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Fixed Point of Demicontinuous Pseudocontractive Mapping in Hilbert Space

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ARTICLE INFO	ABSTRACT
Published Online: 18 April 2020 Corresponding Author: **Chetan Kumar Sahu**	In this paper, we study some fixed point theorem for pseudo-contractive demicontinuous mappings in a Hilbert space, it is shown that the same algorithm converges to a fixed point of a pseudo-contractractive mapping under suitable hypotheses on the coefficients. Here the assumptions on the coefficients are different, as well as the techniques of the proof.

KEYWORDS: Fixed Point, Hilbert Space, Contraction mapping, Common fixed points, Pseudo contractive mapping, Nonself mapping, Demi continuouity.

INTRODUCTION

In this paper, our aim is to study a common fixed for the finite family of pseudocontractive mappings in Hilbert spaces. The class of pseudo contractive mappings will play a great role in many of the existence for solutions for nonlinear problems and hence the class of strictly pseudo contractive mappings as a subclass has a significant role.

Preliminaries: This section is devoted to some basic definitions and theorems which are needed for the further study of this paper.

Notation: H will denote the Hilbert space,E will denote the Babach space and B will always denote a closed unit ball centred at origin

Definition 1.1:Let E be a Banach space with its dual E^* and K be non empty, closed and convex subset of E .Then a mapping T: K→E is said to be strongly pseudo contractive if there exists a positive constant k and j(x-y) ∈J(x-y) such that $\langle Tx-Ty, j(x-y)\rangle \le K\|x-y\|^2$

Definition 1.2 A mapping T: K→E is called pseudo contractive if there exists

j(x-y)∈J(x-y)such that $\langle Tx-Ty, j(x-y)\rangle \le K\|x-y\|^2$ for all x , y∈K where $J:E\to 2^{E^*}$ is normalized duality mapping given by $Jx=\left\{f\in E: \langle f, x\rangle = \|x\|^2 = \|f\|^2\right\}$ where ⟨.,.⟩denotes inner product in Hilbert space.

Definition 1.3 A mapping T: K→E is called Lipschitzian if and only if

$$\|Tx-Ty\|\le L\|x-y\|$$

For x,y∈K some 0≤ L<1

Definition 1.4: A mapping T: K→E, is called non expansive if

$$\|Tx-Ty\|\le L\|x-y\|$$

For x,y∈K some 0≤L≤1

Definition 1.5: A mapping A: K→E is said to be accretive if for all

for all x,y∈K and t>0, $\|x-y\|\le\|x-y+t(Ax-Ay)\|$ holds

Definition 1.6: Let (E, ‖.‖) be a Banach space and T:E→E be an operator then f is said to be demi continuous in x if for any sequence $\{x_n\}_{n\in N}$ in E that converges strongly to x, the sequence $\{Tx_n\}_{n\in N}$ converges weakly to T(x)

Definition 1.7:A mapping T:C→H is said to be Hemi continuous if the mapping

$t\to T[(1-t)x+ty]$ is continuous from [0,1] in to weak topology of Hilbert space H.

Lemma 1: Let B be a closed unit ball ball in H and let T:B→B be demi continuous then ∃ a sequence $\{x_n\}$⊂B such that

(a) $\langle Tx_n - x_n, Tx_m - x_m\rangle$= 0 for m≠n

(b) $\langle Tx_n - x_n, x_n\rangle$= 0 and

(c) $\{Tx_n - x_n\}$ converges weakly to 0

Proof. If the mapping T has a fixed point and x is fixed point of t then T(x)=x. By replacing x by $x_n$we get T(x_n) =x_n and $\langle Tx_n - x_n, Tx_m - x_m\rangle$ =$\langle x_n - x_n, Tx_m - x_m\rangle$

= $\langle x_n - x_n, Tx_m - x_m\rangle$
= 0

Similarly we can prove result (b) and (c) now assuming there is no fixed point of T, let M_1be the 1-dimensional sub space of H and T_1: B→M_1⋂B be the well defined orthogonal projection

Then by famous Brouwer theorem T_1oT: M_1⋂B→M_1⋂B is continuous and has a fixed point let it be x_1 then ($T_1 oT$) x_1=x_1.Now $T_1(Tx_1 - x_1)$ =($T_1 oT$)x_1-$T_1(x_1)$

= x_1-x_1
= 0

Which implies $Tx_1 - x_1 \in M_1^{\perp}$.Let M_2=M_1⊕(Tx_1-x_1) and T_2: B→M_2⋂B be orthogonal projection then as above the mapping $T_2 oT$ has a fixed point x_2 in M_2⋂B. Generally we construct a sequence $\{M_n\}$ of subspaces and a sequence $\{x_n\}$ with $x_n \in M_n$⋂B such that

$Tx_n - x_n \in M_n^{\perp}$, M_{n+1}=M_n⊕(Tx_n-x_n) and ($T_n oT$)x_n=x_n.Now we observe that

$\langle Tx_n - x_n, h\rangle$= 0 ∀ h∈$M_n$ and $(Tx_n - x_n)$∈M_n for m< n thus $\langle Tx_n - x_n, x_n\rangle$= 0 and by the symmetry of inner product $\langle Tx_n - xn, Txm - xm$= 0 for m≠n. Since $Txn - xn$ is orthogonal set of vector in H, by Bessels inequality we have $Txn - xn, h$→ 0 ∀ h∈H.

Lemma 2: Let C be a convex subset of H and T:C→C be hemi continuous pseudo contractive mapping. Suppose $\{x_n\}$ is sequence in C such that $(Tx_n - x_n)$ → 0(weakly),$\langle Tx_n - x_n, x_n\rangle$→0 and x_n→x(weakly). Then T has a fixed point.

Proof. We know that the definition of pseudo contractive mapping is equivalent to the mapping T satisfying $\langle Tv - Ty, v - y\leq\| v - y\| 2$+ max$\{0, Tv - v, Ty - y\}$∀ v,y∈C.

Let z = (1-b)x+bTx for some b, 0<b<1. Since T is hemicontinuous, we can find number b such that $\langle Tz - z, Tx - x\rangle \geq \frac{1}{2}\langle Tx - x, Tx - x\rangle$. For any ε >0 ∃ n large enough such that

$|\langle Tx_n - x_n, z - x_n\rangle| <$ ε, $|\langle Tx_n - x_n, Tz - z\rangle| <$ ε and $|\langle Tz - z, x - x_n\rangle| <$ ε thus

$\langle Tz - Tx_n, z - x_n\rangle = \langle Tz - z + z - x_n - Tx_n, z - x_n\rangle$

$= \langle Tz - z, z - x_n\rangle + \langle z - x_n, z - x_n\rangle + \langle x_n - Tx_n, z - x_n\rangle$

$= \|z - x_n\|^2 + \langle x_n - Tx_n, z - x_n\rangle$

$+ \langle Tz - z, x - x_n\rangle + \langle Tz - z, b(Tx - x)\rangle$

$+ \langle Tz - z, x - x_n\rangle$

$\geq \|z - x_n\|^2 + \frac{b}{2}\|TX - x\|^2$-2ε

But T is pseudo contractive therefore

$$\langle Tz - Tx_n, z - x_n\rangle \leq \|z - x_n\|^2 + \max\{0, \langle Tz - z, Tx_n - x_n\rangle\}$$

$$\leq \|z - x_n\|^2 + ε$$

Thus with the help of above result $\|z - x_n\|^2 + ε \geq \|z - x_n\|^2 + \frac{b}{2}\|Tx - x\|^2$-2ε

$\Rightarrow 3ε \geq \frac{b}{2}\|Tx - x\|^2$

⇒Tx=x

Lemma 3: Let C be a closed, bounded convex subset of H and let T:C→C be a hemicontinuous pseudocontinuous type mapping then operator I-T is demiclosed at zero.

For the proof we can see [2]

Lemma 4: (Ramsey's theorem**):** Let V⊆N×N be set of all ordered pair (m ,n) such that m>n. Let ∅:V→{0,1} be any function then there exist A⊆N such that ∅ restricted to V⋂(A×A) is a constant.

For the proof can see [3]

Lemma 5: Given an infinite set of vectors W in a Hilbert space and any δ>0 there exist V, an infinite subset of W such that for all $y_n, y_m \in$V, $\|y_n - y_m\|^2 \leq (1+\delta)\|y_n\|^2 + (1+\delta)\|y_m\|^2$

Proof. To prove this lemma we use the Ramsey theorem which states that if V⊆N×N be set of all ordered pair (m ,n) such that m>n. Let ∅:V→{0,1} be any function then there exist A⊆N such that ∅ restricted to V⋂(A×A) is a constant. We define ∅ in this manner

$$\emptyset(m,n) = \begin{cases} 1 \text{ if } y_n \text{and } y_m \text{satisfy the inequility} \\ 0 \qquad\qquad\qquad \text{if they do not} \end{cases}$$

Thus either there exists the infinite subset V we desire or there exists an infinite subset S such that for all $y_n, y_m \in$S, $\|y_n - ym\| 2 \leq (1+\delta)\|yn\| 2 + (1+\delta)\|ym\| 2$which is equivalent to

$$-2\langle y_n, y_m\rangle \geq \delta\|y_n\|^2 + \delta\|y_m\|^2$$

This implies that the ratio of norm is bounded by 2/ δ and without loss of generality, we can say that $1 \geq \|y_m\| \geq \delta/2$ for any $y_m \in$S

We renumber the element of S as $\{z_n\}$, We then construct an orthogonal basis $\{u_n\}$ so that

$z_1 = a_1 u_1$, $z_2 = b_{12} u_1 + a_2 u_2$, ..., $z_n = b_{1n} u_1 + b_{2n} u_2 + \ldots\ldots a_n u_n$

Where all a_j's are positive. It is clear that $b_{ij} \leq -(\delta/2)\|z_j\|$ for all i and j

Thus when j>[(2/ δ)²+1}, we get $\|z_j\| > 1$, which is contradiction and this contradiction proves the lemma.

OUR MAIN RESULT IS AS FOLLOWS

Theorem: If C is a closed, bounded, convex subset of Hilbert space and T:C→C is a demicontinuous pseudo contraction,then T has a fixed point.

Proof: To show that the operator T has at least one fixed point in C we will use the result of Lemma 2 and weak compactness of C, to show that ∃ a sequence $\{x_n\}$ in C such that

(1) $\langle Tx_n - x_n\rangle$ converges weakly to zero and

(2) $\langle Tx_n - x_n\rangle \to 0$

Let B be a ball containing C and let ψ:H→C be the retraction sending x∈H to the closest point x in C then we know that ψ will satisfy

$$\|\psi(x) - \psi(y)\| \leq \|x-y\| \ \forall\ x, y \in H \text{ and}$$

$$\|x - \psi(x)\|^2 \leq \|x - y\|^2 - \|\psi(x) - y\|^2 \ \forall\ x \in C \text{ and } y \in H$$

For the demicontinuous mapping Toψ:B→B exist, by Lemma1∃ a sequence $\{z_n\}$having the property (a)-(c) given in Lemma 1. Set $x_n = \psi(z_n)$ then $Tx_n - x_n = T o\psi z_n - z_n + z_n - \psi z_n$

Which implies $\langle Tx_n - x_n, x_n\rangle = \langle T o\psi z_n - z_n, z_n\rangle + \langle T o\psi z_n - z_n, \psi z_{\square} - z_n\rangle$

$$+\langle z_n - \psi z_n, z_n - \psi z_n\rangle + \langle z_n - \psi z_{n,} z_n\rangle,$$

Thus if lim inf$\|\psi z_n - z_n\| = 0$ then a subsequence $\{x_n\}$will satisfies (1) and (2) and this completes the proof.

Now we assume contradiction that ∃ b> 0 such that $\|\psi z_n - z_n\| \geq$b for all n. By Lemma 3 ∃ V, an infinite subset of { $T o\psi z_n - \psi z_n$} such that for two elements of V

$$\|T o\psi z_n - T o\psi z_j\|^2 \leq \|T o\psi z_n - \psi z_n\|^2 + \|T o\psi z_j - \psi z_j\|^2 + b^2$$

Let W denote the corresponding subset of $\{z_n\}$ Since T is pseudocontractive therefore for $z_n, z_j \in$W, $\|T o\psi z_n - T o\psi z_j\|^2 \leq \|\psi z_n - \psi z_j\|^2 + \|To\psi z_n - \psi z_n\|^2$

$$+ \|To\psi z_j - \psi z_j\|^2 + b^2$$

$$\leq \|z_n - z_j\|^2 + \|To\psi z_n - \psi z_n\|^2$$

$$+ \|\psi z_n - z_n\|^2 + \|To\psi z_j - \psi z_j\|^2$$

$$- \|\psi z_j - z_j\|^2 + b^2$$

$$\leq \|z_n - z_j\|^2 + \|To\psi z_n - \psi z_n\|^2$$

$$+ \|To\psi z_j - \psi z_j\|^2 + b^2 - 2b^2$$

$$\leq \|z_n - z_j\|^2 + q\|To\psi z_n - \psi z_n\|^2$$

$$+ q\|To\psi z_j - \psi z_j\|^2 \text{ for some } q<1$$

Now by low of cosines we have

$$\|T o\psi z_n - T o\psi z_j\|^2 \geq \|z_n - z_j\|^2 + \|To\psi z_n - \psi z_n\|^2$$

$$+ \|To\psi z_j - \psi z_j\|^2$$

$$-2\left|\langle T o \psi z_n - z_n, T o \psi z_j - \psi z_j \rangle\right|$$
$$-2\left|\langle T o \psi z_n - z_n, z_n - z_j \rangle\right|$$
$$-2\left|\langle T o \psi z_j - z_j, z_n - z_j \rangle\right|$$

Combining above two inequality and using the notation

$$\left|\langle T o \psi z_n - z_n, T o \psi z_j - \psi z_j \rangle\right| = \begin{cases} 0 \; for \; j \neq i \\ 1 \; for \; j > i \end{cases} \text{ we have}$$

$$2\left|\langle T o \psi z_n - z_n, z_n - z_j \rangle\right| \geq (1-q)\left\| T o \psi z_n - z_n \right\|^2$$
$$+(1-q)\left\| T o \psi z_j - z_j \right\|^2$$
$$\left| \geq (1-q)\left\| \psi z_n - z_n \right\|^2 \geq (1-q)\, b^2\right.$$

Let S= $\{z_n\}$be subsequence of W and le z be weak limit of S. Now we observe that

$$\langle T o \psi z_n - z_n, z_n - z_j \rangle = \langle T o \psi z_n - z_n, z_n - T_n z \rangle$$
$$+ \langle T o \psi z_n - z_n, T_n z - z \rangle$$
$$+ \langle T o \psi z_n - z_n, z - z_n \rangle$$

Also $\langle T o \psi z_n - z_n, z_n - T_n z \rangle$ and for sufficiently large n, $\| T_n z - z \|$is as small as necessary and for fixed n, $|\langle T o \psi z_n - z_n, z - zn|$ is also small as desired. This givers the required contradiction.

CONCLUSION

Finding fixed points of nonlinear mappings (especially, nonexpansive mappings) has received vast investigations due to its extensive applications in, partial differential equations, nonlinear differential equations. It is well known that pseudocontractive mappings have more powerful applications than nonexpansive mappings in solving problems. In this paper, we devote to construct the methods to finding the fixed points of demicontinuous pseudocontractive mappings and hemicontinuous pseudo contractive mappings.

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