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Root of a Transcendental Equation: Geometric View of Taylor's Approximation

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ABSTRACT

Newton-Raphson method and iteration method are widely used to solve non-algebraic or transcendental equation. In this paper we use first three terms of Taylor's series to find the equivalent quadratic equation. Solving this quadratic equation we can easily find an iterative formula for the solution which gives better approximation than that of Newton-Raphson method. Here we present comparison of the roots and its convergency in geometric view.

Keywords: Transcendental equations; Newton-Raphson method; quadratic equation; Taylor's series.

INTRODUCTION

There are several types of transcendental equations containing trigonometric, algebraic, exponential, logarithmic, etc. terms. Not all the cases transcendental equations have analytic solutions. To find the analytical solution in closed form of some families of transcendental equations was studied in [1]. The mathematical origin of the analytical closed-form solution was offered there. A special technique with clamped cubic spline formula was used to find a root of a transcendental equation in [2] whereas the transformation of a transcendental equation into an equivalent quadratic form and then solved in [3]. In the paper [4] authors shows a general interval analysis method ensures existence and uniqueness while simultaneously providing error bounds. The paper [5] presents a mixture of global iterative methods, based on the Fatou-Julia theory, and local methods to find selected roots of simple transcendental equations generated in complex Sturm-Liouville eigenvalue problems.

This work is based on the simple idea of quadratic approximation. The essential part of this method is to suppose that x_0 is an approximate root of $f(x)=0$ and to expand $f(x)$ with Taylor's series about $x=x_0$ [6]. After simplification, we can form a quadratic polynomial $g(x)$ which is a representative of $f(x)$. A solution of $g(x)=0$ will be an approximate root of $f(x)=0$. We can establish an algorithm whose iterations give better approximation than that of Newton-Raphson method for the root of $f(x)=0$. Here the procedure is interpreted geometrically as well as numerically along with Newton-Raphson method.

FORMULATION

To find the equivalent quadratic expression of $f(x)$ we use first three terms of Taylor's series:

$$g(x)= f(x_0)+(x-x_0)f'(x_0)+\frac{(x-x_0)^2}{2!}f''(x_0) \quad (1)$$

The simplified form of (1) is

$$g(x)=\frac{f''(x_0)}{2}x^2+\{f'(x_0)-x_0f''(x_0)\}x+\left\{f(x_0)-x_0f'(x_0)+\frac{x_0^2}{2}f''(x_0)\right\} \quad (2)$$

So from (2) we get the quadratic equation

$$\frac{f''(x_0)}{2}x^2+\{f'(x_0)-x_0f''(x_0)\}x+\left\{f(x_0)-x_0f'(x_0)+\frac{x_0^2}{2}f''(x_0)\right\}=0 \quad (3)$$

Solving the quadratic equation (3) we have the solutions

$$x=\frac{-f'(x_0)+x_0f''(x_0)+\sqrt{(f'(x_0))^2-2f(x_0)f''(x_0)}}{f''(x_0)} \quad (4)$$

and

$$x=\frac{-f'(x_0)+x_0f''(x_0)-\sqrt{(f'(x_0))^2-2f(x_0)f''(x_0)}}{f''(x_0)} \quad (5)$$

One (check for which $|f(x)|$ is smaller) of the above formulas will be act as the iterative formula for the working rule.

Algorithm

To find a solution to $f(x)=0$ given the continuous function f: [7]

INPUT; tolerance TOL; maximum number of iterations m

OUTPUT approximate solution x_0 or message of failure

Step 1 Set $i=1$

Step 2 while $i\le$ m do Step 3 to Step 7

Step 3 $P_1=-f'(x_0)+x_0f''(x_0)$

$P_2=\sqrt{(f'(x_0))^2-2f(x_0)f''(x_0)}$

Step 4 Set $x_1=\frac{P_1+P_2}{f''(x)}$

and $x_2=\frac{P_1-P_2}{f''(x)}$

Step 5 If $|f(x_1)|<|f(x_2)|$ then set $p=x_1$ else set $p=x_2$

Step 6 If $|x_0-p|<TOL$ then

OUTPUT (p) (Procedure completed successfully)

STOP

Step 7 Set $i=i+1$

Step 8 OUTPUT (Method failed after m iterations)

(Procedure completed unsuccessfully)

STOP

Example 1

To find a zero of [8]

$$f(x) = e^x + 2^{-x} + \cos x - 6 \qquad (6)$$

we use a trial solution $x_0 = 1$ and tolerance 0.0001.

Using $x_0 = 1$ in equation (3) we get the equivalent quadratic equation of $f(x)$ as

$$0.938952\,x^2 - 1.18914\,x - 1.45093 = 0 \qquad (7)$$

Solving equation (7) we get the values $x_1 = -0.761851$ and $x_2 = 2.0283$. Since $f(x_1) = -2.39041$ and $f(x_2) = 0.962877$ that is $|f(x_1)| > |f(x_2)|$.

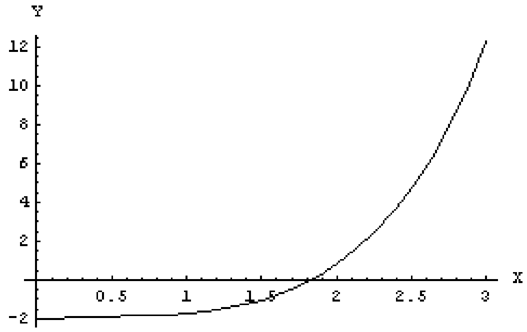


Figure 1: Graph of $f(x)$

So we conclude that the second approximation is nearer to a zero of the function (6). So we replace the value $x_2 = 2.0283$ in place of the value of x_0. Again we use $x_0 = 2.0283$ in equation (3) to get a new quadratic equation and its solutions. Continuing this procedure we can find a root of the transcendental equation (6) up to the desired accuracy.

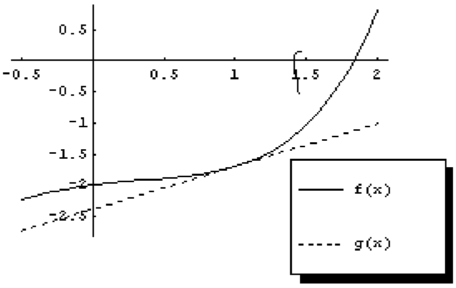
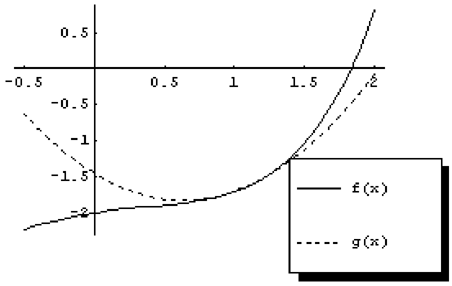
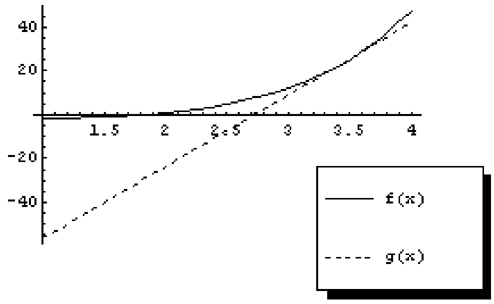
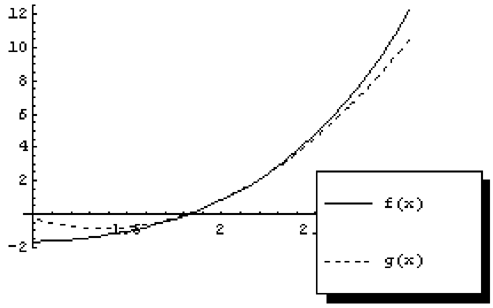
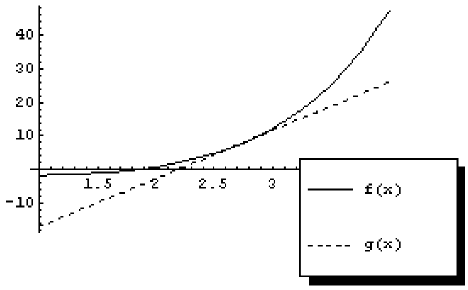
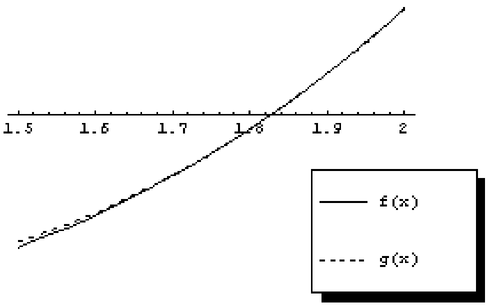
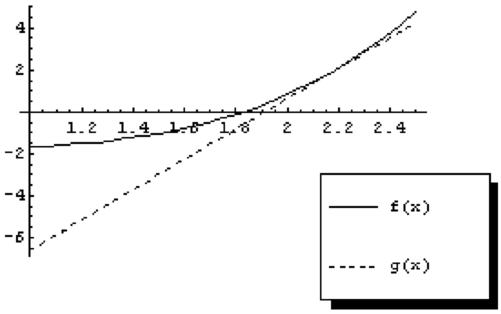
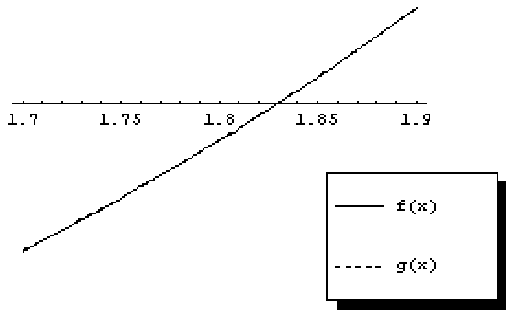
Table 1: Iterative approximations for the present method with trial solution $x = 1$ and tolerance = 0.0001.

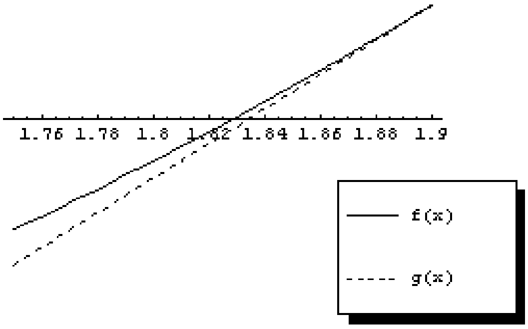
No. of iteration	Present Method	Tolerance
1	2.028302546537848	1.0283025465
2	1.826370289101804	$2.0193225744\times 10^{-1}$
3	1.829383610891708	$3.0133217899\times 10^{-3}$
4	1.829383601933849	$8.9578588991\times 10^{-9}$

Geometric Interpretation

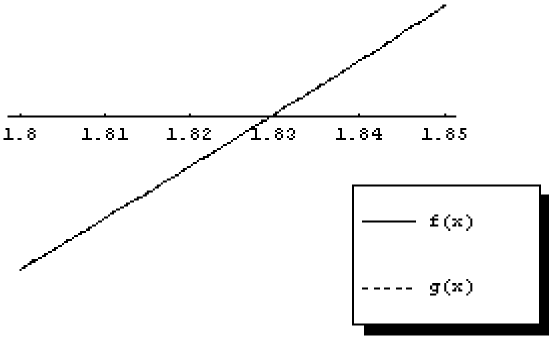
We know that the quadratic equation represents a parabola while the coefficient of x^2 is not zero. Actually in this method we draw a parabola passing through the point $(x_0, f(x_0))$ which cuts a point from x-axis and we can find again the new approximation and so on. The following figures are the iterations of Newton-Raphson's showing method and our present method with comparison.

Table 2: Geometric comparison of Newton-Raphson's method with the present method.

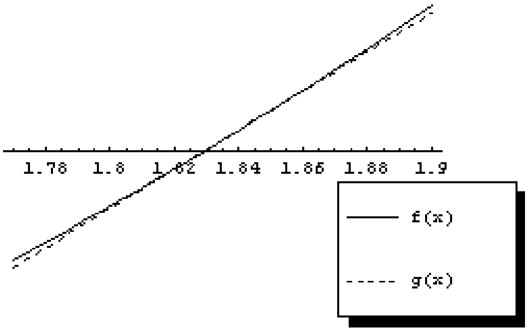
Newton-Raphson's Method	Present Method
 Initial approximation	 Initial approximation
 First iteration	 First iteration
 Second iteration	 Second iteration
 Third iteration	 Third iteration



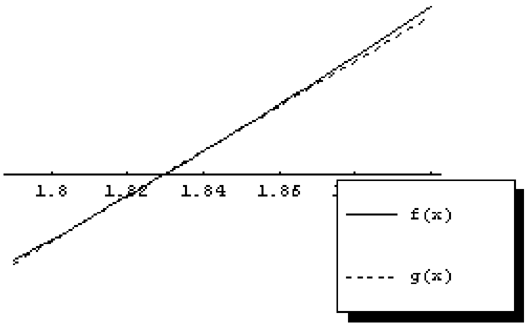
Fourth iteration



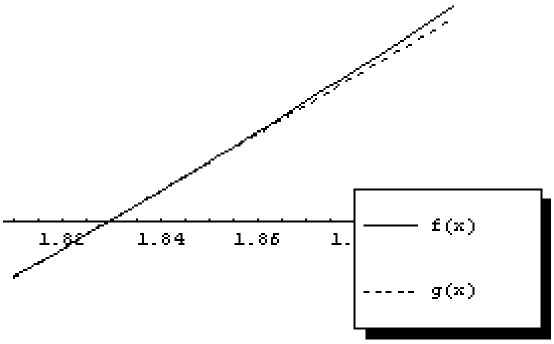
Fourth iteration



Fifth iteration



Sixth iteration



Seventh iteration

Numerical Comparison

Table 3: Iterative approximations for Newton-Raphson's and the present method with trial solution $x_0 = 1$ and tolerance 0.0001.

No. of iteration	Newton-Raphson's Method	Tolerance	Present Method	Tolerance
1	3.4697980105110	2.4697980105	2.0283025465379	1.0283025465
2	2.7261264691777	$7.4367154133\times10^{-1}$	1.8263702891018	$2.0193225744\times10^{-1}$
3	2.1972944842253	$5.2883198495\times10^{-1}$	1.8293836108917	$3.0133217899\times10^{-3}$
4	1.9142730842143	$2.8302140001\times10^{-1}$	1.8293836019338	$8.9578588991\times10^{-9}$
5	1.8349957966534	$7.9277287561\times10^{-2}$		
6	1.8294098740816	$5.5859225718\times10^{-3}$		
7	1.8293836025125	$2.6271569115\times10^{-5}$		

Here we observe that Newton-Raphson's method takes 7 iterations but our present method takes only 4 iterations to reach the result with desired tolerance.

Example 2

To find a zero of

$$f(x) = e^x + 2x^2 - 3 \quad (8)$$

we use a trial solution $x_0 = -0.5$ and tolerance 0.00001.

Table 4: Iterative approximations for Newton-Raphson's and the present method with trial solution $x_0 = -0.5$ and tolerance 0.00001.

No. of iteration	Newton-Raphson's Method	Tolerance	Present Method	Tolerance
1	−0.8588166495983	1.3588166496	−1.1533181985823	$6.5331819858\times10^{-1}$
2	−1.3002197540611	$5.5859689554\times10^{-1}$	−1.1589162814825	$5.5980829002\times10^{-3}$
3	−1.1675978514443	$1.3262190262\times10^{-1}$	−1.1589162836146	$2.1321162613\times10^{-9}$
4	−1.1589535598540	$8.6442915903\times10^{-3}$		
5	−1.1589162843081	$3.7275545886\times10^{-5}$		
6	−1.1589162836146	$6.9344374687\times10^{-10}$		

Here we also observe that Newton-Raphson's method takes 6 iterations but our present method takes only 3 iterations to reach the result with desired tolerance.

CONCLUSION

The quadratic equation approximation provides better approximation than the Newton-Raphson's method for any transcendental equation. Actually, Newton-Raphson's method works with straight line but the present quadratic approximation method works with a parabola (a bending line) so that the root is achieved earlier up to the desired accuracy.

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