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On Conformal Transformation of Lagrange Space with (γ, β)-Metric

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Abstract

The present paper is a study of the conformal transformation of the Lagrange space with (γ, β)-metric.The conformal transformation of the spray coefficient and Riemann curvature are express in Lagrange space with (γ, β)-metric. Further find out the condition that a conformal transformation of Lagrange space with (γ, β)-metric is locally dually flat if and only if the transformation is a homothety.Moreover, the conditions for the transform metrics to be Einstein and isotropic mean Berwald curvature are also find.

Key words: Conformal transformation, Homothety, Lagrange space, Locally dually flat, (γ, β)-metric.
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1 Introduction

M. Matsumoto has been studied the non-Riemannian Finsler space with a cubic metric in 1979 [4]. A cubic metric is defined as,[7]

$$L(x, y) = \left\{ a_{ijk}(x) y^i y^j y^k \right\}^{\frac{1}{3}}. \tag{1.1}$$

where $a_{ijk}(x)$ are components of a symmetric tensor field of $(0,3)$-type depending on the position x alone.
The β metric defined as,

$$\beta(x,y) = b_i(x)y^i, \tag{1.2}$$

where $b_i(x)$ are components of a covariant vector in space.

Pandey and Chaubey introduced the concept of (γ,β)-metric In 2011, where $\gamma = \left\{a_{ijk}(x)y^iy^jy^k\right\}^{\frac{1}{3}}$ is cubic metric and $\beta = b_i(x)y^i$ is a one-form. P.N. Pandey and S.K. Shukla studied Lagrange space with (γ,β) metric and obtained metric tensor in L^n Lagrange space. In 2006 L. Tamassy studied the relation between Lagrange and Finsler space and established that the Hessians $H_{ij}(x,y)$ are positive definite if and only if the hyper-surface $z^{I(x)}$ are convex.

Matsomoto has been developed the theory of Finsler spaces [9] and the theory of conformal transformation of Finsler spaces has been developed by M. Hashiguchi[10] based on the theory of Finsler space. Let F and $\overline{F}$ be two Finsler metrics on manifold M^n then we say that Finsler metric $\overline{F}$ conformally transformed Finsler metric[11] if $\overline{F} = e^{\alpha(x)}F$ where α is a scalar function on M^n. The conformal change is said to be a homothety if α is a constant.

2 Preliminaries

Let M be an n-dimensional smooth manifold and let TM be its tangent bundle. Let (x^i) and (x^i,y^i) be local coordinates on M and TM, respectively. A Lagrangian is a function $L : TM \to R$ which is a smooth function on $\tilde{TM} = TM \setminus \{0\}$ and continuous on the null section.

The fundamental metric tensor of Lagrangian $L(x,y)$ is given by $g_{ij}(x,y)$ and define as,

$$g_{ij}(x,y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2, \tag{2.1}$$

where $\dot{\partial}_i = \frac{\partial}{\partial y^i}$.

Definition 2.1. A Lagrange space is a pair $L^n = (M, L(x,y))$. The metric tensor g_{ij} of Lagrangian $L(x,y)$ being a constant signature on $\tilde{TM}$ is called a regular Lagrangian.

In the present paper, we study a Lagrange space whose Lagrangian L is a function of $\gamma(x,y)$ and $\beta(x,y)$ only, where

$$\begin{cases} a)\ \gamma^3(x,y) = a_{ijk}(x,y)y^iy^jy^k \\ b)\ \beta(x,y) = b_i(x)y^i. \end{cases} \tag{2.2}$$

Let us denote the conformal transform Lagrangian by $\overline{L}$. Thus the conformal transform Lagrange metric given by

$$\overline{L}(\gamma, \beta) = e^{\alpha(x)} L(x, y). \tag{2.3}$$

The space $L^n = (M, L(x, y))$ is called space with (γ, β)-metric [2]. The fundamental metric [1] of $L^n = (M, L(x, y))$ defined as,

$$g_{ij}(x, y) = 2\rho a_{ij} + \rho_{-2} a_i a_j + \rho_{-1}(a_i b_j + b_i a_j) + \rho_0 b_i b_j. \tag{2.4}$$

where ρ, $\rho_0$and ρ_{-1}, ρ_{-2} given as

$$\begin{cases} a)\ \rho = \frac{1}{2}\gamma^{-}2L_{\gamma}, \\ b)\ \rho_0 = \frac{1}{2}L_{\beta\beta}, \\ c)\ \rho_{-1} = \frac{1}{2}\gamma^{-2}L_{\gamma\beta}, \\ d)\ \rho_{-2} = \frac{1}{2}\gamma^{-4}(L_{\gamma\gamma} - 2\gamma^{-1}L_{\gamma}). \end{cases} \tag{2.5}$$

The normalize supporting element l_i and angular metric tensor h_{ij} of L are defined respectively as: $l_i = \frac{\partial L}{\partial y^i}$, $h_{ij} = L\frac{\partial^2 L^2}{\partial y^i \partial y^j}$.

Definition 2.2. A Finsler metric $L(x, y)$ is called a (γ, β)-metric. If L is a positive homogeneous function of first degree in two variable γ and β . Where $\gamma^3 = a_{ijk}y^i y^j y^k$ is a cubic metric and $\beta = b_i(x)y^i$ is a one form.

Definition 2.3. The Lagrangian metric $L(\gamma, \beta)$-metric will be Finsler metric if following conditions hold [8]

$$p_{-1} + q_{-2}\beta + q_{-1}\gamma^3 = 0, \tag{2.6}$$

$$q_0\beta + q_{-2}\gamma^3 = 0, \tag{2.7}$$

where $p_{-1} = \frac{LL_{\gamma}}{2}$, $q_0 = LL_{\beta\beta}$, $q_{-2} = \frac{LL_{\gamma\beta}}{\gamma^2}$, $q_{-1} = \frac{L}{\gamma^4}(L_{\gamma\gamma} - \frac{2L_{\gamma}}{\gamma})$.

The subscripts of coefficient p_{-1}, q_0, q_{-2}, q_{-1} indicate respectively degree of homogeneity.

3 Conformal transformation Lagrange space with (γ, β)-metric

The fundamental metric tensor g_{ij} of Lagrange space L^n is given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j} = LL_{y^i y^j} - L_{y^i}L_{y^j}. \tag{3.1}$$

From equation (2.4) the metric tensor g_{ij} simplifies as $g_{ij} = 2\rho a_{ij} + c_i c_j$.The conformal transform metric $\overline{g}_{ij}$ is derived as:

$$\overline{g}_{ij} = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j \overline{L}^2 = e^{2\alpha} g_{ij}. \tag{3.2}$$

It is also derived that

$$\overline{g}^{ij} = e^{-2\alpha} g^{ij}. \tag{3.3}$$

Theorem 3.1. *The covariant metric tensor* $\overline{g}_{ij}$ *and contravariant metric tensor* $\overline{g}^{ij}$ *of transformed Lagrange space* L^n *with* (γ, β)*-metric are given as*

$$\overline{g}_{ij} = e^{2\alpha} g_{ij} = e^{2\alpha} \left(2\rho a_{ij} + c_i c_j\right), \tag{3.4}$$

$$\overline{g}^{ij} = e^{-2\alpha} g^{mh} = e^{-2\alpha} \frac{1}{2\rho} \left(a^{mh} - \frac{1}{2\rho + c^2} c^m c^h \right). \tag{3.5}$$

The transformed Christoffel symbols of the first kind written for (γ, β)-metric as,

$$[jk, h] = 2e^{2\alpha} \left\{ \alpha_j g_{kh} + \alpha_k g_{jh} - \alpha_h g_{jk} \right\} + e^{2\alpha} [jk, h], \tag{3.6}$$

Hence the conformal transformed Chirstoffel's symbols connected with metric tensor as,

$$\overline{\begin{Bmatrix} m \\ jk \end{Bmatrix}} = \begin{Bmatrix} m \\ jk \end{Bmatrix} + \left\{ \alpha_j \delta_k^m + \alpha_k \delta_j^m - \alpha_h g^{mh} g_{jk} \right\}, \tag{3.7}$$

The index $m \to i$ and h is dummy index so take $h = k$. Then the spray coefficient G^i conformal transformed following equation,

$$\overline{G}^i = \alpha_j + G^i. \tag{3.8}$$

If $\alpha_j = 0$, then $\frac{\partial \alpha}{\partial x^j} = 0 \Rightarrow \alpha = constant$.

Theorem 3.2. *Let* L *and* $\overline{L}$ *conformally related metrics in Lagrange space with* (γ, β)*-metric. Then the geodesic curves are identical if and only if a conformal change is a homothety.*

Proof. The geodesic equation in Lagrange space is given by

$$\frac{d^2 x^i}{ds^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0. \tag{3.9}$$

Let conformal change is homothety, then $\alpha = constant$, hence $\frac{\partial\alpha}{\partial x^j} = 0$. The equation (3.8) gives that the spray coefficient $\overline{G}^i = G^i$, so Geodesic equation (3.9) is identical after transformation.So the geodesic curve also will be identical. □

The conformal transform spray G expression is given by

$$\overline{G} = G - 2\alpha_j e^{\alpha}\left(L_\gamma \gamma y^i + L_\beta b_i\right). \tag{3.10}$$

4 Locally dually flat conformally transformed in Lagrange space with (γ,β)-metric

In the study of information geometry on Riemannian manifolds, Amari-Nagaoka developed the notion of dually flat Riemannian metrics[12]. Locally dually flatness for Finsler metrics notion developed by Shen [13].

A transform Finsler metric $\overline{F}$ on a manifold M^n is said to be locally dually flat if $\left[\overline{F}^2\right]_{x^k y^l} y^k = 2\left[\overline{F}\right]_{x^l}$ at any point with the coordinate system (x^i, y^i) in TM.

Let the conformal transformation $\overline{L} = e^{\alpha}L$, where L is Lagrange metric with (γ,β)-metric.Since $\overline{L}^2_{x^k} = e^{2\alpha}\left[L^2_{x^k} + 2L^2\alpha_k\right]$, where $\alpha_k = \frac{\partial\alpha}{\partial x^k}$, we have $\overline{L}^2_{x^k y^l} y^k = e^{2\alpha}\left[L^2_{x^k y^l} y^k + 2L l_l \alpha_k y^k\right]$.
Hence

$$\begin{aligned} L^2_{x^k y^l} y^k &= 2L^2_\gamma \frac{a_{ij}(y^i y^j y^k)^2}{3\gamma^2} A_k + 2L_\gamma L_\beta b_l A_k y^i y^j (y^k)^2 + 2LL_{\gamma\gamma} \\ &\times \frac{a_{ij}(y^i y^j y^k)^2}{3\gamma^2} A_k + 2LL_{\gamma\beta} b_l A_k y^i y^j (y^k)^2 - \frac{4}{3\gamma^3} A_k (y^i y^j y^k)^2 y^k \\ &+ 2L_\gamma L_\beta \frac{a_{ij}(y^i y^j y^k)^2}{3\gamma^2} + 2L^2_\beta b_l B_k (y^k)^2 + 2LL_\beta \frac{a_{ij} y^i y^j (y^k)^2}{3\gamma^2} B_k \\ &+ 2LL_{\beta\beta} b_l B_k (y^k)^2 + 2LL_\beta B_l y^k, \end{aligned} \tag{4.1}$$

$$2L^2_{x^l} = 4LL_\gamma y^i y^j y^k A_l + 4LL_\beta B_l y^l, \tag{4.2}$$

$$2LL_{y^l}\alpha_k y^k = \left(\frac{2}{3\gamma^2} LL_\gamma a_{ij} y^i y^j y^k + 2LL_\beta b_l y^k\right)\alpha_k. \tag{4.3}$$

We have

$$2\overline{L}^2_{x^l} - \overline{L}^2_{x^k y^l} y^k = e^{2\alpha}\left[2L^2_{x^l} + 4L^2\alpha_l - L^2_{x^k y^l} y^k - 2Ll_l\alpha_k y^k\right]. \tag{4.4}$$

If Lagrange space is locally dually flat then $2L^2_{x^l} - L^2_{x^k y^l} y^k = 0$.Using equation (4.1),(4.2) and (4.3) in equation (4.4) we find out that,

$$2\overline{L}^2_{x^l} - \overline{L}^2_{x^k y^l} y^k = 4L^2\alpha_l - \left(\frac{2}{3\gamma^2} LL_\gamma a_{ij} y^i y^j + 2LL_\beta b_l\right)\alpha_k y^k. \tag{4.5}$$

Where terms used in equation (4.1) to (4.4) are defined in the following way,

1. $\frac{\partial\gamma}{\partial x^k} = A_k y^i y^j y^k$,
2. $\frac{\partial\beta}{\partial x^k} = B_k y^k$,
3. $\frac{\partial^2\beta}{\partial y^l \partial x^l} = B_l$,
4. $\frac{\partial\gamma}{\partial y^l} = \frac{1}{3\gamma^2} a_{ij} y^i y^j$.

Theorem 4.1. *If $\overline{L}$ be a conformal transformed metric with (γ, β)-metric on a manifolds M^n in Lagrange space. Then, $\overline{L}$ is locally dually flat metric if and only if* $2L\alpha_l - \left(\frac{1}{3\gamma^2} L_\gamma a_{ij} y^i y^j + L_\beta b_l\right)\alpha_k y^k = 0$.

Corollary 4.1. *If L is locally dually flat metric then the conformally transformed Lagrangian metric $\overline{L}$ is also locally dually flat if and only if conformal transformation homothetic.*

Proof. Form equation (4.5) the L is locally dually flat if and only $2L\alpha_l - \left(\frac{1}{3\gamma^2} L_\gamma a_{ij} y^i y^j + L_\beta b_l\right)\alpha_k y^k = 0$. Hence $\overline{L}$ is locally dually flat if and only if

$$\alpha_l L - \left(\frac{1}{3\gamma^2} L_\gamma a_{ij} y^i y^j + L_\beta b_l\right)\alpha_0 = 0 \tag{4.6}$$

Contracting equation (4.6) with y^l,.We have

$$\alpha_0 L - \left(\frac{1}{3}\gamma L_\gamma + L_\beta \beta\right)\alpha_0 = 0$$

This gives $\alpha_0 = 0$. Hence from equation (4.6) , $\alpha_l = 0$, i.e. $\frac{\partial\alpha}{\partial x^l} = 0$. So α is constant. Therefore the transformation is homothetc. The converse is also true. □

5 Conformally transformed Lagrangian (γ,β)-metric with isotropic E-curvature

The Berwald curvature of $\overline{L}$ is defined as

$$\overline{B}^i_{jkl} = \frac{\partial^3 \overline{G}^i}{\partial y^j \partial y^k \partial y^l}. \tag{5.1}$$

Where $\overline{G}^i$ are spray coefficients of a Lagrange space $\overline{L}$. The trace of the Berwald curvature is called the E-curvature.So $\overline{E}_{ij} = \frac{1}{2}\overline{B}^m_{mij}$. Let $\overline{L}$ is the Lagrangian metric on an n-dimensional manifold M^n. Then The isotropic mean Berwald curvature or of isotropic E-curvature defined as

$$\overline{E}_{ij} = \frac{c(n+1)}{2\overline{L}}\overline{h}_{ij}. \tag{5.2}$$

where $\overline{h}_{ij} = \overline{g}_{ij} - \overline{l}_i\overline{l}_j$ is the angular metric and $c = c(x)$ is a scalar function on M^n. Now $\overline{L}$ will be weakly Berwald metric if scalar function $c = 0$. In view of equation (3.4) the angular metric is given by

$$\begin{aligned}\overline{h}_{ij} = e^{2\alpha}\Big\{&\rho a_{ij} + \rho_{-2}a_ia_j + \rho_{-1}(a_ib_j + a_jb_i) + \rho_0 b_ib_j - \frac{L_\gamma^2}{9\gamma^4}a_ia_j \\ &- \frac{L_\gamma L_\beta}{3\gamma^2}(a_ib_j + a_jb_i) - L_\beta^2 b_ib_j\Big\}.\end{aligned} \tag{5.3}$$

From equation (5.2) and (5.3), we have

$$\begin{aligned}\overline{E}_{ij} = \frac{(n+1)c}{2\overline{L}}e^{2\alpha}\Big\{&\rho a_{ij} + \rho_{-2}a_ia_j + \rho_{-1}(a_ib_j + a_jb_i) + \rho_0 b_ib_j \\ &- \frac{L_\gamma^2}{9\gamma^4}a_ia_j - \frac{L_\gamma L_\beta}{3\gamma^2}(a_ib_j + a_jb_i) - L_\beta^2 b_ib_j\Big\},\end{aligned} \tag{5.4}$$

After simplification the equation (5.4) we have

$$\begin{aligned}\overline{E}_{ij} = \frac{(n+1)c}{2L}e^{\alpha}\Big\{&\rho a_{ij} + \left(\rho_{-2} - \frac{L_\gamma^2}{9\gamma^4}\right)a_ia_j + \left(\rho_{-1} - \frac{L_\gamma L_\beta}{3\gamma^2}\right) \\ &(a_ib_j + a_jb_i) + \left(\rho_0 - L_\beta^2\right)b_ib_j\Big\}.\end{aligned} \tag{5.5}$$

The equation (5.5) shows that $c = 0$, because neither $e^\alpha = 0$ nor $\rho a_{ij} + \left(\rho_{-2} - \frac{L_\gamma^2}{9\gamma^4}\right)a_ia_j + \left(\rho_{-1} - \frac{L_\gamma L_\beta}{3\gamma^2}\right)(a_ib_j + a_jb_i) + \left(\rho_0 - L_\beta^2\right)b_ib_j = 0$, i.e. $h_{ij} \neq 0$. Hence the isotropic E-curvature $\overline{E}_{ij} = 0$.

Theorem 5.1. *If $\overline{L} = e^cL$ be the conformal change of Lagrangian metric L.Suppose $\overline{L}$ has isotropic mean Berwald curvature. Then it reduced to a weakly Berwald metric.*

6 Conclusion

The article starts with a basic definition of cubic and β metrics with the formulation of Lagrange space with (γ, β)-metrics. The next part of the article gives a conformal change of Lagrangian metrics and locally dually flat change in Lagrange space. There are some results on isotropic E-curvature.

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