International Journal of Mathematics and Computer Research

ISSN: 2320-7167

Volume 08 Issue 08 August 2020, Page no. - 2112-2123
Index Copernicus ICV: 57.55
DOI: 10.47191/ijmcr/v8i8.02

Portfolio Optimization in Share Market Using Multi-Objective Linear Programming

Jayanti Nath[1], Susanta Banik[2], Debasish Bhatacharya[3]

[1]Department of Mathematics, National Institute of Technology, Agartala, Tripura, India-799046
[2]Department of Mathematics, National Institute of Technology Agartala, Tripura, India-799046
[3]Department of Mathematics, National Institute of Technology Agartala, Tripura, India-799046

ARTICLE INFO	ABSTRACT
Published Online: 13 August 2020 Corresponding Author: **Debasish Bhattacharya**	Applying multi-objective linear programming formulation for the solution of portfolio optimization in share market, two methods are proposed in this paper. Here the short and long term returns as well as annual dividend received are maximized. Further, the associated risk in the form of semi-absolute deviation below the expected return is minimized. The proposed methods of solution are illustrated by a real life example based on current data collected from Bombay Stock Exchange (BSE).

KEYWORDS: Portfolio Optimization, De Novo Programming, Fuzzy Multi-objective Programming, Return and risk.

I. INTRODUCTION

A Portfolio in finance means the combination of shares or other investments that a particular person or company has. When a potential purchaser visits a share market for purchasing securities/debentures etc. the first task he/she does is to observe the market and gathers experiences and believes regarding future performances of available securities. Having the relevant believes about future performances, the purchaser selects the portfolio of his/her choice within the available budget, Markowitz [1].

In selecting the portfolio, the investor desires to maximize the expected return and at the same time minimize the concomitant risk and hence make a balance between the return and the risk. Now the share market is unpredictable and fluctuation in the prices is a regular phenomenon. Thus the cost prices of shares and its return is random in nature. So, the selection of portfolios, without proper planning and evaluation of the alternatives is a difficult task.

Naturally, question arises how the investor may select portfolio so that his/her expected return is maximized and at the same time risk is minimized. Markowitz [1] in 1952 first considered these aspects and proposed the mean-variance model for portfolio selection and it is considered as one of the best methods for addressing such problem. Markowitz's model describes how an investor can select optimum portfolio taking into consideration the trade-off between the expected return and the market risk. However to remove some shortcomings of the said method, Markowitz [2] used semi-variance in the place of variance in 1959. Markowitz mean variance model may lead to erroneous conclusion, particularly when the security returns are asymmetric in nature. The existence of such asymmetric security return distribution was later indicated in the works of Liu, et.al.[3]; Yan and Li [4]; Guo, Q. et.al. [5], Mansini et. al. [6], Ayub, et.al. (2015)[7] and they proposed some models to minimize the risk in the way of minimizing semi-variance. Their works enriched the process of portfolio selection.

Now to apply mean-variance or mean semi-variance method of optimal portfolio selection, the probability distribution of the returns is required. Again for the application of the probability theory in the portfolio selection process, the decision-maker must be provided with a reasonably large size of statistical data pertaining to the performance of the securities.

Many researchers proposed an alternative way of selecting portfolios based on expert's opinion regarding the subjective valuation of the security and their prospective returns. Their works can be broadly categorized into three ways: using Fuzzy set theory Gupta, et.al.[8] ; using Possibility theory, Carlsson, et.al.[9] ; Zhang, et.al. [10] and using Credibility theory Huang [11]; Qin, et.al.[12] These methods are used in a situation where sufficient data regarding security returns is lacking. However, these fuzzy methods are also subjected to some drawbacks. When a fuzzy variable is used to represent the security returns, a paradox appears [Huang and Ying [13]]. Liu [14] proposed an alternative way to estimate subjective expert's valuation of the security returns using uncertainty theory. Following this theory, many researchers subsequently worked on the problem of portfolio optimization. Some of such works are Yao [15], You [16]

Some recent works in this area show that the Multi-objective Linear Programming technique could be successfully used for solving portfolio selection problem. In 2008 Pankaj Gupta et al. [8] considered asset portfolio optimization using Fuzzy mathematical programming. They used a semi absolute deviation model for risk calculation by transforming the mean-variance model. In the process, they considered not only yearly return for maximization but also maximized long term returns. Another method for optimum portfolio selection using linear programming under a crisp and fuzzy environment can be seen in the work of Hong-Wei Liu [17].

In the present paper a multi-objective linear programming model for portfolio selection has been considered. In the constructed model four objectives (three maximizing and one minimizing) have been set up. It is then solved by two methods viz. Zimmermann [18] technique under fuzzy environment and Min-max Goal programming which is akin to fuzzy method. To illustrate the proposed methods of solution, a real life example has been given. For construction of the problem, current data regarding monthly return, annual dividend etc. offered to the shareholders by twenty renowned companies have been collected from BSE. The parameters relevant to the problem are calculated from the collected data pertaining to the selected twenty companies and are placed in table (2). The real life portfolio selection problem is finally solved by the said methods using Lingo 18 software and the solutions are compared. With these objectives in mind the paper has been arranged as follows:

In section II, the portfolio selection problem has been modelled as a Multi-objective Linear Programming Problem (MOLPP). In section III two methods of solution for the constructed MOLPP model have been proposed. The first one is a fuzzy method using Zimmermann technique discussed under section III (***A***). The second one is based on Min-max Goal Programming technique for solving MOLPP involving both maximizing and minimizing objectives and is placed in section III (***B***). Section IV deals with the solution of the real life portfolio selection problem based on real data obtained from BSE on monthly closing values, annual dividend etc. per share declared by twenty selected companies over a period of ten years. Section IV (***A***) and IV (***B***) contain respectively the solution obtained by Zimmermann fuzzy method and Min-max G.P method of the real life problem undertaken. The solutions obtained are also compared. Section V contains the conclusions about the findings of the present paper and finally some relevant references are placed.

II. PORTFOLIO OPTIMIZATION PROBLEM AS A MULTI-OBJECTIVE LINEAR PROGRAMING PROBLEM

In this section we propose a multi-objective linear programming model for portfolio selection problem.

Let a potential buyer intends to invest his/her wealth amounting (₹) B in share market among n assets. Let, x_i be the proportion of wealth invested in i^{th} asset, $i = 1, 2, ..., n$.

Then, $x_1 + x_2 + ... + x_n = 1$, also $B_i = x_i B$ is the amount (in rupees) invested in the i^{th} asset.

For formulation of portfolio selection model as in [8] we consider three types of monthly average return per unit of each asset purchased and propose here a modified model.

r_i = Expected monthly rate of return per unit of the i^{th} asset (estimated as the average monthly return over a period of 10 years) $= \frac{1}{120}\sum_{t=1}^{120} r_{it}$

r_i^1 = Average monthly return per unit of the i^{th} asset (calculated over a period of 1 year) $= \frac{1}{12}\sum_{t=1}^{12} r_{it}$

r_i^2 = Average long term monthly return per unit of the i^{th} asset (calculated over a period of 3 years) $= \frac{1}{36}\sum_{t=1}^{36} r_{it}$

d_i = Annual dividend received from per unit of the i^{th} asset.

Where r_{it} denotes the return of the i^{th} asset for the t^{th} month of the period considered.

We set the following objectives in respect of the portfolio $x = (x_1, x_2, ..., x_n)$

$$\max f_1(x) = \sum_{i=1}^{n} r_i^1 x_i, \quad \text{where } r_i^1 = \frac{1}{12}\sum_{t=1}^{12} r_{it} \quad \text{(maximizing the annual return)},$$

$$\max f_2(x) = \sum_{i=1}^{n} r_i^2 x_i \quad \text{where } r_i^2 = \frac{1}{36}\sum_{t=1}^{36} r_{it} \quad \text{(maximizing three-yearly return)}.$$

$$\max f_3(x) = \sum_{i=1}^{n} x_i d_i \quad \text{[maximizing the annual dividend for all the assets purchased]}$$

$$\min f_4(x) = \frac{1}{T}\sum_{t=1}^{T} w_t(x), \text{ where } w_t(x) = \frac{\left|\sum_{i=1}^{n}(r_{it}-r_i)x_i\right| + \sum_{i=1}^{n}(r_i - r_{it})x_i}{2}$$

measures the absolute semi deviation of the returns for the t^{th} month below the expected return comprising of all the assets. The objective $f_4(x)$ stands for the average absolute semi deviation of the returns for the time period T. For a fixed month t, let us partition the set $\{1, 2,...n\}$ in two disjoint subsets A and B such that $i \in A$ implies $r_{it} \geq r_i$ i.e. the return for the t^{th} month of i^{th} asset is at least equal to the expected return for the asset and $i \in B$ implies $r_{it} < r_i$ i.e. the return for the t^{th} month of the i^{th} asset is less than its expected return. Then $w_t(x) = \sum_{i=1}^{n}(r_i - r_{it}), i \in B$ and $w_t(x) = 0$ for $i \in A$.

Here T is the period over which the semi absolute deviation below the expected return is considered. For our purpose T = 36 months has been taken. The objective $f_4(x)$ gives a measure of risk in selecting the portfolio $\boldsymbol{x}$ (for a long term i.e. 3 years). Hence this objective aims at minimizing the risk of getting a return below the expected return for a long term (3 years). If $r_{it} \geq r_i$ for some i, $i=1,2, ... n$, then $w_t(x) = 0$. On the other hand if $r_{it} < r_i$ for some i, $i=1, 2, ... n$, then $w_t(\boldsymbol{x}) = \sum_{i=1}^{n}(r_i - r_{it})x_i$

Thus for the t^th month of the period T, the sum of the risk associated with the portfolio $\boldsymbol{x} = (x_1, x_2,..., x_n)$ is given by $\sum_{i=1}^{n}(r_i - r_{it})x_i$. For the whole period T, the average risk is represented by $f_4(\mathbf{x}) = \frac{1}{36}\sum_{t=1}^{36}\sum_{i=1}^{20}(r_i - r_{it})x_i$

Our aim is to optimize the above mentioned four objectives subject to the following constraints.

$x_1 + x_2 + ... + x_n = 1$ [Budget constraint, $x_i \geq 0$]

$x_i \leq u_i y_i$ [upper bound constraint for investment in i^{th} asset]

$x_i \geq l_i y_i$ [lower bound constraint for investment in i^{th} asset]

where u_i, l_i respectively denote the highest and lowest proportion of investment in the i^{th} asset and y_i is a binary variable defined by

$$y_i = \begin{cases} 1 & if\ the\ investor\ invests\ in\ the\ i^{th}\ asset\ i.e\ if\ \ x_i \neq 0 \\ 0 & otherwise \end{cases} \tag{1}$$

$\sum_{i=1}^{n} y_i = l$ [constraint representing the highest number assets included in the list.]

In the solution process l will be automatically determined.

Thus the portfolio selection problem as a multi-objective L.P.P is given by

$$\text{Max} f_1(\boldsymbol{x}) = \sum_{i=1}^{n} r_i^1 x_i$$

$$\text{Max} f_2(\boldsymbol{x}) = \sum_{i=1}^{n} r_i^2 x_i$$

$$\text{Max} f_3(\boldsymbol{x}) = \sum_{i=1}^{n} x_i d_i$$

$$\text{Min} f_4(\boldsymbol{x}) = \frac{1}{T}\sum_{t=1}^{T} w_t(\boldsymbol{x}), T = 36\ \ months$$

Where $$w_t(\boldsymbol{x}) = \frac{\left|\sum_{i=1}^{n}(r_{it}-r_i)x_i\right| + \sum_{i=1}^{n}(r_i - r_{it})x_i}{2} = \sum_{i=1}^{n}(r_i - r_{it})x_i\ ,\ r_i > r_{it}$$

subject to,

$$\begin{aligned} & x_1 + x_2 + ... + x_n = 1 \\ & x_i \geq l_i y_i \\ & x_i \leq u_i y_i \\ & u_i \in [0,1] \\ & l_i \in [0,1] \\ & y_1 + y_2 + ... + y_n = l \\ & l \in [1,20] \\ & y_i \in \{0,1\} \\ & x_i \geq 0 \\ & i = 1, 2,...n \end{aligned} \tag{2}$$

In [8], for solving the model (2) the authors kept it open for the decision makers about the choice of 'l', the number of non-zero entries in the portfolio. However in our proposed modified model the constant 'l' will be automatically determined by the system. The benefit of this small but significant change is discussed in section 4 while solving a real life problem.

III. SOLUTION OF THE PORTFOLIO SELECTION PROBLEM UNDER FUZZY ENVIRONMENT

Here we solve the portfolio selection problem modelled in (2) by two methods. The first one is a fuzzy method due to Zimmermann and the second one is Min-max Goal Programming [GP] which is akin to fuzzy method

A. Solution Using Zimmermann Technique

We use Zimmermann's [18] technique for solving multi-objective linear programming problem under fuzzy environment. The same is used for the solution of the multi-objective portfolio selection problem (2). For this we first calculate the max/min values of the objectives separately subject to the given constraints and also note the corresponding solution in each case.

This is done by solving four single objective LPPs taking one objective at a time from $f_k(x)$, $k = 1, 2, 3, 4$ and subject to the constraints of (2) using Lingo (18.0).

Let the optimal values and the optimal solutions of the individual objectives be given by

$$z_1^* = \max_{x\in X} f_1(\boldsymbol{x}) = f_1(\boldsymbol{x}_1^*), \quad z_2^* = \max_{x\in X} f_2(\boldsymbol{x}) = f_2(\boldsymbol{x}_2^*), \quad z_3^* = \max_{x\in X} f_3(\boldsymbol{x}) = f_3(\boldsymbol{x}_3^*),$$
$$w^* = \max_{x\in X} f_4(\boldsymbol{x}) = f_4(\boldsymbol{x}_4^*), x \in X \qquad (3)$$

Here $\boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \boldsymbol{x}_3^*$ and $\boldsymbol{x}^*_4$ are their respective optimal solutions. These max/min values of the maximizing / minimizing objectives are respectively used as their optimistic values. Where X is the feasible space defined by the constraints of (2). Next we fuzzify the objectives of the problem (2) as follows

$$f_1(\boldsymbol{x}) \gtrsim z_1^*, \quad f_2(\boldsymbol{x}) \gtrsim z_2^*, \quad f_3(\boldsymbol{x}) \gtrsim z_3^* \quad and \quad f_4(\boldsymbol{x}) \lesssim w^*$$

where the symbols ($\gtrsim$) and ($\lesssim$) respectively represents essentially greater than or equal to and essentially smaller than or equal to , which are respectively the fuzzified version of $\geq$ and $\leq$ respectively [19].

To construct membership function of the fuzzy objectives defined above with the corresponding ideal values as their fuzzy goals, another set of objective values (pessimistic values) is required. This is obtained by using Luhandjula's [20] comparison technique. The technique is discussed as follows.

We prepare the following Table [1] calculating the values of the objectives at each of the points $\boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \boldsymbol{x}_3^*$ and $\boldsymbol{x}^*_4$ which are respectively the optimal solutions of the individual objectives.

Table 1: Calculation of Pessimistic values of the Objectives

Solution	***Objectives***			
	z_1	z_2	z_3	w
$\boldsymbol{x}_1^*$	$z_1(\boldsymbol{x}_1^*) = z_1^*$	$z_2(\boldsymbol{x}_1^*)$	$z_3(\boldsymbol{x}_1^*)$	$w(\boldsymbol{x}_1^*)$
$\boldsymbol{x}_2^*$	$z_1(\boldsymbol{x}_2^*)$	$z_2(\boldsymbol{x}_2^*) = z_2^*$	$z_3(\boldsymbol{x}_2^*)$	$w(\boldsymbol{x}_2^*)$
$\boldsymbol{x}_3^*$	$z_1(\boldsymbol{x}_3^*)$	$z_2(\boldsymbol{x}_3^*)$	$z_3(\boldsymbol{x}_3^*) = z_3^*$	$w(\boldsymbol{x}_3^*)$
$\boldsymbol{x}_4^*$	$z_1(\boldsymbol{x}_4^*)$	$z_2(\boldsymbol{x}_4^*)$	$z_3(\boldsymbol{x}_4^*)$	$w(\boldsymbol{x}_4^*) = w^*$

From this table we calculate the pessimistic values of the objectives as follows

$$\hat{z}_1 = \text{Min}\ \{z_1^*, z_1(x^*_2), z_1(x^*_3), z_1(x^*_4)\}$$
$$\hat{z}_2 = \text{Min}\ \{z_2(x^*_1), z_2^*, z_2(x^*_3), z_2(x^*_4)\}$$
$$\hat{z}_3 = \text{Min}\ \{z_3(x^*_1), z_3(x^*_2), z_3^*, z_3(x^*_4)\}$$
$$\hat{w} = \text{Max}\ \{w(x^*_1), w(x^*_2), w(x^*_3), w^*\}$$

The method explained above is capable of extension in more involved cases.

Now, returning to the solution of the problem (2), let $\hat{z}_1$, $\hat{z}_2$, $\hat{z}_3$ and $\hat{w}$ are respectively the pessimistic values of the objectives $z_1 \equiv f_1(x)$, $z_2 \equiv f_2(x)$, $z_3 \equiv f_3(x)$ and $w \equiv f_4(x)$ obtained by using Luhandjula's technique explained above. Using these ideals (optimistic) and pessimistic values of the objectives, their linear membership functions with the ideal values as their respective fuzzy goals are constructed as follows.

$$\mu_{z_1}(\boldsymbol{x}) = \begin{cases} 1 & if \;\; z_1 \geq z_1^* \\ \frac{z_1 - \hat{z}_1}{z_1^* - \hat{z}_1} & if \;\; \hat{z}_1 \leq z_1 \leq z_1^* \\ 0 & if \;\; z_1 \leq \hat{z}_1 \end{cases} \tag{4}$$

and

$$\mu_{z_2}(\boldsymbol{x}) = \begin{cases} 1 & if \;\; z_2 \geq z_2^* \\ \frac{z_2 - \hat{z}_2}{z_2^* - \hat{z}_2} & if \;\; \hat{z}_2 \leq z_2 \leq z_2^* \\ 0 & if \;\; z_2 \leq \hat{z}_2 \end{cases} \tag{5}$$

$$\mu_{z_3}(\boldsymbol{x}) = \begin{cases} 1 & if \;\; z_3 \geq z_3^* \\ \frac{z_3 - \hat{z}_3}{z_3^* - \hat{z}_3} & if \;\; \hat{z}_3 \leq z_3 \leq z_3^* \\ 0 & if \;\; z_3 \leq \hat{z}_3 \end{cases} \tag{6}$$

and

$$\mu_{w}(\boldsymbol{x}) = \begin{cases} 1 & if \;\; w \leq w^* \\ \frac{\hat{w} - w}{\hat{w} - w^*} & if \;\; w^* \leq w \leq \hat{w} \\ 0 & if \;\; w \geq \hat{w} \end{cases} \tag{7}$$

Now we use optimality principal of Bellman and Zadeh [21]. It states that the fuzzy set “decision” is a confluence of its fuzzy objectives and constraints. Thus using the linear membership values of the fuzzy objectives given in (4) to (7), the Zimmerman fuzzy model, for solving the portfolio selection problem based on the multi-objective linear program (2) is given by

$$\max \lambda$$

subject to,

$$\begin{aligned}
&\lambda \leq \mu_{z_1}(\boldsymbol{x}) = \frac{z_1 - \hat{z}_1}{z_1^* - \hat{z}_1} \\
&\lambda \leq \mu_{z_2}(\boldsymbol{x}) = \frac{z_2 - \hat{z}_2}{z_2^* - \hat{z}_2} \\
&\lambda \leq \mu_{z_3}(\boldsymbol{x}) = \frac{z_3 - \hat{z}_3}{z_3^* - \hat{z}_3} \\
&\lambda \leq \mu_{w}(\boldsymbol{x}) = \frac{\hat{w} - w}{\hat{w} - w^*} \\
&\sum_{i=1}^{n} x_i = 1 \\
&x_i \geq l_i y_i \\
&x_i \leq u_i y_i \\
&u_i \in [0,1] \\
&l_i \in [0,1] \\
&\sum_{i=1}^{n} y_i = l \\
&y_i \in \{0,1\} \\
&l \in [1,20] \\
&x_i \geq 0 \\
&i = 1, 2, ..., n \\
&\lambda \in [0,1]
\end{aligned} \tag{8}$$

which is a confluence of the fuzzy goals and constraints [21].

$$\text{Here } \lambda = \min_{x} \{\mu_{z1}(x), \mu_{z2}(x), \mu_{z3}(x), \mu_{w}(x), \mathbf{1}\},$$

and **1** stands for the constant function having the value 1. It represents the common membership function of each of the crisp constraints in (8). Here l_i and u_i are respectively the lower and upper bounds of $x_i \in [0, 1]$ and y_i is a binary variable defined

$$y_i = \begin{cases} 1 & if\ x_i \neq 0 \\ 0 & if\ x_i = 0 \end{cases}$$

B. Solution Using Min-max Goal Programming

We next solve the portfolio selection problem (2) using Min-max Goal Programming [GP] [22] technique. Min-max GP is an important method to solve MOLPP involving both maximizing and minimizing objectives. This method is akin to the fuzzy method of solution of an MOLPP. Now for solving MOLPP (2) using Min-max GP, we consider the following system

$$\begin{aligned} &\min d \\ &\text{subject to,} \\ &f_k(x) + n_k - p_k = \omega_k \quad k = 1,2,3,4 \qquad (9) \\ &\beta_k n_k + \gamma_k p_k \leq d \\ &x \in X \end{aligned}$$

Where X is the feasible space defined by the constraints of (2). Here d is the maximum deviation between the achievement of the goals and their aspiration levels; ω_k is the specified aspiration level for the k^{th} objective function $f_k(x)$; n_k (resp.p_k) is the negative (resp. positive) deviation from the aspiration level of the objective $f_k(x)$; and β_k, γ_k are the non-negative weights attached to the deviation variables as per decision makers choice such that $\sum_{k=1}^{3} \beta_k + \gamma_4 = 1$. For the maximizing objectives $f_k(x)$, $k = 1, 2, 3$.

$\omega_k = \max_{x \in X} f_k(x) = z_k^*$ and $\omega_4 = w^* = \max_{x \in X} f_4(x)$. The values $z_k^*, k = 1,2,3$ and w^* are the ideal values of the objectives. Since the ideal values have been used as aspiration levels for the maximizing objectives, we must have $f_k(x) \leq z_k^*$, and hence $p_k = 0$ for $k = 1, 2, 3$. Similarly for the minimizing objective $f_k(x) \geq w^*$, and so $n_k = 0$ *for* $k = 4$.

Next we restrict the goal deviations to unit-less numbers, for this normalization of the deviation variables is necessary. This is done by dividing the deviation constraints (9) respectively by $t_k = z_k^* - \hat{z}_k,\ k = 1,2,3$ and by $t_4 = \hat{w} - w^*$. Here z_k^* ($k = 1, 2, 3$) and w^* are the ideal values (optimistic values) of the objectives. Also $\hat{z}_k$ ($k = 1, 2, 3$) and $\hat{w}$ are their pessimistic values.

Thus we have the following modified system.

$$\begin{aligned} &\min \bar{d} \\ &\text{subject to,} \\ &f_k(\boldsymbol{x}) + n_k = z_k^*, \quad k = 1,2,3,4 \\ &\beta_k \frac{n_k}{t_k} \leq \bar{d} \\ &f_4(\boldsymbol{x}) - p_4 = w^* \\ &\gamma_4 \frac{p_4}{t_4} \leq \bar{d} \qquad (10) \\ &x \in X \end{aligned}$$

Where X is the feasible space defined by the constraints of (2). The weights of β_k, $k = 1, 2, 3$ and γ_k, $k = 4$ are chosen by the decision maker such that $\sum_{k=1}^{3} \beta_k + \gamma_4 = 1$.

Here $\bar{d}$ is the maximum normalized weighted deviation between the achievements of the goals and their aspiration levels. The linear program (10) can now be solved by using Lingo software 18.

IV. REAL LIFE EXAMPLE TO ILLUSTRATE THE METHOD OF SOLUTION

Here we construct a real life example to illustrate the proposed techniques (8) and (10) discussed in previous sections for the solution of portfolio selection problem. For this secondary data pertaining to the monthly closing values and annual dividend announced by twenty renowned companies over a period of last ten years (2009-2019) has been collected from the BSE, India (cf. http://in.finance.yahoo.com ; http://www.moneycontrol. com).

The data collected are then used to calculate the parameters viz. monthly return (percent) (r_{it}); average monthly return over a period of 1 year (r_i^1) ; and over a period of 3 years (r_i^2) ; expected return over a period of 10 years(r_i) .

$$r_{it} = \frac{\text{closing value of present month - closing value of previous month}}{\text{Closing value of previous month}} \times 100 = \text{average monthly return percent}$$

$$r_i^1 = \frac{1}{12}\sum_{t=1}^{12} r_{it}$$

$$r_i^2 = \frac{1}{36}\sum_{t=1}^{36} r_{it}$$

$$r_i = \frac{1}{120}\sum_{t=1}^{120} r_{it}$$

Also for the calculation of semi-absolute deviation of r_{it} below the expected return r_i, the expression $\sum_{t=1}^{36}(r_{it} - r_i),\ r_{it} \le r_i,\ \ i = 1, 2, ..., 20$ is needed to be evaluated using the parameters defined above.

All these parameters evaluated using the collected data are presented in Table 2

Table 2: Calculation of Parameters Involved

Sl. No.	Name of the Company	r_i^1	r_i^2	r_i	d_i	$\sum_{t=1}^{36}(r_{it} - r_i), r_{it} \le r_i$
1	ABB	0.04	0.9	0.94	4.8	117.29
2	ACC	-0.16	0.48	0.72	14	110.26
3	ALBK	-6.48	-2.47	-0.71	0	207.34
4	ASHOK LEY	-1.36	0.52	1.71	3.1	140.39
5	BAJAJ AUTO	1.57	0.69	0.69	60	71.23
6	BEL	1.6	-0.05	1.03	1.7	172.22
7	BHEL	-3.59	-1.13	-0.99	1.2	159.12
8	BPCL	3.2	1.2	1.78	8	173.37
9	CIPLA	-0.5	-0.26	0.95	3	115.29
10	DR REDDY	0.89	0.08	0.97	20	120.19
11	INFOSYSTCH	1.1	1.22	0.98	8	81.52
12	ITC	-1.3	0.12	1.05	5.75	104.05
13	MTNL	-3.18	-1.32	-0.59	0	187.4
14	SIEMENS	3.51	1.14	1.24	7	114.61
15	TATAPOWER	-2.27	-0.55	-0.33	1.3	111.55
16	TITAN	2.54	4.06	2.84	5	99.28
17	VOLTAS	1.69	2.26	1.84	4	106.09
18	VSNL	-1.65	-0.97	0.99	4.5	147.93
19	HINDMOTOR	-2.07	0.16	1.91	0	188.57
20	WIPRO	0.22	1.09	0.64	1	88.59

Therefore using the values of the parameters displayed in table-2, the objectives of our stated model are respectively given by

Max $f_1(x) = \sum_{i=1}^{20} r_i^1 x_i = (0.04x_1 - 0.16x_2 - 6.48x_3 - 1.36x_4 + 1.57x_5 + 1.6x_6 - 3.59x_7 + 3.2x_8 - 0.5x_9 + 0.89x_{10} + 1.1x_{11} - 1.3x_{12} - 3.18x_{13} + 3.51x_{14} - 2.27x_{15} + 2.54x_{16} + 1.69x_{17} - 1.65x_{18} - 2.07x_{19} + 0.22x_{20})$

Max $f_2(x) = \sum_{i=1}^{20} r_i^2 x_i = (0.9x_1 + 0.48x_2 - 2.47x_3 + 0.52x_4 + 0.69x_5 - 0.05x_6 - 1.13x_7 + 1.2x_8 - 0.26x_9 + 0.08x_{10} + 1.22x_{11} + 0.12x_{12} - 1.32x_{13} + 1.14x_{14} - 0.55x_{15} + 4.06x_{16} + 2.26x_{17} - 0.97x_{18} + 0.16x_{19} + 1.09x_{20})$ (11)

Max $f_3(x) = \sum_{i=1}^{20} x_i d_i = (4.8x_1 + 14x_2 + 0x_3 + 3.1x_4 + 60x_5 + 1.7x_6 + 1.2x_7 + 8x_8 + 3x_9 + 20x_{10} + 8x_{11} + 5.75x_{12} + 0x_{13} + 7x_{14} + 1.3x_{15} + 5x_{16} + 4x_{17} + 4.5x_{18} + 0x_{19} + 1x_{20}$

$$\text{Min} f_4(\boldsymbol{x}) = \frac{1}{36}\sum_{t=1}^{36} w_t(\boldsymbol{x})$$
$$= (1/36)(117.29x_1 + 110.26x_2 + 207.34x_3 + 140.39x_4 + 71.23x_5 + 172.22x_6 + 159.12x_7$$
$$+ 173.37x_8 + 115.29x_9 + 120.19x_{10} + 81.52x_{11} + 104.05x_{12} + 187.4x_{13} + 114.61x_{14}$$
$$+ 111.55x_{15} + 99.28x_{16} + 106.09x_{17} + 147.93x_{18} + 188.57x_{19} + 88.59x_{20})$$

A. Solution of the Portfolio Selection Problem

Here we solve the portfolio selection problem as a MOLPP modelled in (2) and the objectives are explicitly represented in (11). We do it in two methods viz. Zimmermann fuzzy method and min-max GP method [22].

Solution of the Portfolio Selection Problem using Zimmerman Fuzzy Method

For the solution of the portfolio selection problem modelled in (2) using Zimmerman fuzzy technique detailed in section III(***A***), we consider the system (8) and solve it by Lingo software 18.

The four objectives $z_1 \equiv f_1(x)$, $z_2 \equiv f_2(x)$, $z_3 \equiv f_3(x)$ and $w \equiv f_4(x)$ in (8) are explicitly given in (11). Now, to affect the solution, we need the optimistic and pessimistic values of the objectives. Considering the objectives of model (2), given in (11), one by one together with the constraints of (2) four LPPs are formed. We solve these four LPPs separately by Lingo software and obtain the following four solutions:

$$z_1^* = \max z_1 = 3.51,\ x_i = 0,\ i = 1,2,....,20;\ i \neq 14;\ x_{14} = 1$$
$$z_2^* = \max z_2 = 4.06,\ x_i = 0,\ i = 1,2,....,20;\ i \neq 16;\ x_{16} = 1$$
$$z_3^* = \max z_3 = 60,\ x_i = 0,\ i = 1,2,....,20;\ i \neq 5;\ x_5 = 1$$
$$w^* = \min w = 1.98,\ x_i = 0,\ i = 1,2,....,20;\ i \neq 5;\ x_5 = 1$$

These optimal solution of the individual objective are respectively denoted by $\boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \boldsymbol{x}_3^*, \boldsymbol{x}_4^*$.

Next we calculate the pessimistic values $\hat{z}_1, \hat{z}_2, \hat{z}_3$ and $\hat{w}$ of the objectives using Luhandjula's comparison technique. For this we compute all the objective values at each of these four individual optimal solution $\boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \boldsymbol{x}_3^*, \boldsymbol{x}_4^*$. The calculations are placed in Table (3). Thus from the table, by Luhandjula's comparison technique the pessimistic values of the objectives are given by $\hat{z}_1$ = min {3.51, 2.54, 1.57, 1.57} = 1.57, similarly $\hat{z}_2$ = 0.69, $\hat{z}_3$ = 5, $\hat{w}$ = 3.18.

Now substituting the values of $\boldsymbol{z}_1^*, \boldsymbol{z}_2^*, \boldsymbol{z}_3^*, \boldsymbol{z}_4^*$ and $\hat{z}_1, \hat{z}_2, \hat{z}_3$ and $\hat{w}$ in system (8) we get the following Zimmermann fuzzy model for the solution of portfolio selection problem.

$$\max \lambda$$

subject to,

$$\lambda \le \frac{z_1 - 1.57}{3.51 - 1.57} = \frac{z_1 - 1.57}{1.94}$$
$$\lambda \le \frac{z_2 - 0.69}{4.06 - 0.69} = \frac{z_2 - 0.69}{3.37}$$
$$\lambda \le \frac{z_3 - 5}{60 - 5} = \frac{z_3 - 5}{55}$$
$$\lambda \le \frac{3.18 - w}{3.18 - 1.98} = \frac{3.18 - w}{1.2} \qquad (12)$$
$$x_1 + x_2 + ... + x_n = 1$$
$$x_i \ge l_i y_i$$
$$x_i \le u_i y_i$$
$$u_i \in [0,1]$$
$$l_i \in [0,1]$$
$$y_1 + y_2 + ... + y_n = l$$
$$l \in [1,20]$$
$$y \in \{0,1\}$$
$$x_i \ge 0$$
$$i = 1,2,...,20$$
$$\lambda \in [0,1]$$

$$= \min_x \{\frac{z_1 - 1.57}{1.94}, \frac{z_2 - 0.69}{3.37}, \frac{z_3 - 5}{55}, \frac{3.18 - w}{1.2}, 1\}$$

Solving the model (12) by Lingo 18 software the solution obtained is

$$\lambda = 0.4090404$$
$$z_1 = 2.363538$$
$$z_2 = 2.068466$$
$$z_3 = 27.49722$$
$$w = 2.538599$$
$$x_5 = 0.4010712$$
$$x_{14} = 0.2191520$$
$$x_{16} = 0.3797768$$
$$x_i = 0, \text{for other values of } i = 1, 2, \ldots 20$$

The obtained solution shows that for an investor, it is beneficial to purchase the shares of the three companies corresponding to the non-zero values of the decision variables for overall satisfaction of his objectives. The solution actually trade-offs among the interests of the objectives. This can be seen by comparing the individual optimal values of the objectives and their values obtained by the proposed methods.

Solution of the Portfolio Selection Problem using Min-max GP method

For the solution of the portfolio selection problem modelled in (2) using Min-max GP technique detailed in section III(***B***) we consider the system (10) and solve it by Lingo software 18.

Table 3: Calculation of Pessimistic values of the Objectives

Solution	***Values of the Objectives***			
	z_1	z_2	z_3	w_1
x_1^*	3.51	1.14	7	3.18
x_2^*	2.54	4.06	**5**	2.76
x_3^*	1.57	0.69	60	1.98
x_4^*	1.57	0.69	60	1.98

In section IV(***A***) the calculated optimistic and pessimistic values of the four objectives are as follows:

$$z_1^* = 3.51, \quad z_2^* = 4.06, \quad z_3^* = 60, \quad w^* = 1.98$$
$$\hat{z}_1 = 1.57, \ \hat{z}_2 = 0.69, \ \hat{z}_3 = 5, \ \hat{w} = 3.18$$

Now substituting the values of z_1^*, z_2^*, z_3^*, w^* and $t_k = z_k^* - \hat{z}_k$; $k = 1, 2, 3$; and $t_4 = \hat{w} - w^*$ in system (10) we get the following Min-max GP model for the solution of portfolio selection problem.

$$\min d^{-}$$

subject to,

$$f_1(x) + n_1 = z_1^*$$
$$f_2(x) + n_2 = z_2^*$$
$$f_3(x) + n_3 - z_3^*$$

$$\beta_1 \frac{n_1}{t_1} \le \bar{d}$$
$$\beta_2 \frac{n_2}{t_2} \le \bar{d}$$
$$\beta_3 \frac{n_3}{t_3} \le \bar{d}$$

$$
\begin{aligned}
& f_4(x) - p_4 = w^* \\
& \gamma_4 \frac{p_4}{t_4} \leq \bar{d} \\
& x_1 + x_2 + \ldots + x_n = 1 \\
& x_i \geq l_i y_i \\
& x_i \leq u_i y_i \\
& u_i \in [0,1] \\
& l_i \in [0,1] \\
& y_1 + y_2 + \ldots + y_n = l \\
& l \in [1,20] \\
& y \in \{0,1\} \\
& x_i \geq 0 \\
& i = 1,2,\ldots,20 \qquad (13)
\end{aligned}
$$

The explicit expressions for $f_k(x)$ are given in (11). The weights $n_k \geq 0$, $k = 1, 2, 3$ and $p_4 \geq 0$ are chosen so that $\beta_1 + \beta_2 + \beta_3 + \gamma_4 = 1$. These weights are chosen by the decision maker according to his/her priority for the objectives. We can also take the null hypothesis of equality of all the weights. The solution obtained by Lingo software for some typical choice of the weights is displayed in Table [4]. From the table we see that for the choice of equal weights, the solution obtained is exactly the same as obtained by the Zimmermann fuzzy technique.

In [8], as mentioned earlier, the constant 'l' is to be decided by the decision maker. In the present paper it has been left for the system to determine 'l', so that λ in model (8) and $\bar{d}$ in model (10) are respectively maximized and minimized. It is also observed that the system is capable of determining such 'l' optimally. The benefits of this change can be seen from the solution of the real life problem undertaken in section 4.

Table 4: Solution obtained in Min-max GP method by varying weights

Weights Chosen				***Solution obtained***					
β_1	β_2	β_3	γ_4	x_i	$\bar{d}$	z_1	z_2	z_3	w
0.25	0.25	0.25	0.25	x_5 = 0.4010 x_{14} = 0.2190 x_{16} = 0.3798	0.1477	2.3635	2.0684	27.4972	2.5385
0.1	0.5	0.25	0.15	x_5 = 0.3333 x_{16} = 0.6667	0.1667	2.2167	2.9367	23.3333	2.4980
0.2	0.1	0.4	0.3	x_5 = 0.6584 x_{14} = 0.3416	0.1317	2.2327	0.8437	41.8944	2.3902
0.4	0.1	0.2	0.3	x_5 = 0.4291 x_{14} = 0.5709	0.1716	2.6775	0.9469	29.7400	2.6665
0.1	0.15	0.25	0.5	x_5 = 0.6828 x_{16} = 0.3172	0.1024	1.8778	1.7591	42.5518	2.2257

A synopsis of the system generated solutions for models (12) and (13) (for equal weightage case) is placed in table 5 .

Table 5: Abridged solutions of the models (12) and (13)

Solution of model (12)	***Solution of model (13) for equal weightage***
$\lambda = 0.4090$	$\bar{d} = 0.1477$
$x_5 = 0.4010$	$x_5 = 0.4010$
$x_{14} = 0.2191$	$x_{14} = 0.2191$
$x_{16} = 0.3797$	$x_{16} = 0.3798$

On the other hand choosing the constant 'l', as was proposed in [8], we get the solution displayed in Table-6 for $l = 3$

Table 6: Abridged solutions of the models (12) and (13) choosing 'l'= 3

Solution of model (12)	***Solution of model (13) for equal weightage***
$\lambda = 0.1323$	$\bar{d} = 0.2430$
$x_5 = 0.1471$	$x_{14} = 0.8436$
$x_8 = 0.3131$	$x_{17} = 0.1564$
$x_{11} = 0.5399$	-

Comparing the Table-5 and Table-6 we see that for the system generated solutions the values of the parameters λ and $\bar{d}$ are respectively greater and smaller than their corresponding values obtained by choosing the constant 'l'. Now λ denotes the overall satisfaction of the decision maker towards the achievement of the objective goals. Also, $\bar{d}$ stands for the maximum normalized weighted deviation between the achievements of the goals and their aspiration levels. This indicates that, keeping 'l' to be determining by the system we get better result. This is because the overall satisfaction of the decision maker is maximized and the deviation between the aspiration level and the achievement of the objectives are minimized in this case. Similar findings could be observed by choosing l = 2, 4, 5 etc.

V. CONCLUSION

A modified method for solving portfolio optimization problem using multi-objective linear programming technique under fuzzy environment has been proposed and solved in this paper. The modification lies in the determination of number of non-zero entries in the optimal portfolio. Here instead of deciding in advance the number of such non-zero entries, it has been obtained from the optimum solution of the system. The benefits of such change have been elaborated in section IV. The method of solution has been illustrated with the help of a real life problem of investing wealth among twenty prospective shares of companies listed in table 2. Four objectives are considered, viz., maximizing short and long term returns and annual dividend received; and also minimizing the risk of investment. For the construction of short and long term return objectives, average monthly return over a period of 12 months and 36 months have been considered respectively. Actual annual dividend offered by the companies to shareholders is taken for the construction of annual dividend objective. The risk has been measured as the absolute semi-deviation of the return earned below the expected return. Here the expected return is taken as an average of monthly return over a period of 10 years. With these objectives a modified MOLPP (2) is constructed for the solution of the portfolio selection problem.

Now solving an MOLPP we get some compromised or trade off solution. For an MOLPP involving both maximizing and minimizing type of objectives, Zimmermann fuzzy and Min-max GP are two important methods of eliciting efficient solutions. In the present paper the constructed MOLPP has been solved by both of these methods and the solution obtained are compared. For Min-max GP method of solution, it is manifested from table 4, that the more we attach importance to the objective $f_4(x)$ (minimizing risk), the dividend received increases and risk diminishes. Also, with the increase in the sum of the weights attached to $f_3(x)$ and $f_4(x)$, the dividend also increases, and is independent of the weightage attached to the objective $f_3(x)$ (maximizing dividend). It has been seen that both the methods of solution yielded the same portfolio where 40%, 22% and 38% of the investment are to be made respectively to the shares of Bajaj Auto, Simens and Titan. The nobility of the proposed methods of solution of the modified MOLPP is to convert the same into a single objective fuzzy linear programming problem. For the first method the overall satisfaction of the decision maker is moderate (λ = 41%) and for the second method the maximum of the deviations between achievement of the goals and their respective aspiration levels is $\bar{d} = 0.148$ which is very small.

As a future scope of study more number of relevant objectives could be annexed (e.g. Capital growth, Liquidity, Security of principal amount invested, market availability etc.) to make the construction more realistic. Another aspect which can be looked into is to consider the objectives as fuzzy. They may be represented by triangular, trapezoidal fuzzy numbers.

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