[Volume 4 issue 1 January 2016]
PageNo.1253-1270 ISSN : 2320-7167

INTERNATIONAL JOURNAL OF MATHEMATICS AND COMPUTER RESEARCH

Common Fixed Point Theorems For Nonexpansive Type Single Valued Mappings

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Abstract. In this article, we establish two common fixed point theorems for non-expansive type single valued mappings which include both continuous and discontinuous mappings by relaxing the condition of continuity by weak reciprocally continuous mapping.

Keywords: Non-expansive mapping, common fixed point, reciprocal continuous, weak reciprocal continuous.

2010 MSC: Primary 47H10; Secondary 54H25.

1. Introduction

Fixed point theory is one of the most dynamic research subjects in nonlinear sciences. Regarding the feasibility of application of it to the various disciplines, a number of authors have contributed to this theory with a number of publications. The most impressing result in this direction was given by Banach, called the Banach contraction mapping principle: Every contraction in a complete metric space has a unique fixed point. In fact, Banach demonstrated how to find the desired fixed point by offering a smart and plain technique. This elementary technique leads to increasing of the possibility of solving various problems in different research fields. Fixed point theorems for contractive, non-expansive, contractive type and non-expansive type mappings provide techniques for solving a variety of applied problems in mathematical and engineering sciences. It is one of the reason that many authors have studied various classes of contractive type or non-expansive type mappings. Let (X,d) be a metric space. If T is such that for all x,y in X

$$d(Tx,Ty) \leq \lambda d(x,y) \tag{1.1}$$

where $0 < \lambda < 1$, then T is said to be a contraction mapping. If T satisfies (1.1) with $\lambda = 1$, then T is called a non-expansive mapping. If T satisfies any conditions of type

$$d(Tx,Ty) \leq a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty) + a_4 d(x,Ty) + a_5 d(y,Tx) \tag{1.2}$$

where a_i $(i = 1,2,3,4,5)$ are nonnegative real numbers such that $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, then T is said to be a contractive type mapping. If T satisfies (1.2) with $a_1 + a_2 + a_3 + a_4 + a_5 = 1$, then T is said to be a non-expansive type mapping.

Bogin [4] proved the following result:

Theorem 1.1 Let X be a nonempty complete metric space and $T: X \to X$ a mapping satisfying

$$d(Tx,Ty) \leq ad(x,y) + b[d(x,Tx) + d(y,Ty)] + c[d(x,Ty) + d(y,Tx)] \tag{1.3}$$

where $a \geq 0, b > 0, c > 0$ and

$$a + 2b + 2c = 1 \quad (1.4)$$

Then T has a unique fixed point.

This result was generalized by Rhoades [22], Ciric [7, 9]. Iseki [15] studied a family of commuting mappings $T_1, T_2, \dots T_n$ which satisfy (1.3) with $a \geq 0, b \geq 0, c \geq 0$ and $a + 2b + 2c = 1$. For Banach spaces the famous is Gregus's Fixed Point Theorem [11] for non-expansive type single-valued mappings, which satisfy (1.3) with $c = 0, a < 1$.

Gregus [11] proved the following result:

Theorem 1.2 Let C be a nonempty closed convex subset of a Banach space B and $T : C \to C$ a mapping that satisfies:

$$\|Tx - Ty\| \leq a\|x - y\| + b[\|x - Tx\| + \|y - Ty\|] \quad (1.5)$$

where $a \geq 0, b > 0$ and

$$a + 2b = 1 \quad (1.6)$$

Then T has a unique fixed point.

The class of mappings T satisfying the following non-expansive type condition:

$$d(Tx, Ty) \leq a(x, y)\max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Tx) + d(y,Ty)}{2}\right\}$$
$$+ b(x, y)max\{d(x, Tx), d(y, Ty) + c(x, y)[\, d(x, Ty) + d(y, Tx)] \quad (1.7)$$

for all $x, y \in X$, where a, b, c are nonnegative real numbers such that $b > 0, c > 0$ and $a + b + 2c = 1$, was introduced and investigated by Ciric [9]. Ciric proved that in a complete metric space such mappings have a unique fixed point. This result was generalized by Jhade et al.[13-14]. Ciric [7] introduced and investigated a new class of self-mappings T on X which satisfy an inequality of type (1.3) with $b \geq 0$ and still have a fixed point. Also proved that by an example if the mapping T satisfies (1.3) with $b = 0$ and if a and c are such that (1.4) holds, then T need not have a fixed point. Therefore, a contractive condition for T , which shall guarantee a fixed point of T in the case $b = 0$ and $a + 2c = 1$, must be stricter then (1.3). Chandra et al [4] consider the following generalization of (1.7), let $T, f: X \to X$ satisfying:

$$d(Tx, Ty) \leq a(x, y)d(fx, fy) + b(x, y)max\{d(fx, Tx), d(fy, Ty)\}$$
$$+ c(x, y)[\, d(fx, Ty) + d(fy, Tx)] \quad (1.8)$$

where

$$a(x, y) \geq 0, \;\; \beta = inf_{x,y\in X}\, b(x, y) > 0, \;\; \gamma = inf_{x,y\in X}\, c(x, y) > 0$$

with

$$sup_{x,y\in X}\big(a(x, y) + b(x, y) + 2c(x, y)\big) = 1.$$

In 1982, Sessa [24] introduced the weak commutativity condition for a pair of single valued maps. Jungck [16] generalized the concept of weak commutativity condition by introducing compatibility of maps. Two self-mappings f and g of a metric space (X, d) are said to be compatible (see Jungck [16]) if,

$\lim_{\to+\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t \in X$. It is easy to see that compatible mappings commute at their coincidence points. The study of non-compatibilty was initiated by Pant [20] by introducing point wise R-weakly commutativity of maps. Al-Thagafi and Shahzad [1] introduced the notion of occasionally weakly compatible maps and employed the new notion to prove fixed point theorem under new condition. Here it seems important to mention that weak commutativity implies compatibility but the converse is not true. Weak commutativity implies R-weakly commutativity but R-weakly commutativity implies weak commutativity only when $R \leq 1$. Self-mappings f and g of a metric space (X, d) are called R-weakly commuting of type-(A_g) [21] if there exists some positive real number R such that $d(ffx, gfx) \leq Rd(fx, gx)$ for all x in X. Similarly, two self-mappings f and g of a metric space (X, d) are called R-weakly commuting of type-(A_f) [21] if there exists some positive real number R such that $d(fgx, ggx) \leq Rd(fx, gx)$ for all x in X . It is to be noted that pointwise R-weakly commuting maps commute at their coincidence points and pointwise R-weak commutativity is equivalent to commutativity at coincidence points. Compatible and non-compatible maps can be R-weakly commuting of type-(A_g) or (A_f) In 1998, Pant [19] introduced reciprocal continuity for the pair of single-valued maps which states that maps f and g are reciprocal continuity if and only if $\lim_{n\to+\infty} gfx_n = gt$ and $\lim_{n\to+\infty} fgx_n = ft$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$ for some t in X. They also established some common fixed point theorems for reciprocally continuous maps. It is also proved that a pair of maps which is reciprocally continuous need not to be continuous even on their common fixed point.

In this paper, we shall introduce and investigate following:

(1) a new class of self-mappings T, f on X which satisfy the following non-expansive type conditions:

$$\begin{aligned} d(Tx, Ty) \leq a(x, y) max\{&d(fx, fy), d(fx, Tx), d(fy, Ty) \\ &, \tfrac{1}{2}[M(x, y) + m(x, y)]\} \\ &+ c(x, y)[M(x, y) + hm(x, y)] \end{aligned} \tag{1.9}$$

for all $x, y \in X$, where

$$M(x, y) = max\{d(fx, Ty), d(fy, Tx)$$
$$m(x, y) = min\{d(fx, Ty), d(fy, Tx)$$

and $0 < h < 1$,

with
$$a(x, y) \geq 0,\ \beta = inf\{c(x, y) : x, y \in X\} > 0 \tag{1.10}$$
$$sup_{x,y\in X}\big(a(x, y) + 2c(x, y)\big) = 1. \tag{1.11}$$

(2) a class of self-mappings T, f on X which satisfy the following non-expansive type conditions:

$$\begin{aligned} d(Tx, Ty) \leq\ & a(x, y)d(fx, fy) + b(x, y)max\{d(fx, Tx), d(fy, Ty)\} \\ & + c(x, y)max\{d(fx, fy), d(fx, Tx), d(fy, Ty)\} \\ & + e(x, y)max\{d(fx, fy), d(fx, Tx), d(fy, Ty)\ d(fx, Ty)\} \end{aligned}$$

$$+h(x,y)max\{d(fx,fy),d(fx,Tx),d(fy,Ty),d(fx,Ty),d(fy,Tx)\} \tag{1.12}$$

where

$$a(x,y),b(x,y),c(x,y),e(x,y),h(x,y)\geq 0,$$
$$\beta = inf_{x,y\in X}\big(e(x,y)+h(x,y)\big) > 0$$
$$\gamma = inf_{x,y\in X}\big(b(x,y)+e(x,y)+h(x,y)\big) > 0 \tag{1.13}$$

with

$$sup_{x,y\in X}\big(a(x,y)+b(x,y)+c(x,y)+2e(x,y)+2h(x,y)\big)=1. \tag{1.14}$$

We also establish two common fixed point theorems for non-expansive type single valued mappings satisfying inequities (1.9) and (1.12), respectively, which include both continuous and discontinuous mappings by relaxing the condition of continuity by weak reciprocally continuous mapping.

2. Main Results

Now, we give our first main results.

Theorem 2.1 Let T and f be two weakly reciprocally continuous self maps of a complete metric space (X,d) satisfying (1.9) with $T(X)\subseteq f(X)$, where a and c satisfying (1.10) and (1.11), then T and f have a common fixed point in X if either (a) T and f are compatible or (b) T and f are R-weakly commuting of type$-(A_f\,)$ or (c) T and f are R-weakly commuting of type-(A_T).

Proof. Pick $x_0\in X$. We construct two sequences $\{x_n\}$ and $\{y_n\}$ as follows: Since $T(X)\subseteq f(X)$, choose x_1 so that $y_1 = fx_1 = Tx_0$. In general, choose x_{n+1} so that $y_{n+1} = fx_{n+1} = Tx_n$.

Applying (1.8), we have

$$\begin{aligned} d(Tx_n,Tx_{n+1}) &\leq a\, max\{d(fx_n,fx_{n+1}),d(fx_n,Tx_n),d(fx_{n+1},Tx_{n+1}) \\ &,\tfrac{1}{2}[M(x_n,x_{n+1})+m(x_n,x_{n+1})]\} \\ &+c[M(x_n,x_{n+1})+hm(x_n,x_{n+1})] \\ &= a\, max\{d(fx_n,Tx_n),d(fx_n,Tx_n),d(fx_{n+1},Tx_{n+1}) \\ &,\tfrac{1}{2}[M(x_n,x_{n+1})+m(x_n,x_{n+1})]\} \\ &+b\,[M(x_n,x_{n+1})+hm(x_n,x_{n+1})] \end{aligned} \tag{2.1}$$

where a and b are evaluated at (x_n,x_{n+1}). Since $m(x_n,x_{n+1})=0$ and $M(x_n,x_{n+1})=d(fx_n,Tx_{n+1})$. If we suppose that for some n, $d(fx_{n+1},Tx_{n+1})>d(fx_n,Tx_n)$. Then $M(x_n,x_{n+1})\leq 2d(fx_{n+1},Tx_{n+1})$ and the inequality (2.1) gives

$$d(fx_{n+1},Tx_{n+1})\leq (a+2b)d(fx_{n+1},Tx_{n+1})$$

a contradiction. Therefore, for all n we have

$$d(fx_{n+1},Tx_{n+1})\leq d(fx_n,Tx_n). \tag{2.2}$$

Again from (1.9), we have

$$d(y_{n-1}, Tx_n) = d(Tx_{n-2}, Tx_n)$$
$$\leq a\, max\{d(fx_{n-2}, fx_n), d(fx_{n-2}, Tx_{n-2}), d(fx_n, Tx_n)$$
$$, \frac{1}{2}[M(x_{n-2}, x_n) + m(x_{n-2}, x_n)]\}$$
$$+b[M(x_{n-2}, x_n) + hm(x_{n-2}, x_n)]$$
$$= a\, max\{d(fx_{n-2}, fx_n), d(fx_{n-2}, Tx_{n-2}), d(fx_n, Tx_n)$$
$$, \frac{1}{2}[M(x_{n-2}, x_n) + m(x_{n-2}, x_n)]\}$$
$$+b[M(x_{n-2}, x_n) + hm(x_{n-2}, x_n)] \qquad (2.3)$$

where a and b are evaluated at (x_{n-2}, x_n).

Since

$$d(fx_{n-2}, fx_n) \leq d(fx_{n-2}, fx_{n-1}) + d(fx_{n-1}, fx_n)$$
$$= d(fx_{n-2}, Tx_{n-2}) + d(fx_{n-1}, Tx_{n-1})$$
$$\leq 2d(fx_{n-2}, Tx_{n-2})$$
$$d(fx_{n-2}, Tx_n) \leq d(fx_{n-2}, fx_{n-1}) + d(fx_{n-1}, Tx_n)$$
$$\leq d(fx_{n-2}, fx_{n-1}) + d(fx_{n-1}, fx_n) + d(fx_n, Tx_n)$$
$$= d(fx_{n-2}, Tx_{n-2}) + d(fx_{n-1}, Tx_{n-1}) + d(fx_n, Tx_n)$$
$$\leq 3d(fx_{n-2}, Tx_{n-2})$$

and

$$d(fx_n, Tx_{n-2}) = d(Tx_{n-1}, fx_{n-1}) \leq d(fx_{n-2}, Tx_{n-2})$$

Hence

$$m(x_{n-2}, x_n) = min\{d(fx_{n-2}, Tx_n), d(fx_n, Tx_{n-2})\}$$
$$\leq min\{3d(fx_{n-2}, Tx_{n-2}), d(fx_{n-2}, Tx_{n-2})\}$$
$$= d(fx_{n-2}, Tx_{n-2})$$

and

$$M(x_{n-2}, x_n) = max\{d(fx_{n-2}, Tx_n), d(fx_n, Tx_{n-2})\}$$
$$\leq max\{3d(fx_{n-2}, Tx_{n-2}), d(fx_{n-2}, Tx_{n-2})\}$$
$$= 3d(fx_{n-2}, Tx_{n-2})$$

Using (2.2), the inequality (2.3) gives

$$d(Tx_{n-2}, Tx_n) \leq 2ad(fx_{n-2}, Tx_{n-2}) + b[3d(fx_{n-2}, Tx_{n-2}) + hd(fx_{n-2}, Tx_{n-2})]$$
$$= [2a + b(3 + h)]d(fx_{n-2}, Tx_{n-2})$$
$$= [2 - b(1 - h)]d(fx_{n-2}, Tx_{n-2}) \qquad (2.4)$$

Again from (1.9), we have

$$d(y_n, y_{n+1}) = d(Tx_{n-1}, Tx_n)$$
$$\leq a\, max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n)$$
$$, \frac{1}{2}[M(x_{n-1}, x_n) + m(x_{n-1}, x_n)]\}$$

$$+b[M(x_{n-1},x_n)+hm(x_{n-1},x_n)]$$
$$=a\max\{d(fx_{n-1},Tx_{n-1}),d(fx_{n-1},Tx_{n-1}),d(fx_n,Tx_n)$$
$$,\frac{1}{2}[M(x_{n-1},x_n)+m(x_{n-1},x_n)]\}$$
$$+b[M(x_{n-1},x_n)+hm(x_{n-1},x_n)] \qquad (2.5)$$

where a and b are evaluated at (x_{n-1},x_n).

Since $m(x_{n-1},x_n)=0$ and $M(x_{n-2},x_n)=d(Tx_{n-2},Tx_n)$. Using (2.2) and (2.4), the inequality (2.5) gives

$$d(y_n,y_{n+1})\le ad(fx_{n-2},Tx_{n-2})+bd(Tx_{n-2},Tx_n)$$
$$\le ad(fx_{n-2},Tx_{n-2})+b[2-b(1-h)]d(fx_{n-2},Tx_{n-2})$$
$$=[1-b^2(1-h)]d(fx_{n-2},Tx_{n-2})$$

Hence

$$d(y_n,y_{n+1})\le[1-\beta^2(1-h)]d(y_{n-2},y_{n-1})$$

Proceeding in this manner we obtain

$$d(Tx_{n-1},Tx_n)\le\left(1-\beta^2(1-h)\right)^{\left[\frac{n}{2}\right]}d(y_0,y_1) \qquad (2.6)$$

where $\left[\frac{n}{2}\right]$ stands for the greatest integer not exceeding $\frac{n}{2}$. Since $\beta=\inf\{b(x,y): x,y\in X\}>0$ and $h\in(0,1)$, $\{y_n\}$ is Cauchy, hence converges to a point p in X and then $fx_n\to p$ and $Tx_n\to p$ as $n\to+\infty$.

Now, we have the following three cases:

Case (a): Let f and T are compatible.

Since f and T are weakly reciprocally continuous, hence either $\lim_{n\to+\infty}Tfx_n=Tp$ or $\lim_{n\to+\infty}fTx_n=fp$. Let $\lim_{n\to+\infty}fTx_n=fp$. Compatibility of f and T implies that $\lim_{n\to+\infty}Tfx_n=fp$ and $\lim_{n\to+\infty}Tfx_{n+1}=\lim_{n\to+\infty}TTx_n=fp$.

Appling (1.9), we obtain

$$d(Tp,TTx_n)\le a\max\{d(fp,fTx_n),d(fp,Tp),d(fTx_n,TTx_n)$$
$$,\frac{1}{2}[M(p,Tx_n)+m(p,Tx_n)]\}$$
$$+c[M(p,Tx_n)+hm(p,Tx_n)]$$

Since $M(p,Tx_n)\to d(Tp,fp)$ and $m(p,Tx_n)\to 0$. On taking limit as $n\to+\infty$, in the above inequality, we get

$$d(Tp,fp)\le\sup_{x,y\in X}(a+c)\,d(Tp,fp)$$

Implies that $Tp=fp$. Hence there is a point $x_*\in X$ such that $fp=Tp=x_*$ Here, p is a coincidence point of f and T. Compatibility of f and T implies commutativity at coincidence point, hence $fTp=Tfp=TTp=ffp=fx_*=Tx_*$.

Appling (1.9) again, we obtain

$$d(x_*,Tx_*)=d(Tp,TTp)$$
$$\le a\max\{d(fp,fTp),d(fp,Tp),d(fTp,TTp)$$

$$,\frac{1}{2}[M(p,Tp)+m(p,Tp)]\}$$
$$+c\,[M(p,Tp)+hm(p,Tp)]$$

Since $M(p,Tp)=d(x_*,Tx_*)=m(p,Tp)$. From above inequality, we have

$$d(x_*,Tx_*)\leq[a+c(1+h)]d(x_*,Tx_*)$$
$$\leq[1-\beta(1-h)]d(x_*,Tx_*)$$

Since $\beta=inf\{b(x,y):x,y\in X\}>0$ and $h\in(0,1)$. Hence $d(x_*,Tx_*)=0$ and so $x_*=Tx_*=fx_*$.

Next, we assume that $lim_{n\to+\infty}Tfx_n=Tp$. Since $T(X)\subseteq f(X)$ implies that $Tp=fz$ for some $z\in X$ and $lim_{n\to+\infty}Tfx_n=fz$. Compatibility of f and T implies that $lim_{n\to+\infty}fTx_n=fz$ and $lim_{n\to+\infty}Tfx_{n+1}=lim_{n\to+\infty}TTx_n=fz$.

From (1.9), we have

$$d(Tz,TTx_n)\leq a\,max\{d(fz,fTx_n),d(fz,Tz),d(fTx_n,TTx_n)$$
$$,\frac{1}{2}[M(z,Tx_n)+m(z,Tx_n)]\}$$
$$+c\,[M(z,Tx_n)+hm(z,Tx_n)]$$

Since $M(z,Tx_n)\to d(Tz,fz)$ and $m(z,Tx_n)\to 0$ as $n\to+\infty$.

On taking limit as $n\to+\infty$, we get

$$d(Tz,fz)\leq sup_{x,y\in X}(a+c)\,d(Tz,fz)$$

This implies that $d(Tz,fz)=0$ and so $Tz=fz$. Hence there is a point $x^*\in X$ such that $fz=Tz=x^*$ Here, z is a coincidence point of f and T. Compatibility of f and T implies commutativity at coincidence point, hence $fTz=Tfz=TTz=ffz=fx^*=Tx^*$.Using (1.9), we obtain

$$d(x_*,Tx_*)=d(Tp,TTp)$$
$$\leq a\,max\{d(fp,fTp),d(fp,Tp),d(fTp,TTp)$$
$$,\frac{1}{2}[M(p,Tp)+m(p,Tp)]\}$$
$$+c\,[M(p,Tp)+hm(p,Tp)]$$

Since $M(p,Tp)=d(x_*,Tx_*)=m(p,Tp)$. From above inequality, we have

$$d(x_*,Tx_*)\leq[a+c(1+h)]d(x_*,Tx_*)$$
$$\leq[1-\beta(1-h)]d(x_*,Tx_*)$$

Since $\beta=inf\{b(x,y):x,y\in X\}>0$ and $h\in(0,1)$. Hence $d(x_*,Tx_*)=0$ and so $x_*=Tx_*=fx_*$.

Case (b): Now suppose that T and f are R-weakly commuting of type-(A_f).

Since f and T are weakly reciprocally continuous, hence either $lim_{n\to+\infty}Tfx_n=Tp$ or $lim_{n\to+\infty}fTx_n=fp$. Let $lim_{n\to+\infty}fTx_n=fp$. R-weakly commutativity of type- (A_f) of f and T implies that $d(TTx_n,fTx_n)\leq Rd(Tx_n,fx_n)$ and then $lim_{n\to+\infty}TTx_n=fp$ for some $p\in X$.

Appling (1.9), we obtain

$$d(Tp,TTx_n)\leq a\,max\{d(fp,fTx_n),d(fp,Tp),d(fTx_n,TTx_n)$$

$$,\frac{1}{2}[M(p,Tx_n)+m(p,Tx_n)]\}$$
$$+c[M(p,Tx_n)+hm(p,Tx_n)]$$

Since $M(z,Tx_n)\to d(Tz,fz)$ and $m(z,Tx_n)\to 0$ as $n\to+\infty$.

On taking limit as $n\to+\infty$, we get

$$d(Tp,fp)\le \sup_{x,y\in X}(a+c)\,d(Tp,fp)$$

This implies that $Tp=fp$. Hence there is a point $x_*\in X$ such that $fp=Tp=x_*$ Here, p is a coincidence point of f and T. Again by R-weakly commutativity of type-(A_f) of f and T, $d(TTp,fTp)\le Rd(Tp,fp)=0$. This gives$fTp=Tfp=TTp=ffp=fx_*=Tx_*$.

Appling (1.9) again, we obtain

$$d(x_*,Tx_*)=d(Tp,TTp)$$
$$\le a\,max\{d(fp,fTp),d(fp,Tp),d(fTp,TTp)$$
$$,\frac{1}{2}[M(p,Tp)+m(p,Tp)]\}$$
$$+c[M(p,Tp)+hm(p,Tp)]$$
$$\le sup_{x,y\in X}(a+c)\,d(x_*,Tx_*)$$

Hence $d(x_*,Tx_*)=0$ and so $x_*=Tx_*=fx_*$, $i.e.$ x_* is a common fixed point of f and T.

Next, we assume that $lim_{n\to+\infty}Tfx_n=Tp$. Since $T(X)\subseteq f(X)$ implies that $Tp=fz$ for some $z\in X$ and $lim_{n\to+\infty}Tfx_n=fz$ and $lim_{n\to+\infty}Tfx_{n+1}=lim_{n\to+\infty}TTx_n=fz$. Then R-weakly commutativity of type-(A_f) of f and T implies that $d(TTx_n,fTx_n)\le Rd(Tx_n,fx_n)$ and then $lim_{n\to+\infty}fTx_n=fz$ for some $p\in X$.

From (1.9), we have

$$d(Tz,TTx_n)\le amax\{d(fz,fTx_n),d(fz,Tz),d(fTx_n,TTx_n)$$
$$,\frac{1}{2}[M(z,Tx_n)+m(z,Tx_n)]\}$$
$$+c[M(z,Tx_n)+hm(z,Tx_n)]$$

On taking limit as $n\to+\infty$, we get

$$d(Tz,fz)\le sup_{x,y\in X}(a+c)\,d(Tz,fz)$$

This implies that $d(Tz,fz)=0$ and so $Tz=fz$. Hence there is a point $x^*\in X$ such that $fz=Tz=x^*$ Here, z is a coincidence point of f and T.Again by R-weakly commutativity of type-(A_f) of f and T implies that $d(TTz,fTz)\le Rd(Tz,fz)=0$ and then $fTz=Tfz=TTz=ffz=fx^*=Tx^*$.

Again from (1.9), we have

$$d(x^*,Tx^*)=d(Tz,TTz)$$
$$\le a\,max\{d(fz,fTz),d(fz,Tz),d(fTz,TTz)$$
$$,\frac{1}{2}[M(z,Tz)+m(z,Tz)]\}$$
$$+c[M(z,Tz)+hm(z,Tz)]$$

Implies that

$$d(x^*,Tx^*) \le sup_{x,y\in X}\big(a + c(1+h)\big)\, d(x^*,Tx^*)$$
$$\le sup_{x,y\in X}\big(1-\beta(1-h)\big)\, d(x^*,Tx^*)$$

Hence $d\big(d(x^*,Tx^*)\big) = 0$ and $x^* = Tx^* = fx^*$, *i.e.* x^* is common fixed point of f and T.

Case (c): Now suppose that T and f are R-weakly commuting of type-(A_T).

Since f and T are weakly reciprocally continuous, hence either

$$lim_{n\to+\infty} Tfx_n = Tp \quad \text{or} \quad lim_{n\to+\infty} fTx_n = fp.$$

Let $lim_{n\to+\infty} fTx_n = fp$. Then $lim_{n\to+\infty} fTx_n = lim_{n\to+\infty} ffx_{n+1} = fp$. R-weakly commutativity of type-(A_T) of f and T implies that

$$d(Tfx_n, ffx_n) \le Rd(Tx_n, fx_n)$$

and then $lim_{n\to+\infty} Tfx_n = fp$ for some $p \in X$.

Also $lim_{n\to+\infty} Tfx_{n+1} = lim_{n\to+\infty} TTx_n = fp$.

Appling (1.9), we obtain

$$d(Tp,TTx_n) \le a\, max\{d(fp,fTx_n), d(fp,Tp), d(fTx_n,TTx_n)$$
$$,\tfrac{1}{2}[M(p,Tx_n) + m(p,Tx_n)]\}$$
$$+c\,[M(p,Tx_n) + hm(p,Tx_n)]$$

Since $M(p,Tx_n) \to d(Tp,fp)$ and $m(p,Tx_n) \to 0$ as $n \to +\infty$.

On taking limit as $n \to +\infty$, we get

$$d(Tp,fp) \le sup_{x,y\in X}(a+c)\, d(Tp,fp)$$

Hence $d(Tp,fp) = 0$ and so $Tp = fp$. Hence there is a point $x_* \in X$ such that $fp = Tp = x_*$ Here, p is a coincidence point of f and T. Again by R-weakly commutativity of type-(A_T) of f and T, $d(Tfp,ffp) \le Rd(Tp,fp) = 0$. This gives $Tfp = ffp = TTp = fTp = fx_* = Tx_*$ and then Applying (1.9) again, we obtain $d(x_*,Tx_*) = 0$ and so $x_* = Tx_* = fx_*$, *i. e.* x_* is a common fixed point of f and T.

Next, we assume that $lim_{n\to+\infty} Tfx_n = Tp$. Since $T(X) \subseteq f(X)$ implies that $Tp = fz$ for some $z \in X$ and $lim_{n\to+\infty} Tfx_n = fz$ and $lim_{n\to+\infty} Tfx_{n+1} = lim_{n\to+\infty} TTx_n = fz$. Then R-weakly commutativity of type-(A_T) of f and T implies that $d(Tfx_n, ffx_n) \le Rd(Tx_n, fx_n)$ and then $lim_{n\to+\infty} ffx_n = fz$ for some $p \in X$. Also $lim_{n\to+\infty} ffx_{n+1} = lim_{n\to+\infty} fTx_n = fz$.

From (1.9), we have

$$d(Tz,TTx_n) \le a\, max\{d(fz,fTx_n), d(fz,Tz), d(fTx_n,TTx_n)$$
$$,\tfrac{1}{2}[M(z,Tx_n) + m(z,Tx_n)]\}$$
$$+c\,[M(z,Tx_n) + hm(z,Tx_n)]$$

Since $M(z,Tx_n) \to d(Tz,fz)$ and $m(p,Tx_n) \to 0$ as $n \to +\infty$.

On taking limit as $n \to +\infty$, we get

$$d(Tz, fz) \leq sup_{x,y \in X}(a + c)\, d(Tz, fz)$$

This implies that $d(Tz, fz) = 0$ and so $Tz = fz$. Hence there is a point $x^* \in X$ such that $fz = Tz = x^*$ Here, z is a coincidence point of f and T.Again by R-weakly commutativity of type-(A_T) of f and T implies that $d(fTz, ffz) \leq Rd(Tz, fz) = 0$ and then $fTz = ffz = TTz = Tfz = fx^* = Tx^*$.Again from (1.4), we have $d(x^*, Tx^*) = 0$ and $x^* = Tx^* = fx^*$, *i.e.* x^* is common fixed point of f and T.

Our next result is:

Theorem 2.2 Let T and f be two weakly reciprocally continuous self maps of a complete metric space (X, d) satisfying (1.12) where a, b, c, e and h with $T(X) \subseteq f(X)$, then T and f have a common fixed point in X if either (a) T and f are compatible or (b) T and f are R-weakly commuting of type$-(A_f)$ or (c) T and f are R-weakly commuting of type-(A_T).

Proof. Pick $x_0 \in X$. We construct two sequences $\{x_n\}$ and $\{y_n\}$ as follows: Since $T(X) \subseteq f(X)$, choose x_1 so that $y_1 = fx_1 = Tx_0$. In general, choose x_{n+1} so that $y_{n+1} = fx_{n+1} = Tx_n$.

Applying (1.12), we have

$$\begin{aligned}
d(y_{n+1}, y_{n+2}) &= d(Tx_n, Tx_{n+1}) \\
&\leq a\, d(fx_n, fx_{n+1}) + b\, max\{d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1})\} \\
&\quad + c\, max\{d(fx_n, fx_{n+1}), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1})\} \\
&\quad + e\, max\{d(fx_n, fx_{n+1}), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1})\ d(fx_n, Tx_{n+1})\} \\
&\quad + h\, max\{d(fx_n, fx_{n+1}), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1}), d(fx_n, Tx_{n+1}) \\
&\quad , (fx_{n+1}, Tx_n)\} \\
&\leq a\, d(fx_n, Tx_n) + b\, max\{d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1})\} \\
&\quad + c\, max\{d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1})\} \\
&\quad + (e + h)\, max\{d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1}) \\
&\quad , d(fx_n, Tx_n) + d(fx_{n+1}, Tx_{n+1})\}
\end{aligned}$$

where $a,\ b,\ c,\ e$ and h are evaluated at (x_n, x_{n+1}).

Suppose that for some n,

$$d(fx_{n+1}, Tx_{n+1}) > d(fx_n, Tx_n).$$

Then substituting in the above inequality, we have

$$d(fx_{n+1}, Tx_{n+1}) < \big(a + b + c + 2(e + h)\big)\, d(fx_n, Tx_n)$$

a contradiction. Therefore, for all n we have

$$d(fx_{n+1}, Tx_{n+1}) \leq d(fx_n, Tx_n) \qquad (2.7)$$

Again from (1.12), (2.7) and triangle inequality we have

$$\begin{aligned}
d(y_{n-1}, Tx_n) &= d(Tx_{n-2}, Tx_n) \\
&\leq a\, d(fx_{n-2}, fx_n) + b\, max\{d(fx_{n-2}, Tx_{n-2}), d(fx_n, Tx_n)\} \\
&\quad + c\, max\{d(fx_{n-2}, fx_n), d(fx_{n-2}, Tx_{n-2}), d(fx_n, Tx_n)\}
\end{aligned}$$

$$+ e\ max\{d(fx_{n-2}, fx_n), d(fx_{n-2}, Tx_{n-2}), d(fx_n, Tx_n)$$
$$, d(fx_{n-2}, Tx_n)\}$$
$$+ h\ max\{d(fx_{n-2}, fx_n), d(fx_{n-2}, Tx_{n-2}), d(fx_n, Tx_n)$$
$$, d(fx_{n-2}, Tx_n), d(fx_n, Tx_{n-2})\}$$

where $a,\ b,\ c,\ e$ and h are evaluated at (x_{n-2}, x_n). Since

$$d(fx_{n-2}, fx_n) \leq d(fx_{n-2}, Tx_{n-2}) + d(Tx_{n-2}, fx_n)$$
$$= d(fx_{n-2}, Tx_{n-2}) + d(fx_{n-1}, Tx_{n-1})$$
$$\leq 2d(fx_{n-2}, Tx_{n-2})$$

Hence the last inequality gives

$$d(y_{n-1}, Tx_n) \leq 2a\ d(fx_{n-2}, Tx_{n-2}) + b\ d(fx_{n-2}, Tx_{n-2}) + 2c(fx_{n-2}, Tx_{n-2})$$
$$+ e\ max\{2d(fx_{n-2}, Tx_{n-2}), d(fx_{n-2}, Tx_{n-2}) + d(fx_{n-1}, Tx_n)\}$$
$$+ h\ max\{2d(fx_{n-2}, Tx_{n-2}), d(fx_{n-2}, Tx_{n-2}) + d(fx_{n-1}, Tx_n)$$
$$, (fx_n, Tx_{n-2})\}$$
$$\leq 2a\ d(fx_{n-2}, Tx_{n-2}) + b\ d(fx_{n-2}, Tx_{n-2}) + 2c(fx_{n-2}, Tx_{n-2})$$
$$+ e\ max\{2d(fx_{n-2}, Tx_{n-2}), d(fx_{n-2}, Tx_{n-2}) + d(fx_{n-1}, Tx_{n-1})$$
$$+ d(fx_n, Tx_n)\}$$
$$+ h\ max\{2d(fx_{n-2}, Tx_{n-2}), d(fx_{n-2}, Tx_{n-2}) + d(fx_{n-1}, Tx_{n-1})$$
$$+ d(fx_n, Tx_n), (Tx_{n-1}, fx_{n-1})\}$$
$$\leq 2a\ d(fx_{n-2}, Tx_{n-2}) + b\ d(fx_{n-2}, Tx_{n-2}) + 2c(fx_{n-2}, Tx_{n-2})$$
$$+ 3e\ d(fx_{n-2}, Tx_{n-2}) + \ 3h\ d(fx_{n-2}, Tx_{n-2})$$

This implies that

$$d(y_{n-1}, Tx_n) \leq (2a + b + 2c + 3e + 3h)d(fx_{n-2}, Tx_{n-2})$$
$$\leq (2 - b - e - h)d(fx_{n-2}, Tx_{n-2}) \qquad (2.8)$$

Using (1.12), (2.7) and (2.8), we have

$$d(y_n, Tx_n) = d(Tx_{n-1}, Tx_n)$$
$$\leq a\ d(fx_{n-1}, fx_n) + b\ max\{d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n)\}$$
$$+ c\ max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n)\}$$
$$+ e\ max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n)$$
$$, d(fx_{n-1}, Tx_n)\}$$
$$+ h\ max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n)$$
$$, d(fx_{n-1}, Tx_n), d(fx_n, Tx_{n-1})\}$$

where $a,\ b,\ c,\ e$ and h are evaluated at (x_{n-1}, x_n). Hence

$$d(y_n, Tx_n) \leq a\ d(fx_{n-2}, Tx_{n-2}) + b\ d(fx_{n-2}, Tx_{n-2}) + c\ d(fx_{n-2}, Tx_{n-2})$$
$$+ (e + h)\ (2 - b - e - h)d(fx_{n-2}, Tx_{n-2})$$
$$\leq \big(a + b + c + (e + h)\ (2 - b - e - h)\big)d(fx_{n-2}, Tx_{n-2})$$

$$\leq \left(1-(e+h)\,(b+e+h)\right)d(fx_{n-2},Tx_{n-2})$$
$$\leq (1-\beta\gamma)d(fx_{n-2},Tx_{n-2})$$
$$= (1-\beta\gamma)d(y_{n-1},y_{n-2})$$
$$\leq (1-\beta\gamma)^{\left[\frac{n}{2}\right]}d(y_0,y_1) \qquad (2.9)$$

where

$$\beta = inf_{x,y\in X}\left(e(x,y)+h(x,y)\right) > 0$$
$$\gamma = inf_{x,y\in X}\left(b(x,y)+e(x,y)+h(x,y)\right) > 0$$

where $\left[\frac{n}{2}\right]$ stands for the greatest integer not exceeding $\frac{n}{2}$.From (2.9), $\{y_n\}$ is Cauchy, hence converges to some point p in X and $lim_{n\to+\infty} fx_{n+1} = lim_{n\to+\infty} Tx_n = p \in X$.

Now, we have the following three cases:

Case (a): Let f and T are compatible.

Since f and T are weakly reciprocally continuous, hence either $lim_{n\to+\infty} Tfx_n = Tp$ or $lim_{n\to+\infty} fTx_n = fp$. Let $lim_{n\to+\infty} fTx_n = fp$. Compatibility of f and T implies that $lim_{n\to+\infty} Tfx_n = fp$ and $lim_{n\to+\infty} Tfx_{n+1} = lim_{n\to+\infty} TTx_n = fp$.

Appling (1.12), we obtain

$$d(Tp,TTx_n) \leq a\, d(fp,fTx_n) + b\, max\{d(fp,Tp), d(fTx_n,TTx_n)\}$$
$$+ c\, max\{d(fp,fTx_n), d(fp,Tp), d(fTx_n,TTx_n)\}$$
$$+ e\, max\{d(fp,fTx_n), d(fp,Tp), d(fTx_n,TTx_n)\ d(fp,TTx_n)\}$$
$$+ h\, max\{d(fp,fTx_n), d(fp,Tp), d(fTx_n,TTx_n), d(fp,TTx_n)$$
$$, d(fTx_n,Tp)\}$$

On taking limit as $n \to +\infty$, we get

$$d(Tp,fp) \leq \sup_{x,y\in X}(b+c+e+h)\, d(Tp,fp)$$

Since $\gamma = inf_{x,y\in X}(b+e+h) > 0$ and $sup_{x,y\in X}(a+b+c+2e+2h) = 1$ implies that $sup_{x,y\in X}(b+c+e+h) < 1$. Hence $d(Tp,fp) = 0$ and so $Tp = fp$. Hence there is a point $x_* \in X$ such that $fp = Tp = x_*$ Here, p is a coincidence point of f and T. Compatibility of f and T implies commutativity at coincidence point, hence $fTp = Tfp = TTp = ffp = fx_* = Tx_*$.

Appling (1.12) again, we obtain

$$d(x_*,Tx_*) = d(Tp,TTp)$$
$$\leq a\, d(fp,fTp) + b\, max\{d(fp,Tp), d(fTp,TTp)\}$$
$$+ c\, max\{d(fp,fTp), d(fp,Tp), d(fTp,TTp)\}$$
$$+ e\, max\{d(fp,fTp), d(fp,Tp), d(fTp,TTp)\ d(fp,Ty)\}$$
$$+ h\, max\{d(fp,fTp), d(fp,Tp), d(fTp,TTp), d(fp,TTp), d(fTp,Tp)\}$$
$$\leq sup_{x,y\in X}(a+c+e+h)\, d(x_*,Tx_*)$$

Since $\beta = inf_{x,y \in X}(b+e) > 0$, $\gamma = inf_{x,y \in X}(b+e+h) > 0$ and $sup_{x,y \in X}(a+b+c+2e+2h) = 1$ implies that $sup_{x,y \in X}(a+c+e+h) < 1$. Hence $d(x_*, Tx_*) = 0$ and so $x_* = Tx_* = fx_*$, $i.\ e.\ x_*$ is a common fixed point of f and T.

Next, we assume that $lim_{n \to +\infty} Tfx_n = Tp$. Since $T(X) \subseteq f(X)$ implies that $Tp = fz$ for some $z \in X$ and $lim_{n \to +\infty} Tfx_n = fz$. Compatibility of f and T implies that $lim_{n \to +\infty} fTx_n = fz$ and $lim_{n \to +\infty} Tfx_{n+1} = lim_{n \to +\infty} TTx_n = fz$.

From (1.12), we have

$$d(Tz, TTx_n) \leq a\, d(fz, fTx_n) + b\, max\{d(fz, Tz), d(fTx_n, TTx_n)\}$$
$$+ c\, max\{d(fz, fTx_n), d(fz, Tz), d(fTx_n, TTx_n)\}$$
$$+ e\, max\{d(fz, fTx_n), d(fz, Tz), d(fTx_n, TTx_n)\ d(fz, TTx_n)\}$$
$$+ h\, max\{d(fz, fTx_n), d(fz, Tz), d(fTx_n, TTx_n), d(fz, TTx_n)$$
$$, d(fTx_n, Tz)\}$$

On taking limit as $n \to +\infty$, we get

$$d(Tz, fz) \leq sup_{x,y \in X}(b+c+e+h)\, d(Tz, fz)$$

This implies that $d(Tz, fz) = 0$ and so $Tz = fz$. Hence there is a point $x^* \in X$ such that $fz = Tz = x^*$ Here, z is a coincidence point of f and T. Compatibility of f and T implies commutativity at coincidence point, hence $fTz = Tfz = TTz = ffz = fx^* = Tx^*$.

Again from (1.12), we have

$$d(x^*, Tx^*) = d(Tz, TTz)$$
$$\leq a\, d(fz, fTz) + b\, max\{d(fz, Tz), d(fTz, TTz)\}$$
$$+ c\, max\{d(fz, fTz), d(fz, Tz), d(fTz, TTz)\}$$
$$+ e\, max\{d(fz, fTz), d(fz, Tz), d(fTz, TTz)\ d(fz, TTz)\}$$
$$+ h\, max\{d(fz, fTz), d(fz, Tz), d(fTz, TTz), d(fz, TTz), d(fTz, Tz)\}$$

Implies that

$$d(x^*, Tx^*) \leq sup_{x,y \in X}(a+c+e+h)\, d(x^*, Tx^*)$$

Hence $d\big(d(x^*, Tx^*)\big) = 0$ and $x^* = Tx^* = fx^*$, *i.e.* x^* is common fixed point of f and T.

Case (b): Now suppose that T and f are R-weakly commuting of type-(A_f).

Since f and T are weakly reciprocally continuous, hence either $lim_{n \to +\infty} Tfx_n = Tp$ or $lim_{n \to +\infty} fTx_n = fp$. Let $lim_{n \to +\infty} fTx_n = fp$. R-weakly commutativity of type- (A_f) of f and T implies that $d(TTx_n, fTx_n) \leq Rd(Tx_n, fx_n)$ and then $lim_{n \to +\infty} TTx_n = fp$ for some $p \in X$.

Appling (1.12), we obtain

$$d(Tp, TTx_n) \leq a\, d(fp, fTx_n) + b\, max\{d(fp, Tp), d(fTx_n, TTx_n)\}$$
$$+ c\, max\{d(fp, fTx_n), d(fp, Tp), d(fTx_n, TTx_n)\}$$
$$+ e\, max\{d(fp, fTx_n), d(fp, Tp), d(fTx_n, TTx_n)\ d(fp, TTx_n)\}$$

$+ h\, max\{d(fp, fTx_n), d(fp, Tp), d(fTx_n, TTx_n), d(fp, TTx_n)$
$, d(fTx_n, Tp)\}$

On taking limit as $n \to +\infty$, we get

$$d(Tp, fp) \leq \sup_{x,y \in X} (b + c + e + h)\, d(Tp, fp)$$

Hence $d(Tp, fp) = 0$ and so $Tp = fp$. Hence there is a point $x_* \in X$ such that $fp = Tp = x_*$ Here, p is a coincidence point of f and T. Again by R-weakly commutativity of type-(A_f) of f and T, $d(TTp, fTp) \leq Rd(Tp, fp) = 0$. This gives$fTp = Tfp = TTp = ffp = fx_* = Tx_*$.

Appling (1.12) again, we obtain

$d(x_*, Tx_*) = d(Tp, TTp)$
$\leq a\, d(fp, fTp) + b\, max\{d(fp, Tp), d(fTp, TTp)\}$
$+ c\, max\{d(fp, fTp), d(fp, Tp), d(fTp, TTp)\}$
$+ e\, max\{d(fp, fTp), d(fp, Tp), d(fTp, TTp)\ d(fp, Ty)\}$
$+ h\, max\{d(fp, fTp), d(fp, Tp), d(fTp, TTp), d(fp, TTp)$
$, d(fTp, Tp)\}$
$\leq sup_{x,y \in X}(a + c + e + h)\, d(x_*, Tx_*)$

Hence $d(x_*, Tx_*) = 0$ and so $x_* = Tx_* = fx_*$, $i.e.$ x_* is a common fixed point of f and T.

Next, we assume that $lim_{n \to +\infty} Tfx_n = Tp$. Since $T(X) \subseteq f(X)$ implies that $Tp = fz$ for some $z \in X$ and $lim_{n \to +\infty} Tfx_n = fz$ and $lim_{n \to +\infty} Tfx_{n+1} = lim_{n \to +\infty} TTx_n = fz$. Then R-weakly commutativity of type-(A_f) of f and T implies that $d(TTx_n, fTx_n) \leq Rd(Tx_n, fx_n)$ and then $lim_{n \to +\infty} fTx_n = fz$ for some $p \in X$.

From (1.12), we have

$d(Tz, TTx_n) \leq a\, d(fz, fTx_n) + b\, max\{d(fz, Tz), d(fTx_n, TTx_n)\}$
$+ c\, max\{d(fz, fTx_n), d(fz, Tz), d(fTx_n, TTx_n)\}$
$+ e\, max\{d(fz, fTx_n), d(fz, Tz), d(fTx_n, TTx_n)\ d(fz, TTx_n)\}$
$+ h\, max\{d(fz, fTx_n), d(fz, Tz), d(fTx_n, TTx_n), d(fz, TTx_n)$
$, d(fTx_n, Tz)\}$

On taking limit as $n \to +\infty$, we get

$$d(Tz, fz) \leq sup_{x,y \in X}(b + c + e + h)\, d(Tz, fz)$$

This implies that $d(Tz, fz) = 0$ and so $Tz = fz$. Hence there is a point $x^* \in X$ such that $fz = Tz = x^*$ Here, z is a coincidence point of f and T.Again by R-weakly commutativity of type-(A_f) of f and T implies that $d(TTz, fTz) \leq Rd(Tz, fz) = 0$ and then $fTz = Tfz = TTz = ffz = fx^* = Tx^*$.

Again from (1.12), we have

$d(x^*, Tx^*) = d(Tz, TTz)$
$\leq a\, d(fz, fTz) + b\, max\{d(fz, Tz), d(fTz, TTz)\}$

$$+ c\ max\{d(fz, fTz), d(fz, Tz), d(fTz, TTz)\}$$
$$+ e\ max\{d(fz, fTz), d(fz, Tz), d(fTz, TTz)\ d(fz, TTz)\}$$
$$+ h\ max\{d(fz, fTz), d(fz, Tz), d(fTz, TTz), d(fz, TTz), d(fTz, Tz)\}$$

Implies that

$$d(x^*, Tx^*) \leq sup_{x,y\in X}(a + c + e + h)\ d(x^*, Tx^*)$$

Hence $d\big(d(x^*, Tx^*)\big) = 0$ and $x^* = Tx^* = fx^*$, *i.e.* x^* is common fixed point of f and T.

Case (c): Now suppose that T and f are R-weakly commuting of type-(A_T).

Since f and T are weakly reciprocally continuous, hence either

$$lim_{n\to+\infty} Tfx_n = Tp \quad \text{or} \quad lim_{n\to+\infty} fTx_n = fp.$$

Let $lim_{n\to+\infty} fTx_n = fp$. Then $lim_{n\to+\infty} fTx_n = lim_{n\to+\infty} ffx_{n+1} = fp$. R-weakly commutativity of type-(A_T) of f and T implies that

$$d(Tfx_n, ffx_n) \leq Rd(Tx_n, fx_n)$$

and then $lim_{n\to+\infty} Tfx_n = fp$ for some $p \in X$.

Also $lim_{n\to+\infty} Tfx_{n+1} = lim_{n\to+\infty} TTx_n = fp$.

Appling (1.12), we obtain

$$d(Tp, TTx_n) \leq a\ d(fp, fTx_n) + b\ max\{d(fp, Tp), d(fTx_n, TTx_n)\}$$
$$+ c\ max\{d(fp, fTx_n), d(fp, Tp), d(fTx_n, TTx_n)\}$$
$$+ e\ max\{d(fp, fTx_n), d(fp, Tp), d(fTx_n, TTx_n)\ d(fp, TTx_n)\}$$
$$+ h\ max\{d(fp, fTx_n), d(fp, Tp), d(fTx_n, TTx_n), d(fp, TTx_n)$$
$$, d(fTx_n, Tp)\}$$

On taking limit as $n \to +\infty$, we get

$$d(Tp, fp) \leq \sup_{x,y\in X} (b + c + e + h)\ d(Tp, fp)$$

Hence $d(Tp, fp) = 0$ and so $Tp = fp$. Hence there is a point $x_* \in X$ such that $fp = Tp = x_*$ Here, p is a coincidence point of f and T. Again by R-weakly commutativity of type-(A_T) of f and T, $d(Tfp, ffp) \leq Rd(Tp, fp) = 0$. This gives $Tfp = ffp = TTp = fTp = fx_* = Tx_*$.

Appling (1.12) again, we obtain

$$d(x_*, Tx_*) = d(Tp, TTp)$$
$$\leq a\ d(fp, fTp) + b\ max\{d(fp, Tp), d(fTp, TTp)\}$$
$$+ c\ max\{d(fp, fTp), d(fp, Tp), d(fTp, TTp)\}$$
$$+ e\ max\{d(fp, fTp), d(fp, Tp), d(fTp, TTp)\ d(fp, Ty)\}$$
$$+ h\ max\{d(fp, fTp), d(fp, Tp), d(fTp, TTp), d(fp, TTp)$$
$$, d(fTp, Tp)\}$$
$$\leq sup_{x,y\in X}(a + c + e + h)\ d(x_*, Tx_*)$$

Hence $d(x_*, Tx_*) = 0$ and so $x_* = Tx_* = fx_*$, *i. e.* x_* is a common fixed point of f and T.

Next, we assume that $lim_{n\to+\infty} Tfx_n = Tp$. Since $T(X) \subseteq f(X)$ implies that $Tp = fz$ for some $z \in X$ and $lim_{n\to+\infty} Tfx_n = fz$ and $lim_{n\to+\infty} Tfx_{n+1} = lim_{n\to+\infty} TTx_n = fz$. Then R-weakly commutativity of type-(A_T) of f and T implies that $d(Tfx_n, ffx_n) \leq Rd(Tx_n, fx_n)$ and then $lim_{n\to+\infty} ffx_n = fz$ for some $p \in X$. Also $lim_{n\to+\infty} ffx_{n+1} = lim_{n\to+\infty} fTx_n = fz$.

From (1.12), we have

$$\begin{aligned} d(Tz, TTx_n) \leq & \; a\, d(fz, fTx_n) + b\, max\{d(fz, Tz), d(fTx_n, TTx_n)\} \\ & + c\, max\{d(fz, fTx_n), d(fz, Tz), d(fTx_n, TTx_n)\} \\ & + e\, max\{d(fz, fTx_n), d(fz, Tz), d(fTx_n, TTx_n)\ d(fz, TTx_n)\} \\ & + h\, max\{d(fz, fTx_n), d(fz, Tz), d(fTx_n, TTx_n), d(fz, TTx_n) \\ & , d(fTx_n, Tz)\} \end{aligned}$$

On taking limit as $n \to +\infty$, we get

$$d(Tz, fz) \leq sup_{x,y\in X}(b + c + e + h)\, d(Tz, fz)$$

This implies that $d(Tz, fz) = 0$ and so $Tz = fz$. Hence there is a point $x^* \in X$ such that $fz = Tz = x^*$ Here, z is a coincidence point of f and T.Again by R-weakly commutativity of type-(A_T) of f and T implies that $d(fTz, ffz) \leq Rd(Tz, fz) = 0$ and then $fTz = ffz = TTz = Tfz = fx^* = Tx^*$.

Again from (1.12), we have

$$\begin{aligned} d(x^*, Tx^*) = & \; d(Tz, TTz) \\ & \leq a\, d(fz, fTz) + b\, max\{d(fz, Tz), d(fTz, TTz)\} \\ & + c\, max\{d(fz, fTz), d(fz, Tz), d(fTz, TTz)\} \\ & + e\, max\{d(fz, fTz), d(fz, Tz), d(fTz, TTz)\ d(fz, TTz)\} \\ & + h\, max\{d(fz, fTz), d(fz, Tz), d(fTz, TTz), d(fz, TTz), d(fTz, Tz)\} \end{aligned}$$

Implies that

$$d(x^*, Tx^*) \leq sup_{x,y\in X}(a + c + e + h)\, d(x^*, Tx^*)$$

Hence $d\big(d(x^*, Tx^*)\big) = 0$ and $x^* = Tx^* = fx^*$, *i.e.* x^* is common fixed point of f and T.

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