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# A Study of Multiphase Queueing System of Three Servers for Structured Markov Chain under Matrix Geometric Method

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**ABSTRACT**

In this, we are analysing a multiphase queueing system of three heterogeneous servers using matrix – geometric method. Arrival of passengers follows the Poisson's distribution law with parameter λ, while the servers serving their services (parameter μ) using exponential distribution law on the basis of FCFS discipline. The solution of steady state queue length for a continuous Markov chain is derive by Matrix Geometric method. Various performance measurements for this system such as stationary queue length distribution, waiting time distribution and busy period distribution are obtained. Numerical example with graphical arrangement are also discuss.

**KEYWORDS:** Multiphase Queueing System, Matrix Geometric Method, Queue Size, Steady State, Transition Probability Rate Matrix.

## 1. Introduction

Recently queueing systems with multiservers have a wide range of utility. Normally it is suppose that the server is consistent, where the rate of services are same for all the servers which are working in the system. But practically it is possible in those cases where the service process is mechanically or electronically controlled. Most probably, the above assumption is not applicable in human based servers. It is observe that services are server depiction for same jobs at different service rates. Thus, we need a system with different servers where the distribution of service time is different for each server. In this paper, we consider a multiserver queueing model with three assorted servers. Consider a situation of bus stand having three servers, where one server is for bulk arrival, one server is for individual passengers and one server is for VIP passengers. By Matrix Geometric method, we can analyse Quasi Birth Death process (QBD's), continuous time Markov chain (CTMC) whose transition rate matrices have repetitive block structure also known as tridiagonal block structure.

The method of Matrix Geometric method is applied by many researchers for solving different types of queueing system. The tridiagonal block structure was first developed by Marcel F. Neuts in 1975. Neuts [12] defines Matrix Geometric solutions of various stochastic models. Neuts [11] developed a system of Markovian queue having N servers with respect to server breakdown and repair. Madhu Jain and Anamika Jain [10] developed a matrix geometric method for a queueing model with multiple vacations and breakdowns. Wang, Chen and Yang [20] defines the optimal management for machine repair problem with working vacations. Padma, Venkateswara [13] represents the approach of M/M/C/N queueing system by matrix method. Jau- Chuan Ke [6] describes the optimal solution of M/G/1 queueing system with server vacations, startup and breakdowns. Amani, Rayes [2] solving infinite stochastic process using matrix geometric method. Shah, Kumar [19] evaluate the controlled arrival rate system for Quasi Birth Death process. Ramswami, taylor [16] analysed the Quasi Birth Death process with countable number of phases for stochastic models of Markovian distribution. Various performance measures are generated through Markovian distribution. If we apply matrix geometric method for such a queueing system then block matrices can be solved easily without constructing the Markov chain. Matrix geometric method is helpful for constructing explicit formula for the probability distribution of the queue length. Krishna kumar, Madheshwari [7] analysed the queueing system with two heterogeneous servers and multiple vacations with respect to server breakdowns. Similarly, Chandrasekar, Renisagaya Raj [15] proposed the theory for matrix geometric method of a queueing system having *n*- vacations policy with respect to server breakdown and repair. Chandrasekar, Renisagaya Raj [17] also proposed matrix method for queueing system with state dependent arrival of an unreliable server and PH service. R.Aniyeri, R.Nadar [18] gives the model for multiphase queueing system with assorted number of servers. Qi Ming He, discuss some basics of matrix method in his book Fundamentals of Matrix – Analytic methods.

The remaining overview of this paper is as follows – In point 2, we represent the description of mathematical model. Point 3 stands for some proof of theorems used in methodology. Point 4 and 5 stands for numerical illustration and graphical representation. Finally conclusion is drawn in last one.

**1.1 Some basic terms related to matrix – geometric model**

Markov chain is a process of transitions from one state to another under some probabilistic rules. It is the probability of transitioning to a particular state which depends only on current state and time elapsed. Quasi – birth death process is the particular case of infinite state continuous time Markov chain (CTMCs). QBD is like two dimensional strip, which is finite in one dimension and infinite in the other. Now these strips are categorised in levels. First level is called border level (level 0) and remaining are repeating levels. Those transitions which occur between same level are shown by positive entries in Q, whereas repeating levels are in same inter and intra level transition structure. In the figure the interlevel transitions are $B_{01}$, $B_{10}$, $A_0$, $A_2$ whereas the intra-level transitions are $B_{00}$, $B_{11}$ and $A_1$

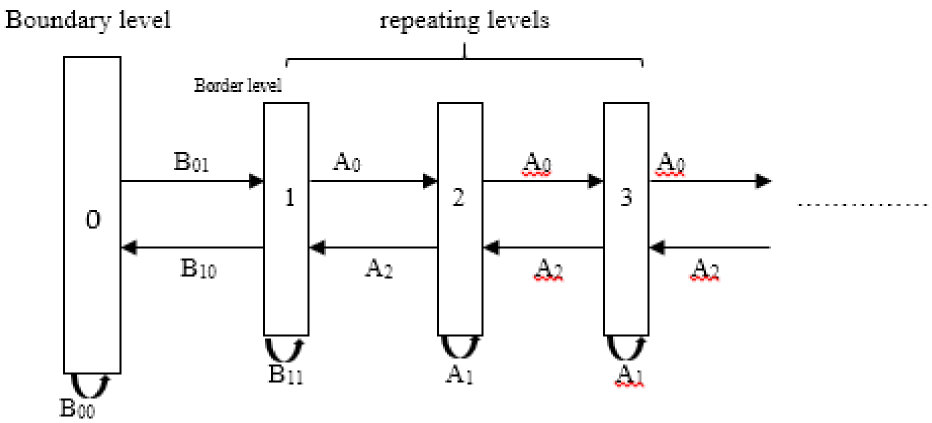


**Figure -** Distribution of levels in QBD

QBD process is the generalisation of birth death process. In this process state transition occurs between adjacent levels, while there is no restriction for phase transition. The transition rate matrix for a QBD process has a tridiagonal block structure.

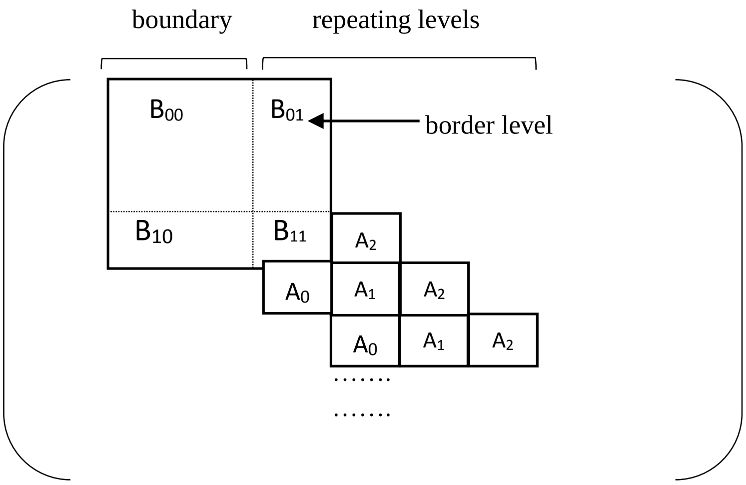


**Figure .** Block Tridiagonal matrix for Q

Where, $B_{OO} \in [\quad]_{M_O \times M_O}$

represents the intra-level transition structure (boundary level),

$B_{O1} \in [\ ]_{N \ X \ M_O}$

represents the inter-level transition structure from the boundary level to the border level,

$B_{1O} \in [\ ]_{M_O \ X \ N}$

represents the inter-level transition structure from the border level to the boundary level,

$B_{11} \in [\ ]_{N \ X \ N}$

represents the intra-level transition structure (border level),

$A_O \in [\ ]_{N \ X \ N}$

represents the transitions from one repeating level to the next higher level,

$A_1 \in [\ ]_{N \ X \ N}$

represents the intra level transitions for the repeating level,

$A_2 \in [\ ]_{N \ X \ N}$

represents the transitions from one repeating level to the next lower.

According to Neuts, their survey are solutions of matrix – geometric for stochastic models. For an irreducible, positive recurrent Markov chains, are block partitioned structure of the type,

$$\begin{pmatrix} B_0 & A_0 & 0 & 0 & 0 & 0 & \ldots\ldots\ldots \\ B_1 & A_1 & A_0 & 0 & 0 & 0 & \ldots\ldots\ldots \\ B_2 & A_2 & A_1 & A_0 & 0 & 0 & \ldots\ldots\ldots \\ & & & \ldots\ldots\ldots & & & \end{pmatrix}$$

which have an invariant probability vector of the form,

$$x_o,\ x_oT,\ x_oT^2,\ x_oT^3,\ \ldots$$

where matrix T is the minimal non – negative solution to a nonlinear matrix equation. Terms related to Markov chain are represent with respect to matrix T and vector $x_o$. The matrix T can be evaluated by numerical methodology. These results are widely used for stochastic models. In a vector space process, states are represented as pairs in the form $(x, y)$ where $x$ = 0, 1, … and $y$ =0, 1, …n. Normally $x$ represents the number of customers in the system and $y$ represents the state of the customer in service.

## 2. The Mathematical Model Description

Let us suppose M/M/3 queuing system. We assume that the service rate of server is different. The setup of the server is such that it works for both batch queue and individual queue. The arrival customers enter the terminal with single queue, now if servers are doing the batch service then batch queue passengers join the queue and if servers are doing the individual service then individual passengers join the queue. The arrival of passengers follows the Poisson's distribution law with parameter $\lambda$. The complete system i.e., total number of passengers and the system capacity are supposed to be fix.

There are three servers and these three server serving their services using exponential distribution law. Let $\mu_1$ stands for service rate of server 1, $\mu_2$ stands for service rate of server 2 and $\mu_3$ stands for service rate of server 3. The arriving customers follows according to first come first serve discipline. Using continuous Markov Chain, we can formulate the multiserver queueing system. The states of the system at any point are represented by $(i, j)$ where $i \geq 0$ stands for number of passengers in the system and $j = 1, 2, 3$ stands for states of the server. For example, state $(i, 1)$ means there are $i$ passengers in the system corresponds to server 1, state $(i, 2)$ means there are $i$ passengers in the system corresponds to server 2.

In a continuous time Markov Chain, if a transition occurs from state $i$ to state $j$ then it is represented by $Q(i, j) \geq 0$, for $i \neq j$, and $Q(i, i)$ denotes the negative sum of the off diagonal entries in the same row of Q.

The transition rate matrix for infinite state markov chain is generally represented by Q with tridiagonal structure and is given by,

$$Q = \begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & \cdots\cdots\cdots \\ B_{10} & C & B & 0 & 0 & 0 & \cdots\cdots\cdots \\ 0 & A & C & B & 0 & 0 & \cdots\cdots\cdots \\ 0 & 0 & A & C & B & 0 & \cdots\cdots\cdots \\ 0 & 0 & 0 & A & C & B & \cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{pmatrix}$$

Where each of $B_{00}$, $B_{10}$, $B_{01}$, B, A and C are matrices, matrix Q is the form of Quasi- birth death process.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & \mu_3 \end{pmatrix}, \qquad B = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix},$$

$$C = \begin{pmatrix} -(\mu_1+\mu_2+\mu_3+\lambda) & \mu_1 & \mu_2 & \mu_3 \\ 0 & -(\mu_1+\lambda) & 0 & 0 \\ 0 & 0 & -(\mu_2+\lambda) & 0 \\ 0 & 0 & 0 & -(\mu_3+\lambda) \end{pmatrix},$$

$$B_{00} = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_{01} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B_{10} = \begin{pmatrix} 0 & 0 & 0 \\ \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}$$

Note that the row sums are 0 *i.e.*,

$(B_{00} + B_{01})e = 0$, $(B_{10} + B + C)e = 0$ and $(B + C + A)e = 0$ ,

where *e* represents the column vector with all its elements equal to one.

Here the continuous time Markov chain with generators B, C, A is reducible with absorbing state (4, 4) and the stationary probability factor is Φ. The sub matrix form which satisfies the condition,

$$\Phi Be < \Phi Ae$$

is the necessary and sufficient condition for the stability of QBD process where e denotes the column vector with all its elements equal to 1(one). We have a well-known theorem / result also known as theorem of Ergodicity.

- ***Theorem 2.1***- The QBD is ergodic (i.e., mean recurrence time of the states is finite) iff

$$\Phi Be < \Phi Ae \quad \text{(mean drift condition)}$$

where e is the column vector of ones and Φ is the equilibrium distribution of the irreducible Markov chain with generator,

$$D = B + C + A \,,\ \Phi D = 0,\ \Phi e = 1$$

where Φ = stationary probability vector

e = column vector (value of each element is 1)

Above condition is also known as necessary and sufficient condition for stability of QBD process.

***Interpretation:*** ΦBe is the mean drift from level *i* to *i + 1,*

ΦAe is the mean drift from level *i* to *i - 1,* ( by using Neut's drift condition [12] )

Generator D decides the behaviour of Quasi Birth Death Chain within the level, where D is the generator of an irreducible Markov chain.

**2.1 Equilibrium distribution of QBD's**

For stability the required condition

$$\lambda < \mu_1 + \mu_2 + \mu_3 \qquad (2.1)$$

under equation (2.1) the stationary probability vector X of the matrix Q , the stationary probability vector $\Phi$ is partioned as follows, let $\Phi_n = ( \Phi(n,0),\dots \Phi(n,m-1))$ and $\Phi = ( \Phi_0, \Phi_1, \Phi_2, \dots)$ which is given by

$$\Phi_0 B_{00} + \Phi_1 B_{10} = 0 \tag{2.2}$$

$$\Phi_0 B_{01} + \Phi_1 C + \Phi_2 A = 0 \tag{2.3}$$

$$\Phi_1 B + \Phi_2 C + \Phi_3 A = 0 \tag{2.4}$$

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$$\Phi_{n-1} B + \Phi_n C + \Phi_{n+1} A = 0 \qquad n \geq 2 \tag{2.5}$$

- ***Theorem 2.2 –*** For the positive recurrent of Quasi Birth Death process, there exists a constant matrix R such that,

$$\Phi_n = \Phi_{n-1} R \quad , n \geq 2 \qquad \Rightarrow \qquad \Phi_n = \Phi_1 R^{n-1}, \quad n \geq 2$$

***Proof :*** See in section 3.

Using result of above theorem,

$$\Phi_n = \Phi_1 R^{n-1} \tag{2.6}$$

and the normal equation is,

$$1 = \Phi_0 + \Phi_1 e + ( I - R )^{-1} \Phi_2 e \tag{2.7}$$

where $\Phi_0$ is the probability for no passengers in the system,

I = identity matrix of order 4,

R = minimal non – negative solution of the matrix quadratic equation

$$B + RC + R^2 A = 0 \tag{2.8}$$

The vector, $\Phi_n = (\Phi_{n0}, \Phi_{n1}, \Phi_{n2})$ and $\Phi_1 = (\Phi_{10}, \Phi_{11}, \Phi_{12})$ for $n \geq 2$ denotes the probability for $n$ passengers in the system in which $\Phi_{ij}$ means the joint probability that there are $i$ customers in the system and $j$ = 0, 1, 2, 3 corresponds to the status of the servers in the system.

- ***Theorem 2.3 (R – Matrix lemma).*** The matrix R is the minimal non – negative solution of the matrix equation, $B + RC + R^2A = 0$

***Proof:*** Substituting $\Phi_n = \Phi_1 R^{n-1}$ , $n \geq 2$ in equation (2.5) we have

$\Phi_{n-1} B + \Phi_n C + \Phi_{n+1} A = 0 \qquad n \geq 2$ gives,

$\Phi_1 R^{n-2} B + \Phi_1 R^{n-1} C + \Phi_1 R^n A = 0$

$\Phi_1 R^{n-1} ( R^{-1} B + C + RA ) = 0$

$\Phi_1 R^{n-1} ( IB + RC + R^2 A ) = 0$

$\Phi_1 R^{n-1} ( B + RC + R^2 A ) = 0$

□ $\Phi_1 \neq 0, \quad R^{n-1} \neq 0, \quad B + RC + R^2 A = 0$

- R is called the *rate matrix* of the Markov process Q.
- The spectral radius of R is less than one *i.e.*, R < 1 , it implies ( I – R ) is invertible.

Now the square matrix A, B, C of order 4 x 4 are upper triangular matrices, R is also a 4 x 4 upper triangular matrix. Now the relation,

$$RBe = Ae \tag{2.9}$$

shows that the rate of transition from a state where there are $n$ passengers to a state with $n + 1$ matches the transition rate from $n$ *to* $n - 1$. To obtain R from equation (2.8),

$$B + RC + R^2 A = 0$$
$$RC = - B - R^2 A$$
$$R = - BC^{-1} - R^2 AC^{-1}$$

Using initial value of R = 0, we can find the value of R and check the accuracy of this approximation by using equation (2.9). Since $(-C^{-1})$ and $(B + R^2 A)$ are positive implies value of R converges. The iteration can be shown to converge to R (fixed point equation), since spectral radius < 1. Thus, for each iteration, the elements of R will increase monotonically. The boundary probabilities $\Phi_0$, $\Phi_1$, $\Phi_2$ and $\Phi_n$, $n > 3$ are obtained from solving equation (2.2) to (2.7).

## 3. Theorems Based on Above Methodology.

**Theorem 3.1** For a Quasi Birth Death process $(X_z, J_z)$ $z = 0, 1, 2..$ is ergodic its limiting probabilities is given by $\Phi_n = \Phi_1 R^{n-1}$ $n = 2, 3, ....$ where Φ = stationary probability vector, R = minimal non – negative solution of the matrix quadratic equation B + RC + $R^2A = 0$.

***Proof:*** Let

$$Q = \begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & \cdots\cdots \\ B_{10} & C & B & 0 & 0 & 0 & \cdots\cdots \\ 0 & A & C & B & 0 & 0 & \cdots\cdots \\ 0 & 0 & A & C & B & 0 & \cdots\cdots \\ \cdots\cdots\cdots & & & & & & \end{pmatrix}$$

and $\Phi = (\Phi_0, \Phi_1, \Phi_2, ....)$ where $\Phi_i = (\Phi_{i1}, \Phi_{i2}, \Phi_{i3}, .... \Phi_{iz})$ then from equilibrium equation ΦQ = 0 we have,

$$\left.\begin{aligned} \Phi_0 B_{00} + \Phi_1 B_{10} &= 0 \\ \Phi_0 B_{01} + \Phi_1 C + \Phi_2 A &= 0 \\ \Phi_1 B + \Phi_2 C + \Phi_3 A &= 0 \\ \Phi_2 B + \Phi_3 C + \Phi_4 A &= 0 \\ &\vdots \\ \Phi_{n-1} B + \Phi_n C + \Phi_{n+1} A &= 0 \end{aligned}\right\} \quad (3.1)$$

which is similar to solution of M/M/1. Therefore $\Phi_n$ is a function only of the transition rates between states with $n - 1$ queued customers and states with $n$ queued customers.

In analogy with the point situation, there exists a constant matrix R such that,

$$\Phi_n = \Phi_{n-1} R \quad , n \geq 2 \qquad (3.2)$$

The sub vectors $\Phi_i$ are geometrically related to each other since,

$$\Phi_n = \Phi_1 R^{n-1} \quad , n \geq 2 \qquad (3.3)$$

By putting equation (3.3) in equation (3.1) we have,

$$\Phi_{n-1} B + \Phi_n C + \Phi_{n+1} A = 0 \qquad n \geq 2 \qquad \text{gives,}$$

$$\Phi_1 R^{n-2} B + \Phi_1 R^{n-1} C + \Phi_1 R^n A = 0$$

$$\Rightarrow \quad B + RC + R^2 A = 0 \qquad (3.4)$$

*Value of R-*

$$\begin{aligned} RC &= - B - R^2 A \\ R &= - BC^{-1} - R^2 AC^{-1} \\ R &= - V - WR^2 \qquad (3.5) \end{aligned}$$

$$\text{where } V = BC^{-1}, \; W = AC^{-1}$$

<u>*Iteration:*</u>

$$R_{(0)} = 0, \qquad R_{(z+1)} = - V - W R_{(z)}^2 \qquad z = 1, 2, \ldots\ldots$$

**Theorem 3.2** The stationary probability vectors $\Phi_0$ and $\Phi_1$ are the unique positive solution of the linear system ΦQ = 0.

***Proof :*** By taking first two equations of system (3.1)

$$\Phi_0 B_{00} + \Phi_1 B_{10} = 0$$

$$\Phi_0 B_{01} + \Phi_1 C + \Phi_2 A = 0$$

$\because$ $\Phi_n = \Phi_{n-1} R$ implies $\Phi_2 = \Phi_1 R$, substituting value of $\Phi_2$

$$\Phi_0 B_{00} + \Phi_1 B_{10} = 0$$

$$\Phi_0 B_{01} + \Phi_1 C + \Phi_1 RA = 0$$

In matrix form,

$$(\Phi_0 \; , \Phi_1) \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & C + RA \end{bmatrix} = (0 \; , \; 0) \qquad (3.6)$$

can be solved for $\Phi_0$ and $\Phi_1$ with condition , $\Phi e = 1$

$$1 = \Phi e = \Phi_0 e + \Phi_1 e + \sum_{i=2}^{\infty} \Phi_i e$$

$$= \Phi_0 e + \Phi_1 e + \sum_{i=2}^{\infty} \Phi_1 R^{i-1} e$$

$$= \Phi_0 e + \sum_{i=1}^{\infty} \Phi_1 R^{i-1} e$$

$$= \Phi_0 e + \sum_{i=0}^{\infty} \Phi_1 R^{i} e$$

This implies the condition,

$$\Phi_0 e + \Phi_1 [\sum_{i=0}^{\infty} R^i ]e = 1$$

The eigen values of R lies within the circle of radius 1, implies that *(I – R)* is non – singular and hence

$$\sum_{i=0}^{\infty} R^i = S \ (let)$$

$$\Rightarrow \quad S = \sum_{i=0}^{\infty} R^i$$

$$S = \sum_{i=1}^{\infty} R^{i-1}$$

$$S = I + R + R^2 + \ldots.$$

$$SR = R + R^2 + R^3 + \ldots.$$

On subtracting $S (I – R) = I$

$$S = I (I – R)^{-1} = (I – R)^{-1} \qquad (3.7)$$

Normalise the vector $\Phi_0$ and $\Phi_1$ by solving ,

$$\alpha = \Phi_0 e + (I – R)^{-1} \Phi_1 e$$

and dividing the computed sub vectors $\Phi_0$ and $\Phi_1$ by α.

**Theorem 3.3** The expected number of customers in the queue is given by,

$$E(n) = \Phi_1 (I – R)^{-2} e$$

***Proof:*** Let there are *n* customers in the queue,

$$E(n) = E \text{ (queued customers)}$$

$$= \sum_{n=1}^{\infty} n \Phi_n e$$

$$= \sum_{n=1}^{\infty} n \Phi_1 R^{n-1} e \qquad \textit{(using equation 3.2)}$$

$$= \Phi_1 [\sum_{n=1}^{\infty} n R^{n-1} ]e$$

Put $\sum_{n=1}^{\infty} n R^{n-1} = S \ (let)$

$$\Rightarrow \quad S = \sum_{n=1}^{\infty} n R^{n-1} = ( I + 2R + 3R^2 + 4R^3 + \ldots..)$$

$$RS = R + 2R^2 + 3R^3 + \ldots..$$

On subtracting, $S (I - R) = I + R + R^2 + R^3 + \ldots..$

$$S = I (I – R)^{-2} = (I – R)^{-2}$$

$$\Rightarrow \quad E (n) = \Phi_1 [\sum_{n=1}^{\infty} n R^{n-1} ]e = \Phi_1 (I – R)^{-2} e$$

## 4. Numerical Illustration

For Matrix – Geometric method, we ensures that it follows following steps-

*Step 1: Verify that the matrix satisfies requirements of QBD structure (for this we check each transition rate matrix row sum equal to zero).*

*Step 2: Verify that stability condition is satisfied (for this we check Markovian is ergodic i.e., $\Phi Be < \Phi Ae$ with $D = B + C + A$ , $\Phi D = 0$, $\Phi e = 1$)*

*Step 3: Use recursion to compute the R-matrix*

*Step 4: Solve the set of equations to calculate $\Phi_0$ and $\Phi_1$*

*Step 5: Use recursion $\Phi_n = \Phi_{n-1}R$ to find all other $\Phi_n$ vectors.*

To understanding the performance of this queueing system, it is necessary to check the effect of parameters λ, $\mu_1$, $\mu_2$, $\mu_3$ on the system. Let us suppose the following values of parameters,

$$\lambda = 1,\ \mu_1 = 2,\ \mu_2 = 3 \text{ and } \mu_3 = 4$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & \mu_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix},$$

$$B = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} -(\mu_1+\mu_2+\mu_3+\lambda) & \mu_1 & \mu_2 & \mu_3 \\ 0 & -(\mu_1+\lambda) & 0 & 0 \\ 0 & 0 & -(\mu_2+\lambda) & 0 \\ 0 & 0 & 0 & -(\mu_3+\lambda) \end{pmatrix}$$

$$= \begin{pmatrix} -10 & 2 & 3 & 4 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -5 \end{pmatrix}$$

$$B_{00} = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B_{01} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B_{10} = \begin{pmatrix} 0 & 0 & 0 \\ \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$Q = \left[\begin{array}{c|c|c|c|c}
\begin{matrix}-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{matrix} & \begin{matrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{matrix} & & & \\ \hline
\begin{matrix}0 & 0 & 0\\ 2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 4\end{matrix} & \begin{matrix}-10 & 2 & 3 & 4\\ 0 & -3 & 0 & 0\\ 0 & 0 & -4 & 0\\ 0 & 0 & 0 & -5\end{matrix} & \begin{matrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{matrix} & & \\ \hline
 & \begin{matrix}0 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 4\end{matrix} & \begin{matrix}-10 & 2 & 3 & 4\\ 0 & -3 & 0 & 0\\ 0 & 0 & -4 & 0\\ 0 & 0 & 0 & -5\end{matrix} & \begin{matrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{matrix} & \text{---------} \\ \hline
 & & \begin{matrix}0 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 4\end{matrix} & \begin{matrix}-10 & 2 & 3 & 4\\ 0 & -3 & 0 & 0\\ 0 & 0 & -4 & 0\\ 0 & 0 & 0 & -5\end{matrix} & \begin{matrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{matrix} \\ \hline
 & & \ddots & \ddots & \ddots
\end{array}\right]$$

***Step 1.*** The matrix obviously has the correct QBD structure.

***Step 2.*** To check the system is stable,

$$D = B + C + A = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} + \begin{pmatrix}-10 & 2 & 3 & 4\\ 0 & -3 & 0 & 0\\ 0 & 0 & -4 & 0\\ 0 & 0 & 0 & -5\end{pmatrix} + \begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 4\end{pmatrix}$$

$$= \begin{pmatrix}-9 & 2 & 3 & 4\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

has probability stationary vector, $\Phi_D = (\, 0.2116, 0.4652, 0.3232, 0\,)$

$$\Phi_D Be = (\, 0.2116 \quad 0.4652 \quad 0.3232 \quad 0\,) \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\\ 1\\ 1\end{pmatrix} = 1.00$$

$$\text{And } \Phi_D Ae = (\, 0.2116 \quad 0.4652 \quad 0.3232 \quad 0\,) \begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 4\end{pmatrix}\begin{pmatrix}1\\ 1\\ 1\\ 1\end{pmatrix} = 1.90$$

- $\Phi Be < \Phi Ae$ , means Markovian is ergodic.

***Step 3.*** Recursion for R – matrix.

$$R = -BC^{-1} - R^2AC^{-1}$$
$$R = -V - WR^2$$

First we find inverse of C.

$$C^{-1} = \begin{pmatrix} -0.1 & -0.066 & -0.075 & -0.08 \\ 0 & -0.33 & 0 & 0 \\ 0 & 0 & -0.25 & 0 \\ 0 & 0 & 0 & -0.2 \end{pmatrix}$$

which allows us to compute,

$$V = BC^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -0.1 & -0.066 & -0.075 & -0.08 \\ 0 & -0.33 & 0 & 0 \\ 0 & 0 & -0.25 & 0 \\ 0 & 0 & 0 & -0.2 \end{pmatrix} = \begin{pmatrix} -0.1 & -0.066 & -0.075 & -0.08 \\ 0 & -0.33 & 0 & 0 \\ 0 & 0 & -0.25 & 0 \\ 0 & 0 & 0 & -0.2 \end{pmatrix}$$

$$\text{and } W = AC^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} -0.1 & -0.066 & -0.075 & -0.08 \\ 0 & -0.33 & 0 & 0 \\ 0 & 0 & -0.25 & 0 \\ 0 & 0 & 0 & -0.2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -0.66 & 0 & 0 \\ 0 & 0 & -0.75 & 0 \\ 0 & 0 & 0 & -0.8 \end{pmatrix}$$

- $R_{(z+1)} = -V - W\,R_{(z)}^2$

$$R_{(z+1)} = \begin{pmatrix} 0.1 & 0.066 & 0.075 & 0.08 \\ 0 & 0.33 & 0 & 0 \\ 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0.2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.66 & 0 & 0 \\ 0 & 0 & 0.75 & 0 \\ 0 & 0 & 0 & 0.8 \end{pmatrix} R_{(z)}^2$$

and iterating successively, starting with $R_{(0)} = 0$,

$$at\ z = 0, \quad R_{(1)} = \begin{pmatrix} 0.1 & 0.066 & 0.075 & 0.08 \\ 0 & 0.33 & 0 & 0 \\ 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0.2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.66 & 0 & 0 \\ 0 & 0 & 0.75 & 0 \\ 0 & 0 & 0 & 0.8 \end{pmatrix} R_{(0)}^2$$

$$R_{(1)} = \begin{pmatrix} 0.1 & 0.066 & 0.075 & 0.08 \\ 0 & 0.33 & 0 & 0 \\ 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0.2 \end{pmatrix}$$

Similarly, we can derive $R_2$, $R_3$, and so on.

***Step 4.*** Calculation of $\Phi_0$ , $\Phi_1$ and $\Phi_2$.

$$(\Phi_0\ ,\Phi_1,\ \Phi_2)\begin{bmatrix} B_{00} & B_{01} \\ B_{10} & C+RA \end{bmatrix} = (0\ ,\ 0\ ,\ 0)$$

$$(\Phi_0\ ,\Phi_1,\ \Phi_2)\left[\begin{array}{ccc|cccc} -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -10 & 2 & 3 & 4 \\ 2 & 0 & 0 & 0 & -234 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & -325 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 42 \end{array}\right] = (0\ ,\ 0\ ,\ 0)$$

Solve this by replacing last equation with $\Phi_{0_1} = 1$ , i.e., set the first component of the sub vector $\Phi_0$ to 1.

$$(\Phi_0, \Phi_1, \Phi_2)\left[\begin{array}{ccc|cccc} -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -10 & 2 & 3 & 4 \\ 2 & 0 & 0 & 0 & -234 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & -325 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & -42 \end{array}\right] = (0, 0, 0, 1)$$

$$(\Phi_0\ ,\Phi_1,\ \Phi_2) = (\ 1,\ 0\ ,\ 0\ |\ 0.5396,\quad 0.9578,\quad 0.5462,\quad 0.3254)$$

Now for normalisation, let α is the normalisation constant then,

$$\alpha = \Phi_0 + \Phi_1 e + (I - R)^{-1}\ \Phi_2 e = 5.6724$$

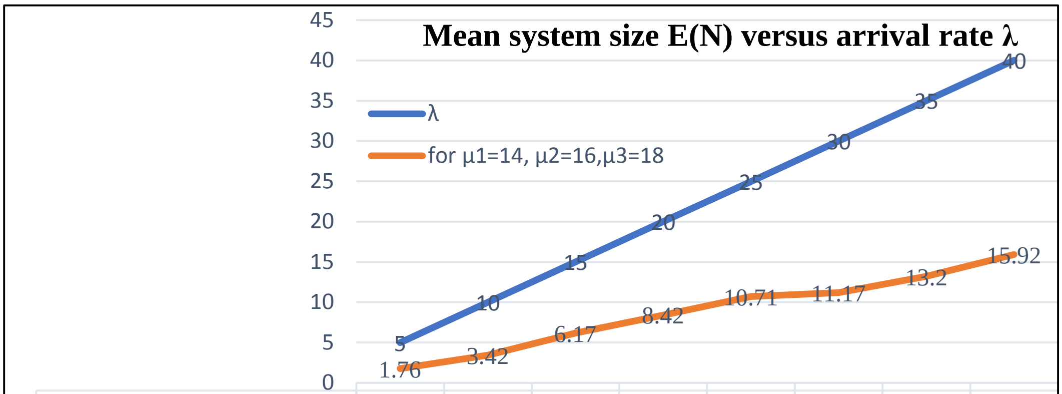
***Step 5.*** Using relation $\Phi_n = \Phi_{n-1}R$, we have the result.

**5. Graphical Illustration of Average Queue Length over Different Values Of Λ.**

In section 6, we are trying to observe the variation of mean system size *E(N)* versus arrival rate λ.

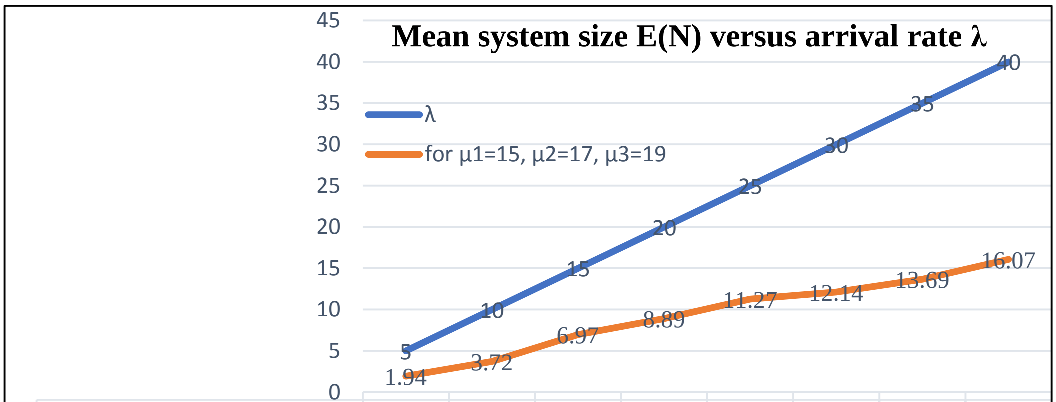
| Mean number of passengers in the system, λ | When $\mu_1 = 14$, $\mu_2 = 16$, $\mu_3 = 18$ | When $\mu_1 = 15$, $\mu_2 = 17$, $\mu_3 = 19$ | When $\mu_1 = 16$, $\mu_2 = 18$, $\mu_3 = 22$ |
|---|---|---|---|
| 5 | 1.76 | 1.94 | 2.01 |
| 10 | 3.42 | 3.72 | 4.10 |
| 15 | 6.17 | 6.97 | 7.12 |
| 20 | 8.42 | 8.89 | 8.99 |
| 25 | 10.71 | 11.27 | 12.12 |
| 30 | 11.71 | 12.14 | 13.16 |
| 35 | 13.20 | 13.69 | 14.11 |
| 40 | 15.92 | 16.07 | 16.14 |

It is clear from the above table is when the capacity of the system increases then the expected queue length also increases.



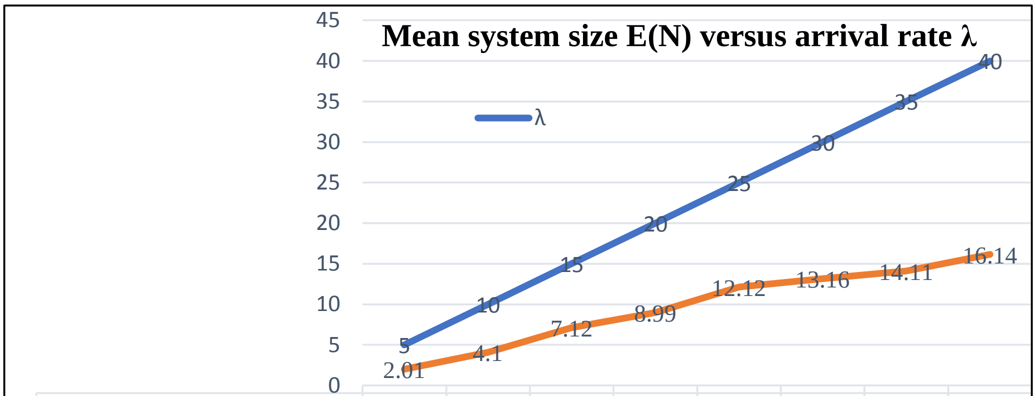

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| λ | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 |
| for μ1=14, μ2=16,μ3=18 | 1.76 | 3.42 | 6.17 | 8.42 | 10.71 | 11.17 | 13.2 | 15.92 |

Average queue length over different values of λ if $\mu_1 = 14$, $\mu_2 = 16$, $\mu_3 = 18$.




| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| λ | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 |
| for μ1=15, μ2=17, μ3=19 | 1.94 | 3.72 | 6.97 | 8.89 | 11.27 | 12.14 | 13.69 | 16.07 |

Average queue length over different values of λ if $\mu_1 = 15$, $\mu_2 = 17$, $\mu_3 = 19$.




| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| λ | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 |
| for μ1=16, μ2=18, μ3=21 | 2.01 | 4.1 | 7.12 | 8.99 | 12.12 | 13.16 | 14.11 | 16.14 |

Average queue length over different values of λ if $\mu_1 = 16$, $\mu_2 = 18$, $\mu_3 = 21$.

## 6. Conclusion

This paper is a study of M/M/3 queueing system. This study is the extension of single server queueing system to multi – server queueing system. Matrix – geometric analytical method is used for derive the steady state probability and transition rate matrix. The distribution of stationary queue length and waiting time of a customer is derive by QBD and Matrix Geometric method. There were three objectives of this paper. Firstly, to understand the nature of queueing models and use in particle life. Secondly, to develop a more appropriate analysis of queueing models and methodology for design and management, and, finally, to create a tool that allows others to understand the nature, problems and necessary components of queueing models of rate matrix.

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