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Negative Factorial

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ARTICLE INFO	ABSTRACT
Published Online: 30 December 2020 Corresponding Author **Phani Srikar**	The main focus of this paper is to present a way to think About the existence of negative factorials and how to extend the idea of non-positive factorials, and stress more on the thought process of realising their existence rather than their application ,and understand deeply about the origin of the idea and then analyse their functional behaviour ,using the LIMITING approach. And focus on the application using the idea of damped oscillations towards increasing the stability of high altitude bridges and apply it to its electrical analogy

KEYWORDS: factorial, combinatorics, predictive analysis, electronics

1.INTRODUCTION

Factorials are a magical discovery in the field of mathematics. Their use has always been in the fields of combinatory, calculus, probability right from their discovery and the actual behaviour of them has been neglected. We have always stressed on their animal application rather than their magical behaviour and left us blinded by their applicative properties and we have forgotten to experiment with their functional behaviour.

2. HISTORY

Multiple scientists have worked on this subject, but the principal inventors are those who gives the asymptotic formula after some work in collaboration with De Moivre, finally and introduces the actual notation n!. Of course other scientists such as Taylor have also worked a lot with this notation. The notation n! was introduced by French mathematician Christian Kramp(1760-1826) in 1808.The term factorial was first coined in French as factorielle by another French math- ematician Louis Francois Antoine Arbogast(1759-1803) and Kramp decided to use the term factorial so as to circumvent printing difficulties incurred by the previous used symbol Ln. (Anglani & Barile, 2007)gave the additive representation of factorials. (Bhargava, 2000) gave an expository account of the factorials, gave several new results and posed certain problems on factorials. Research on the interpolation of factorials started with correspondence among Leonhard Euler, Daniel Bernoulli and Christian Goldbach in the year 1729 (Refer to the correspondence reproduced by (Luschny, 2012)). (Dutka, 1991) gave an account of the early history of the factorial function. (Ibrahim, 2013; Thukral, 2014) defined the factorial of negative integers until a certain extent.

3. THE ORIGIN OF THOUGHT

3.1 Realizing the idea of LIMITING approach with the aid of a^0 proof

In mathematics, the factorial of a non-negative integer n, denoted by $n!$, is the product of all positive integers less than or equal to n.

For example : By definition of factorial ;

$$n!=n(n-1)(n-2).....1 \qquad (1)$$

The value of 0! is 1, according to the convention for a an empty product. It all began with the proof of $a^0 = 1$, where a is in any real number = 0.

This is somewhat similar to a *LIMITING* approach towards $a^0 = 1$, where we try to approach from both the sides towards a^0 by symmetric mathematical operations. i.e. here we multiply by a for negative powers of a and divide by a for positive powers ,where powers from positive and negative numbers tend to reach 0.

3.2 Analogy to Factorial

Now this proof has made me wonder and inspired to approach the proof of 0! in a similar manner.

$$n! = n(n-1)(n-2)\ldots\ldots 1$$

Here we have divided the subsequent factorial value by the factorial number in a descending way and approach towards 0 and hence 0! is found equal to 1.

Now just like a^0 proof what if we try to simulate the existence of factorial of non-negative numbers using the LIMITING approach in such a way that they seem to approach 0!.

$2^{-3} = 1/8$ ↘ multiply by 2

$2^{-2} = 1/4$ ↘ multiply by 2

$2^{-1} = 1/2$ ↘ multiply by 2

$2^0 = ? = 1$

$2^1 = 2$ ↗ divide by 2

$2^2 = 4$ ↗ divide by 2

$2^3 = 8$ ↗ divide by 2

Figure 1: LIMITING approach through the proof of a^0

0! = ? = 1

1! = 1 ↗ divide 1! by 1

2! = 2x1 = 2 ↗ divide 2! by 2

3! = 3x2x1 = 6 ↗ divide 3! by 3

4! = 4x3x2x1 = 24 ↗ divide 4! by 4

5! = 5x4x3x2x1 = 120 ↗ divide n! by n i.e by 5! By 5 here

Figure 2: LIMITING approach to prove 0!

3.3 Failure of Gamma Function extension to negative numbers

By referring to gamma function (Borwein & Corless, 2018) and (Gautschi, 2008) we have $\Gamma(n) = (n - 1)!$,for n >30,where $n \in Z$. Legendre in 1808 gave the notation to the Euler's gamma function (Gautschi, 2008).

It is clear by definition that gamma function fails for non-positive numbers and it value tends to infinity (Lefort, 2002; Lexa, 2002).

3.4 Redefining Gamma Function

But what if we increase the values from a given negative number and make it approach 0!, this is just opposite to what we do with factorials of positive numbers , just like we decrease positive numbers by 1 we increase the negative numbers by 1.

This could potentially lead to some new definition for extension of factorials to negative numbers.

This is also just like the limiting approach where we tend to reach 0! From positive side by decreasing values ,and for the negative side in order to show such similar behaviour , the only thing we could do to them make approach 0! is to increase them i.e add 1 rather subtracting 1 from them just as the gamma function does which fails to halt.

For example , Lets us try to attempt to increase the negative numbers from a given number to approach 0! ,

$$(-5)! = -5 \cdot -4 \cdot -3 \cdot -2 \cdot -1 = -120$$

Here we stop at -1 ,as extending further is of no use as our primary objective was to just approach until 0! .

$$(-4)! = -4 \cdot -3 \cdot -2 \cdot -1 = +24$$

$$(-3)! = -3 \cdot -2 \cdot -1 = -6$$

$$(-2)! = -2 \cdot -1 = 2$$

$$(-1)! = -1$$

By practical observation and rom the approach that we have decided we can formulate this behaviour as

$$(-n)! = (0-n)\cdot(1-n)\cdot(2-n)\cdot \dots\dots\dots\dots\dots\dots - 1.$$

or simplifying this we get

$$(-n)! = (-1)^n \cdot n!.$$

Here this formula completely justifies our vision of increasing the negative numbers to approach 0!.Just like the positive numbers decrease themselves to approach 0!.

One can also compare the functional values of n! and (n)! and see that the values are same in magnitude but differ in their sign alternatively, i.e. neg- ative factorial of odd numbers bear the opposite sing as that of the factorial of positive odd numbers ,but the negative factorial of even numbers are same as that of factorials of even numbers. Among the other well defined functions for the factorials of real negative numbers are, Hadamard's gamma function (Davis, 1959) and Luschny's factorial function (Luschny, 2012), both of which are continuous and positive at all real numbers.

Now this how the LIMITING proof of 0! looks as shown in Figure 3.

(-5)! = -5 x -4 x -3 x -2 x -1 = -120 ↘ dividing by (-n)! in (-n) i.e by -5 here we get

(-4)! = -4 x -3 x -2 x -1 = +24 ↘ divide (-4)! by -4

(-3)! = -3 x -2 x -1 = -6 ↘ divide (-3)! by -3

(-2)! = -2 x -1 = 2 ↘ divide (-2)! by -2

(-1)! = -1 ↘ divide (-1)! by -1

0! = ? = 1 hence approaching from both sides we get the same value for 0!

1! = 1 ↗ divide 1! by 1

2! = 2x1 = 2 ↗ divide 2! by 2

3! = 3x2x1 = 6 ↗ divide 3! by 3

4! = 4x3x2x1 = 24 ↗ divide 4! by 4

5! = 5x4x3x2x1 = 120 ↗ divide n! by n i i.e divide 5! By 5 here

Figure 3: Limiting approach to prove 0! = 1

3.5 Conclusion

The results as shown in Figure 3 support our formula by showing a symmetrical approach (just like a^0 proof) towards 0!.

3.6 Graphical Analysis

while searching for the existence or validation of the negative factorial by some real world function(existing in present without any ambiguity) ,the discrete valued graph of the function $f(t) = e^{-at} \cdot \sin(bt)$ (Oldham, Myland, & Spanier, 2010)seems to have some similarity with the graph of $(-n)!$.

The results of the following graphs plotted using MATLAB show some encouraging results.

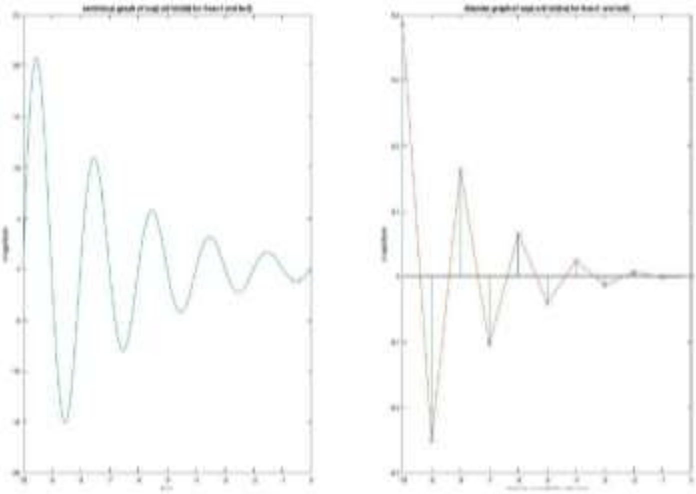

Figure 4: Graph of $f(t) = e^{-at} \cdot \sin(bt)$ in discrete and continuous domain

Now let us compare this with the discrete graph of $n!$ according to out our definition that we formulated.
One can clearly observe that the variation of the the discretegraph of the function and negative factorial is similarand only vary in amplitude and can be easily corrected to match the value of negative factorial at that point.

4.1 Findiing the values of the constants *a,b*

4.1.1 Trial and Error method

But for what values of the constants in the function (a, b, A) would this be possible and what should be nature of the amplitude, constant or variable? well first let’s see for what values of a and b the function varies in the similar manner to negative factorial
In MATLAB I plotted the graph of the function for negative values of t(for the function) and n (for its discrete form).
The graphs for $a, b > 0$ are as follows:

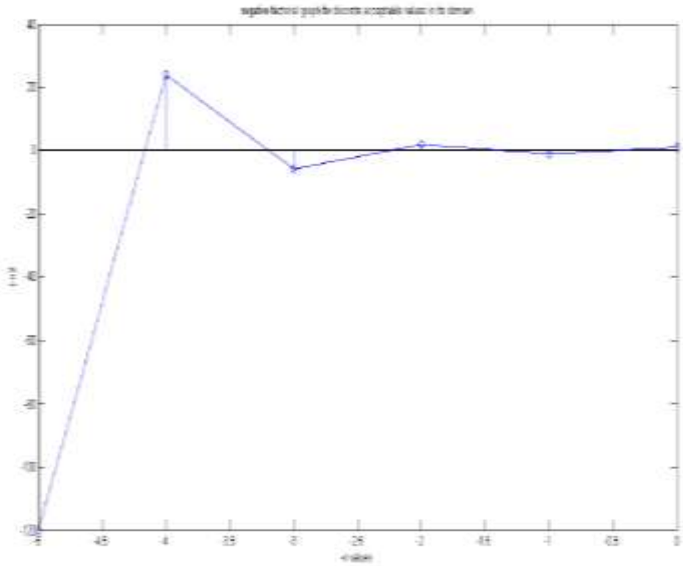

Figure 5: negative factorial graph for negative acceptable values in its domain

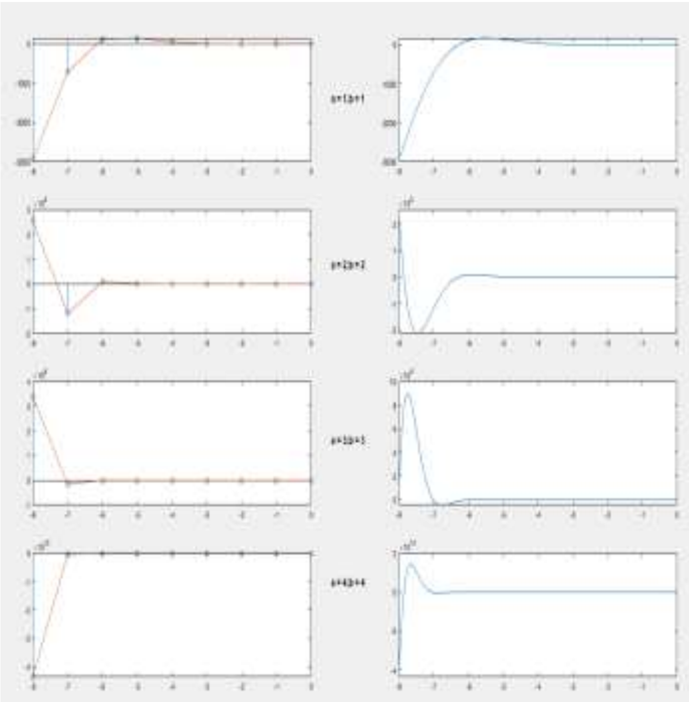

Figure 6: graphs of $f(t)$ for $a, b > 0$

well these graphs does not seem similar to negative factorial graphs. Hence next I choose the values of a as$0 < a < 1$ and $b > 0$, now the results seem promising

I kept changing the values of a from 0.1 to 0.4 and b value from 1 to 4 but the right combination was a=0.3 and b = 3 that provided the right frequency as that of the negative factorial graph.

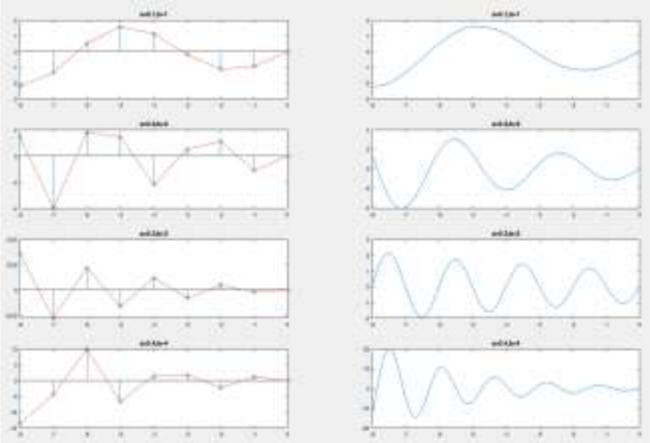

Figure 7: graphs of $f(t)$ for $0 < a < 1$ and $b > 0$

close experimentation and multiple trials has found that the constants de- pend on pi i.e. a = (pi - (1/pi2) /10) = 0.30403 and b = pi - (1/pi2) =3.0403.

4.1.2 Interpolating the amplitude

Now coming to amplitude ,it can be chosen in such a way that the function value attains the value of negative factorial at that particular value of -n .

Since we know the value of (-n)! we can equate it to the function and find for what value of amplitude the function can be equal to the value of negative factorial(Loeb, 1995) at that point. since we have An*e-an *sin(bn) = (-n)!

Hence An = mod((-n)!/e-an*sin(bn)) ,where mod indicates the modulus function i.e mod(x) = x and mod(-x) = x.

Here is a table showing the value of An vs n to help the function reach the value of (-n)! All of these A values were found using he expression An = mod(n!/e-an*sin(bn)) and n = (-n). Where mod is the modulus function.

Now let us choose a value of A and plot the graph to compare with negative factorial graph. Using the following code in MATLAB we can plot the graph upto n = -5 and choosing the appropriate A.

4.2 MATLAB code for comparing the function and −n!

n	mod(A_n)
-1	7.295
-2	5.4101
-3	8.0524
-4	18.04085
-5	54.08325
-6	203.3974
-7	921.33
-8	4887.4
-9	29746.4567

Figure 8: N vs Amplitude

```
b2 = 5:-1:0;\
b1 = ((-1).^b2).*factorial(b2);\
\
a = (pi - (1/(pi*pi)))/10;\
b = pi - (1/(pi*pi));\
\
%here we plot the graph for A5 (fixed amplitude)\
z = -5;\
z1 = exp(-a*z);\
z2 = sin(b*z);\
z3 = abs(z1.*z2); \
A = factorial(-z)/z3;\
\
n = -5:0;\
n1 = exp(-a*n);\
n2 = A.*sin(b*n);\
n3 = n1.*n2;\
stem(n,b1);\
hold on\
stem(n,n3);\
hold on\
\
t = -5:0.01:0;\
y = A.*sin(b*t);\
y2 = exp(-a*t);\
y3 = y2.* y;\
plot(t,y3);\
\
\
hold on\
plot(n,b1,'r',n,n3,'g',t,y3,'b');\
}
```

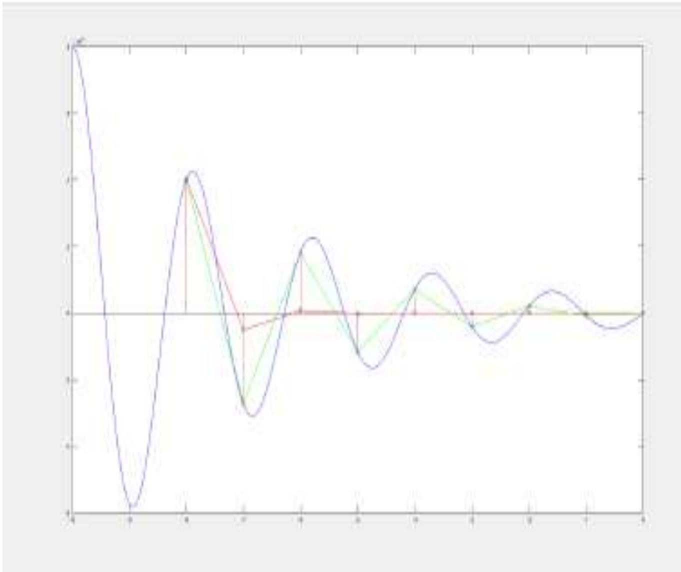

Figure 9: N Fact Comparision with function

The blue curve represents the function f(n) in a continuous domain and the green impulses are the discrete mappings of the negative factorial and red im- pulses represent the discrete form of the continuous function f(t).

We can observe that at n = -5 both function value and negative factorial value are equal this happened because of the value of the amplitude A that we have chosen but we can also see that the function does not equate to the values of negative factorial at points n = -4,-3,-2 and -1.

Hence to make all the values of negative factorial lie on the curve f(t) we can use the amplitudes that would make this

possible at their respective -n and use Lagrange’s interpolation to find A as a function of n ,say A(n) where it satisfies A(-1) = mod(A-1) ,A(-2) = mod(A-2),A(-3) = mod(A-3), ... and A(-n) =
mod(A-n) .

4.3 Interpolating and plotting the graphs

I plotted the graph by finding A as a polynomial of n and satisfying A(i) for I = - 1,-2,-3,-4,-5,-6 in MATLAB and we can see that all the values of -n! for n¡= -6 lie on the curve .

The code for this is below which takes input from user to plot the graphs from the above rules to plot up-to a certain -n. It represents all the 3 graphs (f(t) its discrete(green) , continuous form(blue,and the negative factorial discrete plot(red impulses))

MATLAB code for Interpolating and plotting the graphs

```
clear all;
close all;
clc;
number = input('enter the number :');
b2 = -number:-1:0;
b1 = ((-1).^b2).*factorial(b2);

a = (pi - (1/(pi*pi)))/10;
b = pi - (1/(pi*pi));
z = -1:-1:number;
z1 = exp(-a*z);
z2 = sin(b*z);
z3 = abs(z1.*z2);
for i = 1:-number
    z4(i) = factorial(i)/z3(i);

end
x=1:1:-number;
y=z4;
sum=0;
for i=1:length(x)
    p=1;
    for j=1:length(x)
        if j~=i
            c = poly(x(j))/(x(i)-x(j));
            p = conv(p,c);
        end
    end
    term = p*y(i);
    sum= sum + term;
end
disp(sum);
poly = poly2sym(sym(sum));
```

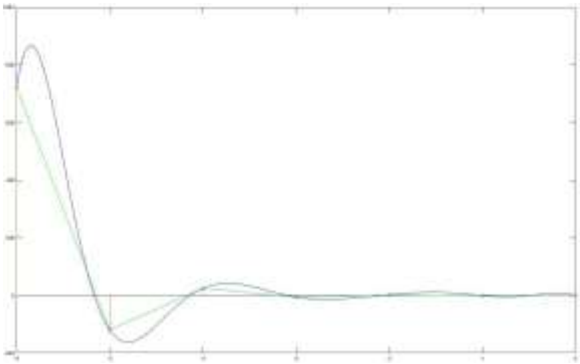

Figure 10: Interpolated negative Factorial

Now we can clearly see that all the values of negative factorial perfectly lie on the curve ,we can clearly say that we have found a perfect function equivalent for the negative factorial that we defined according for their existence.

CONCLUSION

As we have seen from the above discussions and proof that negative factorials ex- ist and show very mystical behaviour both in their mathematical and graphical analysis.
Negative factorial of a number n can be defined as the product of numbers greater than -n up to -1 . We have seen that factorials of non-negative and non-positive numbers have same values with different signs.
As we have also seen the graph on -n vs (-n)! Is found to be damped oscillatory motion.
Hence the graph of negative factorial is similar to that of the function f (t) = e at A sin(bt) for particular values of a, b, A i.e. a = (pi - (1/pi2) /10)
= 0.30403 and b = pi - (1/pi2) = 3.0403. and A is a Lagrange’s interpolated polynomial A(n) of degree n-1 found using -1,-2,-3. . . ..-n as x values and A(-1)
= mod(A-1) ,A(-2) = mod(A-2),A(-3) = mod(A-3), and A(-n) = mod(A-n) as the corresponding y values where An = mod((-n)!/e-an*sin(bn)) ,where mod indicates the modulus function i.e mod(x) = x and mod(-x) = x and n =-1,-2,-3,,,-n.
Hence we can clearly state that negative factorials exist (according to the definition that we have given) and are found to be equal to a function f(t) = e-at * A*sin(bt) for particular values of a, b, A.
We have also understood how to think or develop a thought process to arrive at the existence of factorials for negative numbers at the same time understating the proof of 0! And a0 = 1 ,a is any real number not equal 0 though the idea of limiting approach towards a number from either sides

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