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The Approximation Properties of a Type of Current Basis Functions

Yijun Yu[1], Nailong Guo[2]

[1]Department of Mathematics, Tuskegee University, AL, USA
[2]Department of Mathematics and Computer Science, Benedict College, SC, USA

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Corresponding Author:
Yijun Yu

ABSTRACT
In an integral equation formulation, the electromagnetic fields are expressed in terms of surface currents. The numerical calculation of the integral equation, in many cases, needs a vector basis with continuity of the normal components across the interfaces among adjacent elements. This paper provides and proves the approximation properties of RWG basis function and extended high order basis functions.

KEYWORDS: Basis function, Approximation property, Integral equation, Numerical calculation

I. INTRODUCTION

Various numerical techniques have been developed to carry out the electromagnetic field simulation [1-4]. The integral equation formulation of electromagnetic scattering of conductive surfaces is a very popular approach for many applications [5].

In an integral equation formulation, the electromagnetic fields are expressed in terms of surface currents. To represent the current vector field over conductor's surfaces, in many cases it is important to have a vector basis with continuity of the normal components across the interfaces among adjacent elements. The RWG basis function [6] is the most used basis function with such a property for scattering calculation. The high order basis functions derived in [7] have same property, which are higher order while the lowest order, over flat triangular patches, reduces to the usual RWG basis function. In this paper we will provide and prove the approximation properties of RWG basis function and the high order basis functions.

The remainder of the paper is organized as follows. Sec. II presents the construction of the triangular basis functions. In Sec. III we provide a complete proof of the approximation properties of RWG basis function and the high order basis functions. Sec. IV contains the conclusion of the paper.

II. BASIS FUNCTIONS

A. RWG Basis Function

Let S be the surface of a scatter and $\mathcal{T}_h$ be a flat triangulation of S. Each RWG basis function [6] is associated with an interior edge. Fig. 1 shows two such triangles, T_n^+ and T_n^-, corresponding to the nth edge in triangulation $\mathcal{T}_h$. Points in T_n^+ can be designated by the position vector $\boldsymbol{\rho}_n^+$ defined with respect to the free vertex of T_n^+. Similar remarks apply to the position vector $\boldsymbol{\rho}_n^-$ except that it is directed toward the free vertex of T_n^- ,The plus or minus designation of the triangles is determined by the choice of a positive current reference direction for the nth, for which is assumed to be from T_n^+ to T_n^-.

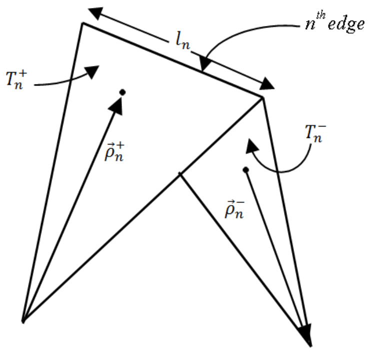

Figure 1: Triangle pair and geometrical parameters associated with interior edge.

Then the RWG vector basis function associated with the nth edge is defined as

$$\mathbf{f}_n(\mathbf{x}) = \begin{cases} \frac{l_n}{2A_n^+}\boldsymbol{\rho}_n^+, & \mathbf{x} \text{ in } T_n^+ \\ \frac{l_n}{2A_n^-}\boldsymbol{\rho}_n^-, & \mathbf{x} \text{ in } T_n^- \\ \mathbf{0}, & \text{otherwise} \end{cases} \quad (2.1)$$

where l_n is the length of the edge and $A_n^\pm$ is the area of triangle $T_n^\pm$.

The current basis function (2.1) has no component normal to the boundary of surface formed by the triangle pair T_n^+ and T_n^-. The component of current normal to the nth edge is constant and continuous across the edge.

B. High Order Basis Functions

The RWG basis function is a popular basis in numerical calculation of electromagnetic scattering problem due to its simplicity. But its approximation order is low. The RWG basis was extended in [7] to high order basis functions over arbitrary curved triangular patches, as defined below.

Let Σ be a curved triangular surface in $\mathcal{R}^3$ and Σ is parameterized by $\mathbf{x}=\mathbf{x}(u_1,u_2)$, $(u_1,u_2)\in T_{sr}$, T_{sr} is a standard reference triangle in (u_1,u_2) plane (see Fig. 2)

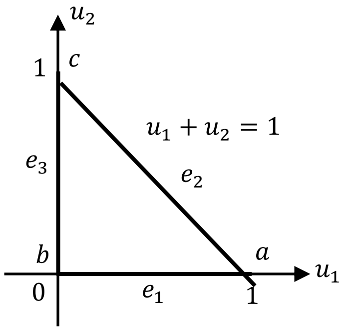

Figure 2: Standard reference triangle T_{sr}

Tangential vectors: $\partial_i\mathbf{x}$, $i=1,2$, are defined as

$$\partial_i\mathbf{x}=\frac{\partial\mathbf{x}}{\partial u_i}\qquad i=1,2 \qquad (2.2)$$

Metric Tensor: The distance between two points on Σ parameterized by (u_1,u_2) and (u_1+du_1,u_2+du_2) is given by

$$(ds)^2=g_{\mu\nu}(u)du_\mu du_\nu \qquad (2.3)$$

where repeated indices imply summation

$$g_{\mu\nu}=\partial_\mu\mathbf{x}\cdot\partial_\nu\mathbf{x}\qquad 1\le\mu,\nu\le 2(2.4)$$

The determinant of $\{g_{\mu\nu}\}$ is denoted by

$$g=\det\{g_{\mu\nu}\}=g_{11}g_{22}-g_{12}^2=\|\partial_1\mathbf{x}\times\partial_2\mathbf{x}\|^2 \quad (2.5)$$

Let T_{sr} be the reference triangle with vertices *a*, *b*, *c* in Fig.2. The polynomials with variables u_1 and u_2, $(u_1,u_2)\in T_{sr}$, can be grouped into three modes: vertex modes, edge modes and internal modes [8].

Vertex modes:

$$g_a(u_1,u_2)=u_1$$
$$g_b(u_1,u_2)=1-u_1-u_2 \qquad (2.6)$$
$$g_c(u_1,u_2)=u_2$$

Each vertex mode will take value 1 at one vertex and zero at other two vertices.

Edge modes: $2\le l\le M$

$$g_l^{ab}(u_1,u_2)=g_a(u_1,u_2)g_b(u_1,u_2)p_{l-2}(g_b-g_a)$$
$$g_l^{bc}(u_1,u_2)=g_b(u_1,u_2)g_c(u_1,u_2)p_{l-2}(g_c-g_b)(2.7)$$
$$g_l^{ca}(u_1,u_2)=g_c(u_1,u_2)g_a(u_1,u_2)p_{l-2}(g_a-g_c)$$

where $p_l(\xi),\xi\in[-1,1]$, is *l*-th order Legendre polynomial.

Each of the edge modes is only nonzero along one edge of the triangle T_{sr}

Internal modes: $0\le k+l\le M-3$

$$g_{l,k}^{int}(u_1,u_2)$$
$$=g_a(u_1,u_2)g_b(u_1,u_2)g_c(u_1,u_2)p_k(2g_c-1)p_l(g_b-g_a) \qquad (2.8)$$

Each of the internal mode will vanish over all edges ofT_{sr}.

Consider two curved triangular patches T^+ and T^- with a common interface *AC* with length ℓ in Fig. 3. Let T^+ and T^-be parameterized, respectively, by

$$\mathbf{x}=\mathbf{x}^+(u_1,u_2):\ T_{sr}\to T^+ \qquad (2.9)$$

$$\mathbf{x}=\mathbf{x}^-(u_1,u_2):\ T_{sr}\to T^- \qquad (2.10)$$

We assume that the interface *AC* in both T^+ and T^- is parameterized by $u_1+u_2=1$ and is labelled as side e_2^+ in T^+ and side e_2^-in T^-.

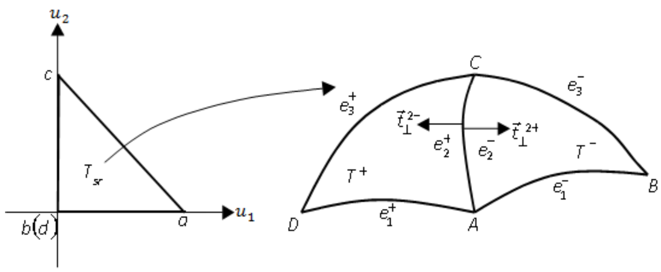

Figure 3: Left: standard reference triangleT_{sr}; Right: curved triangular patches

Then the *M*-th order basis function for a triangular patch in Fig. 3 can be defined as [7]

$$\mathbf{f}(\mathbf{x})=\begin{cases}\dfrac{\ell}{\sqrt{g^+}}(P_1^+(u_1,u_2)\partial_1\mathbf{x}+P_2^+(u_1,u_2)\partial_2\mathbf{x})\\ \qquad\text{if}\mathbf{x}=\mathbf{x}^+(u_1,u_2)\in T^+\\ \dfrac{\ell}{\sqrt{g^-}}(P_1^-(u_1,u_2)\partial_1\mathbf{x}+P_2^-(u_1,u_2)\partial_2\mathbf{x})\\ \qquad\text{if}\mathbf{x}=\mathbf{x}^-(u_1,u_2)\in T^-\end{cases} \qquad (2.11)$$

Based on condition

$$\mathbf{f}^+\cdot\mathbf{t}_\perp^{2+}=-\mathbf{f}^-\cdot\mathbf{t}_\perp^{2-}\text{on}AC$$

$P_1^+(u_1,u_2)$, $P_2^+(u_1,u_2)$, $P_1^-(u_1,u_2)$ and $P_2^-(u_1,u_2)$ can be derived as

$$P_1^+(u_1,u_2)=I_n^a g_A(u_1,u_2)+\sum_{m=2}^{M}\frac{I_n^{(m)}-I_t^{(m)}}{2}g_m^{ca}(u_1,u_2)$$
$$+\sum_{(l,m)\in\mathcal{L}_\Delta}c_{lm}^1 g_{lm}^{int}$$
$$P_2^+(u_1,u_2)=I_n^c g_C(u_1,u_2)+\sum_{m=2}^{M}\frac{I_n^{(m)}+I_t^{(m)}}{2}g_m^{ca}(u_1,u_2)$$
$$+\sum_{(l,m)\in\mathcal{L}_\Delta}c_{lm}^2 g_{lm}^{int}$$

(2.12)

and

$$P_1^-(u_1,u_2) = -I_n^a g_A(u_1,u_2) + \sum_{m=2}^{M} \frac{-I_n^{(m)} - \hat{I}_t^{(m)}}{2} g_m^{ca}(u_1,u_2) + \sum_{(l,m)\in\mathcal{L}_\Delta} d_{lm}^1 g_{lm}^{int}$$

$$P_2^-(u_1,u_2) = -I_n^c g_C(u_1,u_2) + \sum_{m=2}^{M} \frac{-I_n^{(m)} + \hat{I}_t^{(m)}}{2} g_m^{ca}(u_1,u_2) + \sum_{(l,m)\in\mathcal{L}_\Delta} d_{lm}^2 g_{lm}^{int} \quad (2.13)$$

with

$$\mathcal{L}_\Delta = \{(l,m), 0 \le l+m \le M-3\} \quad (2.14)$$

Unknowns for each edge *AC* are

$$I_n^a, I_n^c, I_n^{(m)}, \tilde{I}_t^{(m)}, \hat{I}_t^{(m)}, \quad 2 \le m \le M \quad (2.15)$$

and interior unknowns are

$$c_{lm}^1, c_{lm}^2, d_{lm}^1, d_{lm}^2, \qquad (l,m) \in \mathcal{L}_\Delta \quad (2.16)$$

The high order basis function (2.11) with (2.12) and (2.13) has no component normal to *AB*, *AD*, *CB* and *CD*, and component normal to *AC* is continuous across the edge.

If T^+ and T^- are flat triangular patches, and we take $M = 1$, $I_n^a = I_n^c = I_n$, then basis function (2.11) reduces to usual RWG basis function.

III. Approximation Properties of Basis Functions

In this section, we will analyse the approximation properties of RWG and high order basis functions on a plane polygonal region.

Let Ω be a plane polygonal region with boundary Γ. We introduce the following normed spaces

$$H(div;\Omega) = \left\{\mathbf{u} \in \left(L^2(\Omega)\right)^2 | Div\mathbf{u} \in L^2(\Omega)\right\}$$

withnorm

$$\|\mathbf{u}\|_{H(div;\Omega)} = \|\mathbf{u}\|_{(L^2(\Omega))^2} + \|Div\mathbf{u}\|_{L^2(\Omega)}$$

and

$$H_0(div;\Omega) = \{\mathbf{u} \in H(div;\Omega) | \mathbf{u}\cdot\mathbf{n} = 0 \text{ on} \Gamma\}$$

Where **n** is the unit outward normal vector to Ω.

Let $\mathcal{T}_h$ be a regular triangulation with maximum diameter h over the $\bar{\Omega}$. We have $\bar{\Omega} = \cup_{K\in\mathcal{T}_h} \bar{K}$.

For $K \in \mathcal{T}_h$, let $\mathbf{a}_i, 1 \le i \le 3$, be the vertexes of K, $\mathbf{a}_{ij}, 1 \le i < j \le 3$, be the mid-points of the edges, $\mathbf{n}_{ij}$ and $\mathbf{t}_{ij}$ be the unit outward normal vector and unit tangent vector, respectively (see Fig. 4).

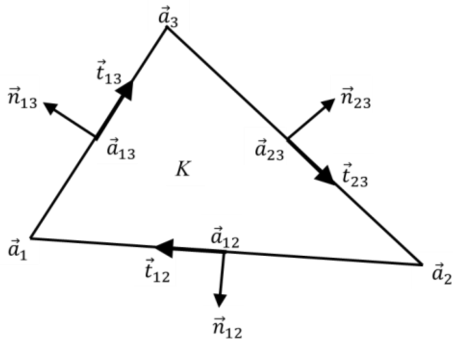

Figure 4: Triangle K on a plane

A. *Approximation Properties of RWG Basis Functions*

Let

$$\bar{P}_1(K) = \left\{\mathbf{p}(\mathbf{x}) \in \left(P_1(K)\right)^2 | \mathbf{p}_{\overline{\mathbf{a}_i\mathbf{a}_j}}(\mathbf{x})\cdot\mathbf{n}_{ij} = const, 1 \le i < j \le 3\right\}$$

where $P_1(K)$ is formed by the restrictions to the K of all polynomials of degree ≤ 1, then $\dim\left(\bar{P}_1(K)\right) = 3$.

We introduce finite element

$$(K, \bar{P}_1(K), \textstyle\sum_K) \quad (3.1)$$

where $\sum_K = \left\{\mathbf{p}(\mathbf{a}_{ij})\cdot\mathbf{n}_{ij}|\mathbf{a}_i - \mathbf{a}_j|, 1 \le i < j \le 3\right\}$.

For $\mathbf{u} \in (C(\bar{\Omega}))^2$, we can define a unique interpolation $\Pi\mathbf{u}$ such that for all $K \in \mathcal{T}_h$,

$$(\Pi\mathbf{u})|_K \in \bar{P}_1(K) \text{and } \alpha_i(\mathbf{u} - \Pi\mathbf{u}) = 0, \forall\alpha_i \in \Sigma_K. \quad (3.2)$$

From the definition, it is directly deduced that finite element (3.1) is conforming in $H(div;\Omega)$.

Now we can define the finite-dimensional subspace S_h of $H(div;\Omega)$

$$S_h = \{\mathbf{v}(\mathbf{x}) | \mathbf{v}_{|K}(\mathbf{x}) \in \bar{P}_1(K), K \in \mathcal{T}_h; \mathbf{v}_{|K_1}(\mathbf{x})\cdot\mathbf{n}_1 = -\mathbf{v}_{|K_2}(\mathbf{x})\cdot\mathbf{n}_2 \text{ on } \bar{K}_1 \cap \bar{K}_2, K_1, K_2 \in \mathcal{T}_h \text{and} \bar{K}_1 \cap \bar{K}_2 \neq \phi\} \quad (3.3)$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ are the unit outward normal vectors to K_1 and K_2 along their common edge, respectively, and the finite-dimensional subspace S_{h_0} of $H_0(div;\Omega)$

$$S_{h_0} = \{\mathbf{v}(\mathbf{x}) | \mathbf{v}(\mathbf{x}) \in S_h, \mathbf{v}(\mathbf{x})\cdot\mathbf{n} = 0 \text{ on} \Gamma\} \quad (3.4)$$

where n is the unit outward normal vector to Ω. RWG basis constructed in [6] is a basis of S_h.

Let family of finite elements $\{(K, \bar{P}_1(K), \sum_K) | K \in \mathcal{T}_h\}$ be regular affine, the corresponding reference finite element be $(\hat{K}, \bar{P}_1(\hat{K}), \sum_{\hat{K}})$ and affine transformation be

$$\mathbf{x} = F_K(\hat{\mathbf{x}}) = B_K\hat{\mathbf{x}} + \mathbf{b}_K \quad (3.5)$$

with B_K an invertible 2×2 matrix and $\mathbf{b}_K$ a vector of $\mathcal{R}^2$. We use $\mathbf{u}$ and $\hat{\mathbf{u}}$ to denote the corresponding functions, respectively, and define the transformation

$$\mathbf{u} = (A^\intercal)^{-1}(B_K^\intercal)^{-1}A^\intercal\hat{\mathbf{u}} \quad (3.6)$$

where $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

In the following, we will use C to indicate various constants not necessarily the same in all instances, use $\sum_{i,j}\alpha_{i,j}$ to express $\sum_{1\le i<j\le 3}\alpha_{i,j}$ and $(\Pi\mathbf{u})|_K=\Pi_K\mathbf{u}$ for all $K\in\mathcal{T}_h$.

Lemma 3.1 *For $\mathbf{u}\in(C(\bar{K}))^2$, if we use formula (3.6) to transform the vector function $\mathbf{u}$, then we have*

$$\widehat{\Pi_K\mathbf{u}}=\hat{\Pi}\hat{\mathbf{u}}$$

where $\hat{\Pi}$ is corresponding interpolation operator on $\hat{K}$.

Proof Let $\xi_{ij}=\mathbf{a}_i-\mathbf{a}_j$, $\hat{\xi}_{ij}=\hat{\mathbf{a}}_i-\hat{\mathbf{a}}_j$,
$(j=i+1,\ 1\le i\le 3,\ \mathbf{a}_4=\mathbf{a}_1, \hat{\mathbf{a}}_4=\hat{\mathbf{a}}_1)$,
then

$$\xi_{ij}=B_K\hat{\xi}_{ij}\text{and }\ \mathbf{n}_{ij}=A\frac{\xi_{ij}}{|\xi_{ij}|},\hat{\mathbf{n}}_{ij}=A\frac{\hat{\xi}_{ij}}{|\hat{\xi}_{ij}|}.$$

By definition, we have

$$\Pi_K\mathbf{u}\in\bar{P}_1(K),\qquad \hat{\Pi}\hat{\mathbf{u}}\ \in\bar{P}_1(\hat{K})$$

and

$$\widehat{\Pi_K\mathbf{u}}=(A^\intercal)^{-1}B_K^\intercal A^\intercal\Pi_K\mathbf{u},$$

$$\hat{\mathbf{u}}=(A^\intercal)^{-1}B_K^\intercal A^\intercal\mathbf{u}.$$

Now we have

$$\widehat{\Pi_K\mathbf{u}}_{|\hat{\mathbf{a}}_i\hat{\mathbf{a}}_j}\cdot\hat{\mathbf{n}}_{ij}|\hat{\xi}_{ij}|=(\Pi_K\mathbf{u})^\intercal_{|\mathbf{a}_i\mathbf{a}_j}AB_KA^{-1}A\hat{\xi}_{ij}$$

$$=(\Pi_K\mathbf{u})^\intercal_{|\mathbf{a}_i\mathbf{a}_j}A\xi_{ij}$$

$$=(\Pi_K\mathbf{u})_{|\mathbf{a}_i\mathbf{a}_j}\cdot\mathbf{n}_{ij}|\xi_{ij}|$$

$$=\mathbf{u}(\mathbf{a}_{ij})\cdot\mathbf{n}_{ij}|\xi_{ij}|. \qquad (3.7)$$

By this equality, we can get $\widehat{\Pi_K\mathbf{u}}\in\bar{P}_1(\hat{K})$.

In addition, we have

$$\hat{\Pi}\hat{\mathbf{u}}(\hat{\mathbf{a}}_{ij})\cdot\hat{\mathbf{n}}_{ij}|\hat{\xi}_{ij}|=\hat{\mathbf{u}}(\hat{\mathbf{a}}_{ij})\cdot\hat{\mathbf{n}}_{ij}|\hat{\xi}_{ij}|$$

$$=\mathbf{u}^\intercal(\mathbf{a}_{ij})AB_KA^{-1}A\hat{\xi}_{ij}$$

$$=\mathbf{u}^\intercal(\mathbf{a}_{ij})A\xi_{ij}$$

$$=\mathbf{u}(\mathbf{a}_{ij})\cdot\mathbf{n}_{ij}|\xi_{ij}|. \qquad (3.8)$$

By (3.7) and (3.8), we obtain $\alpha_i(\widehat{\Pi_K\mathbf{u}}-\hat{\Pi}\hat{\mathbf{u}})=0,\forall\alpha_i\in\Sigma_{\hat{K}}$, so we have conclusion $\widehat{\Pi_K\mathbf{u}}=\hat{\Pi}\hat{\mathbf{u}}$.■

From the proof of Lemma 3.1 above, we also obtain that $\mathbf{u}\in\bar{P}_1(K)$ if and only if $\hat{\mathbf{u}}\in\bar{P}_1(\hat{K})$.

Let

$$\mathbf{a}_{iij}=\frac{1}{3}(2\mathbf{a}_i+\mathbf{a}_j),\ \ \hat{\mathbf{a}}_{iij}=\frac{1}{3}(2\hat{\mathbf{a}}_i+\hat{\mathbf{a}}_j),$$

$i\ne j, 1\le i,j\le 3$.

We use smooth curves to connect the points $\mathbf{a}_{iij}$ and $\mathbf{a}_{iik}$, $\hat{\mathbf{a}}_{iij}$ and $\hat{\mathbf{a}}_{iik}$, $j\ne k$, respectively. The new regions are denoted by K_s and $\hat{K}_s$, respectively. (see Fig. 5).

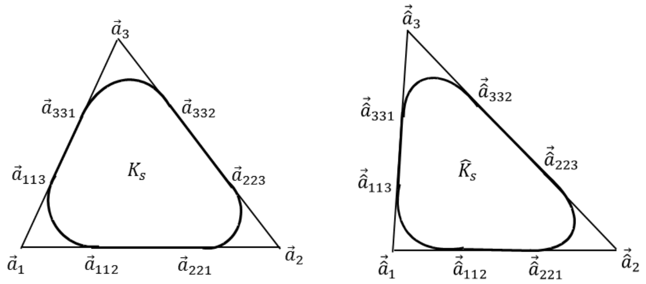

Figure 5: Triangles and corresponding regions with smooth boundary

Lemma 3.2 *There exists a constant $C(\hat{K})$ such that* $\forall\,\mathbf{u}\in\left(H^2(\hat{K})\right)^2$,

$$\inf_{\mathbf{p}\in\bar{P}_1(\hat{K})}\|\mathbf{u}+\mathbf{p}\|_{\left(H^2(\hat{K})\right)^2}$$

$$\le C(\hat{K})\left[|\mathbf{u}|_{\left(H^2(\hat{K})\right)^2}+\sum_{i,j}\left(\|\partial_{\hat{x}_1}\mathbf{u}\|_{\left(L^2(\hat{l}_{ij})\right)^2}\right.\right.$$

$$\left.\left.+\|\partial_{\hat{x}_2}\mathbf{u}\|_{\left(L^2(\hat{l}_{ij})\right)^2}\right)\right].(3.9)$$

where $\hat{l}_{ij}=\overline{\hat{\mathbf{a}}_{iij}\hat{\mathbf{a}}_{jji}}$, $1\le i<j\le 3$.

Proof $\bar{P}_1(\hat{K})$ with the norm of $\left(H^2(\hat{K})\right)^2$ is a normed space and $\dim\left(\bar{P}_1(\hat{K})\right)=3$. Let $f_i, 1\le i\le 3$, be a basis of the dual space of $\bar{P}_1(\hat{K})$. By the Hahn-Banach extension theorem, there exist continuous linear forms over the space $\left(H^2(\hat{K})\right)^2$, again denoted by $f_i, 1\le i\le 3$, such that for any $\mathbf{p}\in\bar{P}_1(\hat{K})$, we have $f_i(\mathbf{p})=0, 1\le i\le 3$, if and only if $\mathbf{p}=0$.

By Sobolev embedding theorem [9], we have

$$\left(H^1(\hat{K})\right)^2\hookrightarrow\left(H^1(\hat{K}_s)\right)^2\hookrightarrow\left(H^{1/2}(\partial\hat{K}_s)\right)^2\hookrightarrow\left(L^2(\partial\hat{K}_s)\right)^2, \qquad (3.10)$$

so

$$\partial_{x_1}\mathbf{u},\ \partial_{x_2}\mathbf{u}\in\left(L^2(\hat{l}_{ij})\right)^2,\quad \forall\,\mathbf{u}\in\left(H^2(\hat{K})\right)^2.$$

Now we show that there exists a constant $C(\hat{K})$ such that $\forall\,\mathbf{u}\in\left(H^2(\hat{K})\right)^2$,

$$\|\mathbf{u}\|_{\left(H^2(\hat{K})\right)^2}\le C(\hat{K})\left(|\mathbf{u}|_{\left(H^2(\hat{K})\right)^2}\right.$$

$$\left.+\sum_{i,j}\left\|\partial_{\mathbf{t}_{ij}}\mathbf{u}\cdot\hat{\mathbf{n}}_{ij}\right\|_{L^2(\hat{l}_{ij})}+\sum_{i=1}^3|f_i(\mathbf{u})|\right).(3.11)$$

If it is true, then inequality (3.9) will be a consequence of inequality (3.11). In fact, given any function $\mathbf{u}\in\left(H^2(\hat{K})\right)^2$, let $\mathbf{q}\in\bar{P}_1(\hat{K})$ be such that $f_i(\mathbf{u}+\mathbf{q})=0, 1\le i\le 3$. Then by (3.11),

$$\inf_{\mathbf{p}\in\bar{P}_1(\hat{K})}\|\mathbf{u}+\mathbf{p}\|_{\left(H^2(\hat{K})\right)^2}$$

$$\leq \|\mathbf{u}+\mathbf{q}\|_{\left(H^2(\hat{K})\right)^2}$$

$$\leq C(\hat{K})\left(|\mathbf{u}|_{\left(H^2(\hat{K})\right)^2}+\sum_{i,j}\left\|\partial_{\mathbf{t}_{ij}}\mathbf{u}\cdot\hat{\mathbf{n}}_{ij}\right\|_{L^2(\hat{l}_{ij})}\right)$$

$$\leq C(\hat{K})\left[|\mathbf{u}|_{\left(H^2(\hat{K})\right)^2}+\sum_{i,j}\left(\|\partial_{\hat{x}_1}\mathbf{u}\|_{\left(L^2(\hat{l}_{ij})\right)^2}\right.\right.$$

$$\left.\left.+\|\partial_{\hat{x}_2}\mathbf{u}\|_{\left(L^2(\hat{l}_{ij})\right)^2}\right)\right].$$

Suppose inequality (3.11) is false, then there exists a sequence $\{\mathbf{u}_l\}_{l=1}^{\infty}$ of functions $\mathbf{u}_l\in\left(H^2(\hat{K})\right)^2$ such that

$$\forall\, l\geq 1,\quad \|\mathbf{u}_l\|_{\left(H^2(\hat{K})\right)^2}=1$$

and

$$\lim_{l\to\infty}\left(|\mathbf{u}_l|_{\left(H^2(\hat{K})\right)^2}+\sum_{i,j}\left\|\partial_{\mathbf{t}_{ij}}\mathbf{u}_l\cdot\hat{\mathbf{n}}_{ij}\right\|_{L^2(\hat{l}_{ij})}+\sum_{i=1}^{3}|f_i(\mathbf{u}_l)|\right)$$

$= 0.$(3.12)

Since the sequence $\{\mathbf{u}_l\}_{l=1}^{\infty}$ is bounded in $\left(H^2(\hat{K})\right)^2$, by Rellich-Kondrachov theorem [9], there exists a subsequence, again denoted by $\{\mathbf{u}_l\}_{l=1}^{\infty}$, and function $\mathbf{u}\in\left(H^1(\hat{K})\right)^2$, such that

$$\lim_{l\to\infty}\|\mathbf{u}_l-\mathbf{u}\|_{\left(H^1(\hat{K})\right)^2}=0.\quad (3.13)$$

By (3.12), we have

$$\lim_{l\to\infty}|\mathbf{u}_l|_{\left(H^2(\hat{K})\right)^2}=0.\quad (3.14)$$

Since the space $\left(H^2(\hat{K})\right)^2$ is complete, we conclude from (3.13) and (3.14) that the sequence $\{\mathbf{u}_l\}_{l=1}^{\infty}$ converges in the space $\left(H^2(\hat{K})\right)^2$. The limit $\mathbf{u}$of this sequence is such that

$$\forall\ \alpha \text{ with } |\alpha|=2,$$

$$\|\partial^{\alpha}\mathbf{u}\|_{\left(L^2(\hat{K})\right)^2}=\lim_{l\to\infty}\|\partial^{\alpha}\mathbf{u}_l\|_{\left(L^2(\hat{K})\right)^2}=0$$

and thus $\partial^{\alpha}\mathbf{u}=0$ for all α with $|\alpha|=2$. Since $\hat{K}$ is connected, we get $\mathbf{u}\in\left(P_1(\hat{K})\right)^2$. Using (3.10), we have

$$\left\|\partial_{\mathbf{t}_{ij}}\mathbf{u}_l\cdot\hat{\mathbf{n}}_{ij}-\partial_{\mathbf{t}_{ij}}\mathbf{u}\cdot\hat{\mathbf{n}}_{ij}\right\|_{L^2(\hat{l}_{ij})}$$

$$\leq C(\hat{K})\left(\|\partial_{x_1}(\mathbf{u}_l-\mathbf{u})\|_{\left(L^2(\partial\hat{K}_s)\right)^2}\right.$$

$$\left.+\|\partial_{x_2}(\mathbf{u}_l-\mathbf{u})\|_{\left(L^2(\partial\hat{K}_s)\right)^2}\right)$$

$$\leq C(\hat{K})\|\mathbf{u}_l-\mathbf{u}\|_{\left(H^2(\hat{K})\right)^2}.(3.15)$$

By (3.12)

$$\lim_{l\to\infty}\sum_{i,j}\left\|\partial_{\mathbf{t}_{ij}}\mathbf{u}_l\cdot\hat{\mathbf{n}}_{ij}\right\|_{L^2(\hat{l}_{ij})}=0$$

and thus we conclude from (3.15) that

$$\sum_{i,j}\left\|\partial_{\mathbf{t}_{ij}}\mathbf{u}\cdot\hat{\mathbf{n}}_{ij}\right\|_{L^2(\hat{l}_{ij})}$$

$$=\lim_{l\to\infty}\sum_{i,j}\left\|\partial_{\mathbf{t}_{ij}}\mathbf{u}_l\cdot\hat{\mathbf{n}}_{ij}\right\|_{L^2(\hat{l}_{ij})}=0\,.$$

So

$$\partial_{\mathbf{t}_{ij}}\mathbf{u}\cdot\hat{\mathbf{n}}_{ij}=0\quad \text{on} \hat{l}_{ij},\quad 1\leq i<j\leq 3,$$

and thus

$$\mathbf{u}\cdot\hat{\mathbf{n}}_{ij}=const\quad \text{on}\hat{l}_{ij},\quad 1\leq i<j\leq 3.$$

Since $\mathbf{u}\in\left(P_1(\hat{K})\right)^2$, we obtain

$$\mathbf{u}\cdot\hat{\mathbf{n}}_{ij}=const\quad \text{on}\overline{\hat{\mathbf{a}}_i\hat{\mathbf{a}}_j},\quad 1\leq i<j\leq 3,$$

and therefore $\mathbf{u}\in\bar{P}_1(\hat{K})$.

Using (3.12), we have

$$f_i(\mathbf{u})=\lim_{l\to\infty}f_i(\mathbf{u}_l)=0,\quad 1\leq i\leq 3,$$

so that we conclude that $\mathbf{u}=0$ from the properties of the linear form f_i. But this contradicts the $\|\mathbf{u}_l\|_{\left(H^2(\hat{K})\right)^2}=1$ for all l. ■

Let $\hat{\mathbf{p}}_{ij}(\hat{\mathbf{x}}), 1\leq i<j\leq 3$, be basis functions of $\bar{P}_1(\hat{K})$, and satisfy

$\hat{\mathbf{p}}_{ij}(\hat{\mathbf{a}}_{kl})\cdot\hat{\mathbf{n}}_{kl}|\hat{\mathbf{a}}_k-\hat{\mathbf{a}}_l|=\delta_{ik}\delta_{jl}$,

$1\leq i<j\leq 3,\ \ 1\leq k<l\leq 3$,

then

$\hat{\Pi}\hat{\mathbf{u}}=\sum_{ij}\hat{\mathbf{p}}_{ij}\cdot\hat{\mathbf{n}}_{ij}|\hat{\mathbf{a}}_i-\hat{\mathbf{a}}_j|\hat{\mathbf{u}}\,(\hat{\mathbf{a}}_{ij})$.

By Sobolev embedding theorem [9]

$H^2(\hat{K})\hookrightarrow C(\bar{\hat{K}})$,

It is easy to check that we have

$\hat{\Pi}\in\mathcal{L}\left(\left(H^2(\hat{K})\right)^2;\left(H^1(\hat{K})\right)^2\right)$.(3.16)

Theorem 3.3*Let* Π *be the interpolation operator defined by (3.2). Then for all* $\mathbf{u}\in\left(H^2(K)\right)^2$, *we have*

$$\|\mathbf{u}-\Pi_K\mathbf{u}\|_{\left(L^2(K)\right)^2}\leq Ch\left(|\mathbf{u}|_{\left(H^1(K)\right)^2}+|\mathbf{u}|_{\left(H^2(K)\right)^2}\right)$$

(3.17)

$$\|Div(\mathbf{u}-\Pi_K\mathbf{u})\|_{L^2(K)}\leq Ch|\mathbf{u}|_{\left(H^2(K)\right)^2}(3.18)$$

where C is a constant independent of h.

Proof Since $\hat{\Pi}\mathbf{p}=\mathbf{p},\forall\,\mathbf{p}\in\bar{P}_1(\hat{K})$, we have

$\hat{\mathbf{u}}-\hat{\Pi}\hat{\mathbf{u}}=(I-\hat{\Pi})(\hat{\mathbf{u}}+\mathbf{p})$,

$\forall\,\hat{\mathbf{u}}\in\left(H^2(\hat{K})\right)^2,\forall\,\mathbf{p}\in\bar{P}_1(\hat{K})$.

By Lemma 3.2, we get

$$\|\hat{\mathbf{u}} - \hat{\Pi}\hat{\mathbf{u}}\|_{(L^2(\hat{K}))^2}$$

$$\leq \|I - \hat{\Pi}\|_{\mathcal{L}\left((H^2(\hat{K}))^2;(H^1(\hat{K}))^2\right)} \inf_{\mathbf{p}\in \bar{P}_1(\hat{K})} \|\hat{\mathbf{u}} + \mathbf{p}\|_{(H^2(\hat{K}))^2}$$

$$\leq C\left[|\hat{\mathbf{u}}|_{(H^2(\hat{K}))^2} + \sum_{i,j}\left(\|\partial_{\hat{x}_1}\hat{\mathbf{u}}\|_{(L^2(\hat{l}_{ij}))^2}\right.\right.$$

$$\left.\left.+\|\partial_{\hat{x}_2}\hat{\mathbf{u}}\|_{(L^2(\hat{l}_{ij}))^2}\right)\right]. (3.19)$$

By Lemma 3.1, we have

$$\|\mathbf{u} - \Pi_K\mathbf{u}\|_{(L^2(K))^2}$$

$$= \|(A^{\intercal})^{-1}(B_k^{\intercal})^{-1}A^{\intercal}(\mathbf{u} - \Pi_K\mathbf{u})^{\wedge}\|_{(L^2(K))^2}$$

$$\leq C\|B_k^{-1}\|\|(\mathbf{u} - \Pi_K\mathbf{u})^{\wedge}\|_{(L^2(K))^2}$$

$$\leq C\|B_k^{-1}\||\det(B_K)|^{\frac{1}{2}}\|\hat{\mathbf{u}} - \hat{\Pi}\hat{\mathbf{u}}\|_{(L^2(\hat{K}))^2}. \qquad (3.20)$$

Since

$$|\hat{\mathbf{u}}|_{(H^2(\hat{K}))^2} = |(A^{\intercal})^{-1}B_k^{\intercal}A^{\intercal}\mathbf{u}|_{(H^2(\hat{K}))^2}$$

$$\leq C\|B_K\||\mathbf{u}|_{(H^2(\hat{K}))^2}$$

and from [10]we have

$$|\mathbf{u}|_{(H^2(\hat{K}))^2} \leq C\|B_K\|^2|\det(B_K)|^{-\frac{1}{2}}|\mathbf{u}|_{(H^2(K))^2},$$

so

$$|\hat{\mathbf{u}}|_{(H^2(\hat{K}))^2} \leq C\|B_K\|^3|\det(B_K)|^{-\frac{1}{2}}|\mathbf{u}|_{(H^2(K))^2}. \quad (3.21)$$

Let $\hat{l}_{ij} = \overline{\hat{\mathbf{a}}_{\iota\iota J}\hat{\mathbf{a}}_{JJ\iota}}$, $1 \leq i < j \leq 3$.

Using

$$D\mathbf{u}(F(\hat{\mathbf{x}}))(\hat{\xi}) = D\mathbf{u}(\mathbf{x})(B_K\hat{\xi})$$

and

$$D^2\mathbf{u}(F(\hat{\mathbf{x}}))(\hat{\xi}_1,\hat{\xi}_2) = D^2\mathbf{u}(\mathbf{x})(B_K\hat{\xi}_1, B_K\hat{\xi}_2)$$

where $D\mathbf{u}$ and $D^2\mathbf{u}$ are the Frechet derivative of $\mathbf{u}$, we can obtain from [10] that

$$|\partial_{\hat{x}_i}\mathbf{u}(F_K(\hat{\mathbf{x}}))| \leq \|D\mathbf{u}(F_K(\hat{\mathbf{x}}))\|$$

$$\leq \|B_K\|\|D\mathbf{u}(\mathbf{x})\|$$

$$\leq C\|B_K\|\max(|\partial_{x_1}\mathbf{u}(\mathbf{x})|, |\partial_{x_2}\mathbf{u}(\mathbf{x})|),$$

$$i = 1,2.$$

and for multi-index $\alpha = (\alpha_1, \alpha_2)$ with $|\alpha| = 2$

$$|\partial^{\alpha}\mathbf{u}(F_K(\hat{\mathbf{x}}))| \leq \|D^2\mathbf{u}(F_K(\hat{\mathbf{x}}))\|$$

$$\leq \|B_K\|^2\|D^2\mathbf{u}(\mathbf{x})\|$$

$$\leq C\|B_K\|^2 \max_{|\alpha|=2}|\partial^{\alpha}\mathbf{u}(\mathbf{x})|.$$

Thus, use Sobolev embedding theorem, and let

$\tilde{\mathbf{u}}(\hat{\mathbf{x}}) = \mathbf{u}(F_K(\hat{\mathbf{x}}))$,

we have

$$\sum_{i,j}\left(\|\partial_{\hat{x}_1}\hat{\mathbf{u}}\|_{(L^2(\hat{l}_{ij}))^2} + \|\partial_{\hat{x}_2}\hat{\mathbf{u}}\|_{(L^2(\hat{l}_{ij}))^2}\right)$$

$$\leq C\|B_K\|\sum_{i,j}\left[\left(\int_{\hat{l}_{ij}}|\partial_{\hat{x}_1}\mathbf{u}(F_K(\hat{\mathbf{x}}))|^2 d\hat{l}\right)^{\frac{1}{2}}\right.$$

$$\left.+\left(\int_{\hat{l}_{ij}}|\partial_{\hat{x}_2}\mathbf{u}(F_K(\hat{\mathbf{x}}))|^2 d\hat{l}\right)^{\frac{1}{2}}\right]$$

$$\leq C\|B_K\|\left(\|\partial_{\hat{x}_1}\tilde{\mathbf{u}}\|_{(L^2(\partial\hat{K}_s))^2} + \|\partial_{\hat{x}_2}\tilde{\mathbf{u}}\|_{(L^2(\partial\hat{K}_s))^2}\right)$$

$$\leq C\|B_K\|\left(|\tilde{\mathbf{u}}|_{(H^1(\hat{K}_s))^2} + |\tilde{\mathbf{u}}|_{(H^2(\hat{K}_s))^2}\right)$$

$$\leq C\|B_K\|^2|\det(B_K)|^{-\frac{1}{2}}\left(|\mathbf{u}|_{(H^1(K_s))^2} + \|B_K\||\mathbf{u}|_{(H^2(K_s))^2}\right)$$

$$\leq C\|B_K\|^2|\det(B_K)|^{-\frac{1}{2}}\left(|\mathbf{u}|_{(H^1(K))^2} + \|B_K\||\mathbf{u}|_{(H^2(K))^2}\right).$$

(3.22)

By (3.19)-(3.22), and inequalities from [10]

$\|B_K\| \leq Ch$ and $\|B_k^{-1}\| \leq Ch^{-1}$,

we obtain

$$\|\mathbf{u} - \Pi_K\mathbf{u}\|_{(L^2(K))^2}$$

$$\leq C\|B_k^{-1}\||\det(B_K)|^{\frac{1}{2}}\left(\|B_K\|^3|\det(B_K)|^{-\frac{1}{2}}\right.$$

$$\left.+\|B_K\|^2|\det(B_K)|^{-\frac{1}{2}}\right)\left(|\mathbf{u}|_{(H^1(K))^2} + |\mathbf{u}|_{(H^2(K))^2}\right)$$

$$\leq Ch\left(|\mathbf{u}|_{(H^1(K))^2} + |\mathbf{u}|_{(H^2(K))^2}\right)$$

which is inequality (3.17).

Now we prove inequality (3.18).

It can be directly checked that

$$Div(\mathbf{u} - \Pi_K\mathbf{u}) = \det(B_k^{-1})Div(\hat{\mathbf{u}} - \widehat{\Pi_K\mathbf{u}}). \quad (3.23)$$

In fact, let

$B_k^{-1} = \begin{bmatrix} b_{11}^K & b_{12}^K \\ b_{21}^K & b_{22}^K \end{bmatrix}$, and $\mathbf{u} = (u_1, u_2)^{\intercal}$and $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2)^{\intercal}$,

then

$$\hat{\mathbf{x}} = F_K^{-1}(\mathbf{x}) = B_k^{-1}\mathbf{x} + \mathbf{d}_K$$

and

$$\mathbf{u} = (A^{\intercal})^{-1}(B_k^{\intercal})^{-1}A^{\intercal}\hat{\mathbf{u}}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} b_{11}^K & b_{21}^K \\ b_{12}^K & b_{22}^K \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\hat{\mathbf{u}}(F_K^{-1}(\mathbf{x}))$$

$$= \begin{bmatrix} b_{22}^K & -b_{12}^K \\ -b_{21}^K & b_{11}^K \end{bmatrix}\hat{\mathbf{u}}(F_K^{-1}(\mathbf{x})),$$

therefore

$$\frac{\partial u_1}{\partial x_1} = b_{22}^K \frac{\partial \hat{u}_1}{\partial \hat{x}_1} b_{11}^K + b_{22}^K \frac{\partial \hat{u}_1}{\partial \hat{x}_2} b_{21}^K - b_{12}^K \frac{\partial \hat{u}_2}{\partial \hat{x}_1} b_{11}^K - b_{12}^K \frac{\partial \hat{u}_2}{\partial \hat{x}_2} b_{21}^K$$

$$\frac{\partial u_2}{\partial x_2} = -b_{21}^K \frac{\partial \hat{u}_1}{\partial \hat{x}_1} b_{12}^K - b_{21}^K \frac{\partial \hat{u}_1}{\partial \hat{x}_2} b_{22}^K + b_{11}^K \frac{\partial \hat{u}_2}{\partial \hat{x}_1} b_{12}^K + b_{11}^K \frac{\partial \hat{u}_2}{\partial \hat{x}_2} b_{22}^K$$

and thus

$$Div\mathbf{u} = b_{11}^K b_{22}^K Div\hat{\mathbf{u}} - b_{12}^K b_{21}^K Div\hat{\mathbf{u}} = \det(B_k^{-1}) Div\hat{\mathbf{u}}.$$

Similarly

$$Div\Pi_K\mathbf{u} = \det(B_K^{-1}) Div\widehat{\Pi_K\mathbf{u}}.$$

So

$$\|Div(\mathbf{u} - \Pi_K\mathbf{u})\|_{L^2(K)} = |\det(B_K)|^{-1} \left\|Div\left(\hat{\mathbf{u}} - \widehat{\Pi_K\mathbf{u}}\right)\right\|_{L^2(K)}$$

$$= |\det(B_K)|^{-\frac{1}{2}} \left\|Div\left(\hat{\mathbf{u}} - \hat{\Pi}\hat{\mathbf{u}}\right)\right\|_{L^2(\hat{K})} \quad (3.24)$$

Since $\forall\, \hat{\mathbf{p}} \in \left(P_1(\hat{K})\right)^2$,

$$Div\left(\hat{\mathbf{p}} - \hat{\Pi}\hat{\mathbf{p}}\right) = \frac{1}{meas(\hat{K})} \int_{\hat{K}} Div\left(\hat{\mathbf{p}} - \hat{\Pi}\hat{\mathbf{p}}\right) d\hat{\mathbf{x}} = \frac{1}{meas(\hat{K})} \int_{\partial\hat{K}} \left(\hat{\mathbf{p}} - \hat{\Pi}\hat{\mathbf{p}}\right) \cdot \hat{\mathbf{n}}\, d\hat{l} = \frac{1}{meas(\hat{K})} \sum_{i,j} \left(\hat{\mathbf{p}} - \hat{\Pi}\hat{\mathbf{p}}\right)(\hat{\mathbf{a}}_{ij}) \cdot \hat{\mathbf{n}}_{ij} |\hat{\mathbf{a}}_i - \hat{\mathbf{a}}_j|$$

$= 0$,

we obtain $\forall\, \hat{\mathbf{p}} \in \left(P_1(\hat{K})\right)^2$

$$Div\left(\hat{\mathbf{u}} - \hat{\Pi}\hat{\mathbf{u}}\right) = Div\left(I - \hat{\Pi}\right)(\hat{\mathbf{u}} + \hat{\mathbf{p}}).$$

From this equality, we deduce that

$$\left\|Div\left(\hat{\mathbf{u}} - \hat{\Pi}\hat{\mathbf{u}}\right)\right\|_{L^2(\hat{K})} \le \left\|I - \hat{\Pi}\right\|_{\mathcal{L}\left(\left(H^2(\hat{K})\right)^2; \left(H^1(\hat{K})\right)^2\right)} \inf_{\hat{\mathbf{p}} \in (P_1(\hat{K}))^2} \|\hat{\mathbf{u}} + \hat{\mathbf{p}}\|_{\left(H^2(\hat{K})\right)^2} \quad (3.25)$$

By (3.21), (3.25) and inequality from [10]

$\forall\, \hat{\mathbf{u}} \in \left(H^2(\hat{K})\right)^2$,

$$\inf_{\hat{\mathbf{p}} \in (P_1(\hat{K}))^2} \|\hat{\mathbf{u}} + \hat{\mathbf{p}}\|_{\left(H^2(\hat{K})\right)^2} \le C|\hat{\mathbf{u}}|_{\left(H^2(\hat{K})\right)^2},$$

we have

$$\left\|Div\left(\hat{\mathbf{u}} - \hat{\Pi}\hat{\mathbf{u}}\right)\right\|_{L^2(\hat{K})} \le C|\hat{\mathbf{u}}|_{\left(H^2(\hat{K})\right)^2} \le C\|B_K\|^3 |\det(B_K)|^{-\frac{1}{2}} |\mathbf{u}|_{\left(H^2(K)\right)^2}. \quad (3.26)$$

It follows from (3.24) and (3.26) that

$$\|Div(\mathbf{u} - \Pi_K\mathbf{u})\|_{L^2(K)} \le C\|B_K\|^3 |\det(B_K)|^{-1} |\mathbf{u}|_{\left(H^2(K)\right)^2} \le Ch|\mathbf{u}|_{\left(H^2(K)\right)^2} \blacksquare$$

Now we can prove following theorem.

Theorem 3.4*Let Ω be a plane polygonal region with boundary Γ. Let S_h and S_{h_0} be defined by (3.3) and (3.4), respectively, Then*

$$\forall\, \mathbf{u} \in \left(H^2(\Omega)\right)^2$$

$$\inf_{\mathbf{u}_h \in S_h} \|\mathbf{u} - \mathbf{u}_h\|_{\left(L^2(\Omega)\right)^2} \le Ch\left(|\mathbf{u}|_{\left(H^1(\Omega)\right)^2} + |\mathbf{u}|_{\left(H^2(\Omega)\right)^2}\right) \quad (3.27)$$

$$\inf_{\mathbf{u}_h \in S_h} \|\mathbf{u} - \mathbf{u}_h\|_{H(Div;\Omega)} \le Ch\left(|\mathbf{u}|_{\left(H^1(\Omega)\right)^2} + |\mathbf{u}|_{\left(H^2(\Omega)\right)^2}\right) \quad (3.28)$$

and

$$\forall\, \mathbf{u} \in \left(H^2(\Omega)\right)^2 \cap H_0(Div;\Omega)$$

$$\inf_{\mathbf{u}_h \in S_{h_0}} \|\mathbf{u} - \mathbf{u}_h\|_{\left(L^2(\Omega)\right)^2} \le Ch\left(|\mathbf{u}|_{\left(H^1(\Omega)\right)^2} + |\mathbf{u}|_{\left(H^2(\Omega)\right)^2}\right) \quad (3.29)$$

$$\inf_{\mathbf{u}_h \in S_{h_0}} \|\mathbf{u} - \mathbf{u}_h\|_{H(Div;\Omega)} \le Ch\left(|\mathbf{u}|_{\left(H^1(\Omega)\right)^2} + |\mathbf{u}|_{\left(H^2(\Omega)\right)^2}\right) \quad (3.30)$$

Proof By Theorem 3.3, we have

$$\inf_{\mathbf{u}_h \in S_h} \|\mathbf{u} - \mathbf{u}_h\|_{\left(L^2(\Omega)\right)^2} \le \|\mathbf{u} - \Pi\mathbf{u}\|_{\left(L^2(\Omega)\right)^2} = \left(\sum_{K \in \mathcal{T}_h} \|\mathbf{u} - \Pi_K\mathbf{u}\|^2_{\left(L^2(K)\right)^2}\right)^{\frac{1}{2}}$$

$$\le Ch\left[\left(\sum_{K \in \mathcal{T}_h} |\mathbf{u}|^2_{\left(H^1(K)\right)^2}\right)^{\frac{1}{2}} + \left(\sum_{K \in \mathcal{T}_h} |\mathbf{u}|^2_{\left(H^2(K)\right)^2}\right)^{\frac{1}{2}}\right] = Ch\left(|\mathbf{u}|_{\left(H^1(\Omega)\right)^2} + |\mathbf{u}|_{\left(H^2(\Omega)\right)^2}\right).$$

Similarly we can prove (3.28).

Taking into account $\Pi\mathbf{u} \in S_{h_0}$ if $\mathbf{u} \in \left(H^2(\Omega)\right)^2 \cap H_0(Div;\Omega)$, (3.29) and (3.30) can be proved completely similar to the proof of (3.27) and (3.28), respectively. ■

B. Approximation Properties of High Order Basis Functions

Let $P_k(K)$ be formed by the restrictions to K of all polynomials of degree $\le k$.

Define the finite-dimensional subspace $S_h^{(k)}$ of $H(Div;\Omega)$

$$S_h^{(k)} = \{\mathbf{v}(\mathbf{x}) | \mathbf{v}_{|K}(\mathbf{x}) \in (P_k(K))^2, \qquad K \in \boldsymbol{\mathcal{T}}_h;$$
$$\mathbf{v}_{|K_1}(\mathbf{x}) \cdot \mathbf{n}_1 = -\mathbf{v}_{|K_2}(\mathbf{x}) \cdot \mathbf{n}_2 \ \text{on} \overline{K}_1 \cap \overline{K}_2,$$
$$K_1, K_2 \in \boldsymbol{\mathcal{T}}_h \text{and } \overline{K}_1 \cap \overline{K}_2 \neq \phi\} \quad (3.31)$$

where$\mathbf{n}_1$ and $\mathbf{n}_2$ are the unit outward normal vectors to K_1 and K_2alongtheir common edge, respectively, and the finite –dimensional subspace $S_{h_0}^{(k)}$ of $H_0(div;\Omega)$

$$S_{h_0}^{(k)} = \{\mathbf{v}(\mathbf{x}) | \mathbf{v}(\mathbf{x}) \in S_h^{(k)},\ \mathbf{v}(\mathbf{x}) \cdot \mathbf{n} = 0 \text{ on} \Gamma\} \quad (3.32)$$

where $\mathbf{n}$ is the unit outward normal vector to Ω. The high order basis constructed in [7] is a basis of $S_h^{(k)}$.

Since $\dim\left((P_k(K))^2\right) = (k+1)(k+2)$, we can define ordinary finite elements as in [10]

$$\left(K, (P_k(K))^2, \Sigma_K^{(k)}\right), \qquad k = 1, 2, \cdots. \quad (3.33)$$

Then the finite element space $X_h^{(k)}$, constructed corresponding to the finite element (3.33), satisfies

$$X_h^{(k)} \subset (C(\overline{\Omega}))^2, \qquad k = 1, 2, \cdots .$$

For $\mathbf{u} \in (C(\overline{\Omega}))^2$, we define a unique interpolation

$$\Pi^{(k)}\mathbf{u} \in X_h^{(k)}$$

such that for all $K \in \boldsymbol{\mathcal{T}}_h$

$$\alpha_i^{(k)}(\mathbf{u} - \Pi^{(k)}\mathbf{u}) = 0, \quad \forall\, \alpha_i^{(k)} \in \Sigma_K^{(k)}.$$

Using notation $\Pi_K^{(k)}\mathbf{u} = (\Pi^{(k)}\mathbf{u})_{|K}$, we have estimation from [10]

$$\forall\, \mathbf{u} \in (H^{k+1}(\Omega))^2, k \geq 1$$
$$\|\mathbf{u} - \Pi_K^{(k)}\mathbf{u}\|_{(H^l(K))^2} \leq Ch^{k+1-l}|\mathbf{u}|_{(H^{k+1}(K))^2}, \qquad l = 0, 1. \quad (3.34)$$

By the definition of $S_h^{(k)}$, $S_{h_0}^{(k)}$ and $\Pi^{(k)}$, we can deduce that for $k \geq 1$

$$\Pi^{(k)}\mathbf{u} \in S_h^{(k)} \cap (H^1(\Omega))^2, \quad \text{if} \mathbf{u} \in (H^{k+1}(\Omega))^2$$

and

$$\Pi^{(k)}\mathbf{u} \in S_{h_0}^{(k)} \cap (H^1(\Omega))^2,$$

if$\mathbf{u} \in (H^{k+1}(\Omega))^2 \cap H_0(Div;\Omega)$,

sonow we can prove following theorem.

Theorem 3.5*Let $S_h^{(k)}$ and $S_{h_0}^{(k)}$ be defined by (3.31) and (3.32), respectively. Then*

$$\forall\, \mathbf{u} \in (H^{k+1}(\Omega))^2, \qquad k \geq 1$$

$$\inf_{\mathbf{u}_h \in S_h^{(k)}} \|\mathbf{u} - \mathbf{u}_h\|_{(L^2(\Omega))^2} \leq Ch^{k+1}|\mathbf{u}|_{(H^{k+1}(\Omega))^2} \quad (3.35)$$

$$\inf_{\mathbf{u}_h \in S_h^{(k)}} \|\mathbf{u} - \mathbf{u}_h\|_{H(Div;\Omega)} \leq Ch^{k}|\mathbf{u}|_{(H^{k+1}(\Omega))^2} \quad (3.36)$$

and$\forall\, \mathbf{u} \in (H^{k+1}(\Omega))^2 \cap H_0(Div;\Omega),\ k \geq 1$

$$\inf_{\mathbf{u}_h \in S_{h_0}^{(k)}} \|\mathbf{u} - \mathbf{u}_h\|_{(L^2(\Omega))^2} \leq Ch^{k+1}|\mathbf{u}|_{(H^{k+1}(\Omega))^2} \quad (3.37)$$

$$\inf_{\mathbf{u}_h \in S_{h_0}^{(k)}} \|\mathbf{u} - \mathbf{u}_h\|_{H(Div;\Omega)} \leq Ch^{k}|\mathbf{u}|_{(H^{k+1}(\Omega))^2} \quad (3.38)$$

Proof For $\mathbf{u} \in (H^{k+1}(\Omega))^2$, by (3.34) and the fact

$$\Pi^{(k)}\mathbf{u} \in S_h^{(k)} \cap (H^1(\Omega))^2,$$

we have

$$\inf_{\mathbf{u}_h \in S_h^{(k)}} \|\mathbf{u} - \mathbf{u}_h\|_{(L^2(\Omega))^2} \leq \|\mathbf{u} - \Pi^{(k)}\mathbf{u}\|_{(L^2(\Omega))^2} = \left(\sum_{K \in \boldsymbol{\mathcal{T}}_h} \|\mathbf{u} - \Pi_K^{(k)}\mathbf{u}\|^2_{(L^2(K))^2}\right)^{\frac{1}{2}} \leq Ch^{k+1}\left(\sum_{K \in \boldsymbol{\mathcal{T}}_h} |\mathbf{u}|^2_{(H^{k+1}(K))^2}\right)^{\frac{1}{2}}$$
$$= Ch^{k+1}|\mathbf{u}|_{(H^{k+1}(\Omega))^2},$$

$$\inf_{\mathbf{u}_h \in S_h^{(k)}} \|\mathbf{u} - \mathbf{u}_h\|_{H(Div;\Omega)} \leq \|\mathbf{u} - \Pi^{(k)}\mathbf{u}\|_{H(Div;\Omega)} \leq \|\mathbf{u} - \Pi^{(k)}\mathbf{u}\|_{(H^1(\Omega))^2}$$
$$\leq Ch^k|\mathbf{u}|_{(H^{k+1}(\Omega))^2}.$$

Similarly prove (3.37) and (3.38). ■

Actually, by definition we have $S_h^{(k)} \supseteq X_h^{(k)}$. Also we notice that $S_h^{(1)} = X_h^{(1)}$.

In fact, let K_1 and K_2 be any two triangles in $\boldsymbol{\mathcal{T}}_h$**,** which have common edge s with ends N_1 and N_2 (see Fig. 6), then $\mathbf{v}(\mathbf{x}) \in S_h^{(1)}$ if and only if

$$\mathbf{v}_{|K_1}(\mathbf{x}) \in (P_1(K_1))^2, \qquad \mathbf{v}_{|K_2}(\mathbf{x}) \in (P_1(K_2))^2$$

and

$$\mathbf{v}_{|K_1}(\mathbf{x}) \cdot \mathbf{n}_1 = -\mathbf{v}_{|K_2}(\mathbf{x}) \cdot \mathbf{n}_2 . \qquad (3.39)$$

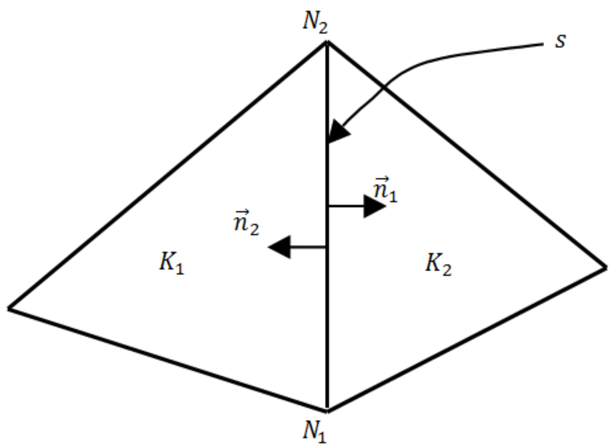

Figure 6: Two triangles with common edge s

(3.39) is equivalent to

$$\begin{cases} \mathbf{v}_{|K_1}(N_1)\cdot\mathbf{n}_1 = \mathbf{v}_{|K_2}(N_1)\cdot\mathbf{n}_1 \\ \mathbf{v}_{|K_1}(N_2)\cdot\mathbf{n}_1 = \mathbf{v}_{|K_2}(N_2)\cdot\mathbf{n}_1 \end{cases}.$$

Let

$$\Sigma_K' = \{\mathbf{v}(\mathbf{a}_1)\cdot\mathbf{n}_{12},\ \mathbf{v}(\mathbf{a}_1)\cdot\mathbf{n}_{13},\ \mathbf{v}(\mathbf{a}_2)\cdot\mathbf{n}_{12},\ \mathbf{v}(\mathbf{a}_2)\cdot\mathbf{n}_{23},\ \mathbf{v}(\mathbf{a}_3)\cdot\mathbf{n}_{13},\ \mathbf{v}(\mathbf{a}_3)\cdot\mathbf{n}_{23}\}$$

(see Fig. 4).

Then $S_h^{(1)}$ is finite element space constructed corresponding to the finite element $\left(K,\left(P_1(K)\right)^2,\Sigma_K'\right)$, but Σ_K' is equivalent to $\Sigma_K^{(1)}$, so $S_h^{(1)} = X_h^{(1)}$.

IV. CONCLUSION

In numerical simulation of electromagnetic scattering problem with integral equation formulation, basis function is very important. In this research we provided and proved the approximation properties of the basis functions, which provided theoretical reference for selection of appropriate basis function in numerical calculation based on the accuracy requirement of the application problem.

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