w_1,v_2),(w_2,v_2),\dots,(w_k,v_2)\}$. Let $x,y \in V(G)-S_1$.
Case (i): Let $x = (x_i,v_1)$, $y = (x_j,v_1)$. Clearly, $x_i,x_j \notin S$. Therefore, x_i,x_j are resolved by some $w_r \in S$. Therefore, $d_H(x_i,w_r) \neq d_H(x_j,w_r)$. Therefore, $d_G((x_i,v_1),(w_r,v_1)) \neq d_G((x_j,v_1),(w_r,v_1))$. Therefore, $(x_i,v_1),(x_j,v_1)$ are resolved in G by $(w_r,v_1) \in S_1$. $deg_G(x_i,v_1) = deg_H(x_i)+1$, $deg_G(x_j,v_1) = deg_H(x_j)+1$. Therefore, $|deg_G(x_i,v_1) - deg_G(x_j,v_1)| = |deg_H(x_i) - deg_(x_j)| \leq 1$ since x_i,x_j are equitable in H. Therefore, S_1 resolves x,y and x,y are equitable in G.
Case (ii): Let $x = (x_i,v_1)$, $y = (x_j,v_2)$. Consider $d_G((x_i,v_1),(w_1,v_2)) = d_G((x_i,v_1),(w_1,v_1))+1 = d_H(x_i,w_1)+1$. $d_G((x_j,v_2),(w_1,v_2)) = d_H(x_j,w_1)$. Therefore, $d_G((x_i,v_1),(w_1,v_2)) \neq d_G((x_j,v_2),(w_1,v_2)))$. Therefore, (x_j,v_1) and (x_j,v_2) are resolved in G by (w_1,v_2) in S_1. Clearly $x_i,x_j \notin S$.
Therefore, $|deg_H(x_i) - deg_H(x_j)| \leq 1$. $|deg_G((x_i,v_1) - deg_G(x_j,v_2))| = |deg_G((x_i,v_1) - deg_G(x_j,v1)| = |deg_H(x_i) - deg_H(x_j)| \leq 1$. Therefore, (x_i,v_1) and (x_j,v_2) are degree equitable in G.
Case (iii): Let $x = (x_i,v_2)$, $y = (x_j,v_1)$. Arguing as in case (ii), we get that x and y are resolved in G by S_1.
Case (iv): Let $x = (x_i,v_2)$, $y = (x_j,v_2)$. Clearly $x_i,x_j \notin S$. There exists $w_r \in S$ such that $d_H(x_i,w_r) \neq d_H(x_j,w_r)$. $d_G((x_i,v_2),(w_r,v_2)) \neq d_G((x_j,v_2),(w_r,v_2))$. Therefore, (x_i,v_2) and (x_j,v_2) are resolved in G by S_1. Since $x_i,x_j \notin S$, $|deg_H(x_i) - deg_H(x_j)| \leq 1$. That is, $|deg_G(x_i,v_2) - deg_G(x_j,v_2)| = |deg_H(x_i) - deg_H(x_j)| \leq 1$. Therefore, (x_i,v_2) and (x_j,v_2) are degree equitable in G. Therefore, S_1 is a co-equitable resolving set of G. $coeqdim(G) = coeqdim(H \square K_2) \leq 2coeqdim(H)$.

Let $G = H \square K_2$. Let H_1 and H_2 be the copies of H in G. Let V_1 be the vertex set of H_1 and V_2 be the vertex set of H_2. Then $V(G) = V_1 \cup V_2$. Let W be a minimum co-equitable resolving set of G. Let $W_1 = W \cap V_1, W_2 = W \cap V_2$. Let $U_1 = W_1 \cup W_2^{'}$ where $W_2^{'}$ consists of those vertices of V_1 corresponding to W_2. $|U_1| = |W_1 \cup W_2^{'}| \leq |W_1| + |W^{'}| = |W| = coeqdim(G)$. Let $x,y \in V(H_1)$. Suppose $x,y \notin W_1 \cup W_2^{'}$. Let $x^{'} = (u_i,v_1)$, $y^{'} = (u_j,v_1)$. There exists $w \in W$ such that $d_G(x^{'},w) \neq d_G(y^{'},w)$. Let $w = (u_r,v_1)$. Therefore, $d_G(x^{'},w) \neq d_G(y^{'},w)$.

That is, $d_G((u_i,v_1),(u_r,v_1)) \neq d_G((u_j,v_1),(u_r,v_1))$. That is, $d_G(u_i,u_r) \neq d_G(u_j,u_r)$ and $u_r \in W_1$. Therefore, x,y

are resolved by W_1. Suppose $w = (u_r, v_2)$. Then $d_G(x', w) \neq d_G(y', w)$. $d_G((u_i, v_1), (u_r, v_2)) \neq d_G((u_j, v_1), (u_r, v_2))$. That is, $d_H(u_i, u_r) + 1 \neq d_H(u_j, u_r) + 1$. $d_H(u_i, u_r) \neq d_H(u_j, u_r)$. Therefore x and y are resolved in H_1 by U_1. Let $x' = (u_i, v_2), y' = (u_j, v_2)$. Then proceeding as in case (i), x and y are resolved in H_1 by U_1. $|deg_G(x') - deg_G(y')| = |deg_H(u_i) - deg_H(u_j)| \leq 1$. Therefore, x and y are degree equitable in H_1. Therefore, U_1 is a co-equitable resolving set of H_1.Therefore, $coeqdim(H_1) \leq |U_1| \leq |W| = coeqdim(H \square K_2)$.

Remark 1.9

(i) Let $\mathrm{H} = \mathrm{K}_3$, $\mathrm{coeqdim(H)} = \mathrm{dim(H)} = 2$, $\mathrm{coeqdim(H \square K_2)} = \mathrm{dim(H \square K_2)} = 2$.

(ii) Let $\mathrm{H} = \mathrm{D}_{3,5}$. $\mathrm{coeqdim(H)} = 8$, $\mathrm{coeqdim(H \square K_2)} = 16 = 2\mathrm{coeqdim(H)}$.

Remark 1.10 *Let* $G = K_{1,n}$. $coeqdim(G) = n, coeqdim(K_2) = 1$. *Therefore,* $\frac{coeqdim(G)}{coeqdim(K_2)} = n$ *and* K_2 *is an induced subgraph of* $K_{1,n}$. *Thus, the ratio* $\frac{coeqdim(G)}{coeqdim(H)}$ *can be made arbitrarily large when* H *is an induced subgraph of* G.

Remark 1.11 Let $n \geq 3$. Let $H = K_{1,2^{n+1}}$. Construct the graph G as in [1]. G is obtained by attaching two vertices x and x' and the $2n+1$ edges xv_i and $x'v_i'$, $1 \leq i, i' \leq 2^n$ where v_i, v_i' are the product vertices of H. Add two sets $W = \{w_1, w_2, \ldots, w_n\}$ and $W' = \{w_1', w_2', \ldots, w_n'\}$ of vertices together with edges $w_i x, w_i' x'$, $1 \leq i \leq n$. Finally, add edges between W and $\{v_1, v_2, \ldots, v_{2^n}\}$ and edges between W' and $\{v_1', v_2', \ldots, v_{2^n}'\}$. The resulting graph is denoted by G. $coeqdim(H) = 2^{n+1}$, $coeqdim(G) \leq 2n + 7$. H is an induced subbgraph of G. Therefore, $\frac{coeqdim(G)}{coeqdim(H)} = \frac{2n+7}{2^{n+1}} \to 0$ as $n \to \infty$. Hence we have the following result.

Theorem 1.12 *For every* $\varepsilon > 0$, *there exists a connected graph* G *and a connected induced subgraph* H *such that* $\frac{coeqdim(G)}{coeqdim(H)} < \varepsilon$.

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